Section 3.7. Higher Order Linear Differential Equations

I. BASIC TERMS (See Section 3.1)

An nth order linear differential equation is an equation of the form:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \qquad \text{(L)}$$

where $p_0, \ p_1, \ \cdots, \ p_{n-1}$ and f are continuous functions on some interval I.

(L) is **homogeneous** if $f(x) \equiv 0$ on I:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \qquad \text{(H)}$$

If f is not identically 0 in I, then (L) is **nonhomogeneous**

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \qquad \text{(N)}$$

$$L[y] = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y$$

is a **linear (differential) operator**:

$$L[y_1 + y_2] = L[y_1] + L[y_2]$$

$$L[Cy] = CL[y], \quad c \quad \text{a constant}$$

Equations (H) and (N) can be written

$$L[y] = 0 \qquad \text{(H)}$$

$$L[y] = f(x) \qquad \text{(N)}$$

Existence and Uniqueness Theorem:

Let a be any point on I. Let

$$\alpha_0,\ \alpha_1,\ \cdots,\ \alpha_{n-1}$$

be any n real numbers. The initial-value problem:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \qquad \text{(N)}$$

$$y(a) = \alpha_0,\ y'(a) = \alpha_1, \cdots,\ y^{(n-1)}(a) = \alpha_{n-1}$$

has a **unique** solution.

II. HOMOGENEOUS EQUATIONS (See Section 3.2)

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \qquad \text{(H)}$$

The zero function, $y(x) = 0$ for all $x \in I$, ($y \equiv 0$) is a solution of (H). The zero solution is called the **trivial solution**. Any other solution is a **nontrivial** solution.

The Theorems:

THEOREM 1: If $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of (H), then

$$u(x) = y_1(x) + y_2(x)$$

is also a solution of (H).

The sum of any two solutions of (H) is also a solution of (H). (Some call this property the *superposition principle*).

THEOREM 2: If $y = y(x)$ is a solution of (H) and if C is any real number, then

$$u(x) = Cy(x)$$

is also a solution of (H).

Any constant multiple of a solution of (H) is also a solution of (H).

THEOREM 3: If

$$y_1,\ y_2,\ \cdots, y_k$$

are solutions of (H) and if

$$C_1,\ C_2,\ \cdots,\ C_k$$

are real numbers, then

$$u = C_1 y_1 + C_2 y_2 + \ \cdots\ + C_k y_k$$

is a solution of (H).

Any linear combination of solutions of (H) is a solution of (H).

General Solution of (H)

Let $y_1(x),\ y_2(x),\ \cdots,\ y_n(x)$ be n solutions of (H). Then, for any choice of constants $C_1,\ C_2,\ \cdots,\ C_n$,

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) \qquad \text{(GS)}$$

is a solution of (H).

Under what conditions is (GS) the general solution of (H)?

The Wronskian

Set

$$W(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-2)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** of $y_1,\ y_2,\ \cdots,\ y_n$.

THEOREM 4: Let $y_1(x),\ y_2(x),\ \cdots,\ y_n(x)$ be n solutions of (H) and let $W(x)$ be their Wronskian. Exactly one of the following holds

1. $W(x) \equiv 0$ on I and $y_1,\ y_2,\ \cdots,\ y_n$ are linearly dependent.

2. $W(x) \neq 0$ for all $x \in I$ and $y_1,\ y_2,\ \cdots,\ y_n$ are linearly independent. In this case

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) \qquad \text{(GS)}$$

is the general solution of (H).

A set of n linearly independent solutions of (H) is called a **fundamental set** or a **solution basis** for (H).

A set of n solutions $\{y_1, y_2, \cdots, y_n\}$ is a fundamental set if and only if their Wronskian $W(x) \neq 0$ for all $x \in I$.

III. HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS (See Section 3.3)

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad \text{(H)}$$

$y = e^{rx}$ is a solution if and only if r is a root of the polynomial equation

$$P(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

$P(r)$ is called the **characteristic polynomial**.

$P(r) = 0$ is called the **characteristic equation**.

Example: $y^{(4)} - 5y'' - 36y = 0$

Char. eqn.: $r^4 - 5r^2 - 36 = 0$

$$\begin{aligned} r^4 - 5r^2 - 36 &= (r^2 - 9)(r^2 + 4) \\ &= (r-3)(r+3)(r^2+4) = 0 \end{aligned}$$

Roots: $r = -3$, $r = 3$, $r = \pm 2i$

Solutions: $y_1 = e^{-3x}$, $y_2 = e^{3x}$, $y_3 = \cos 2x$, $y_4 = \sin 2x$

Linear Independence of Solutions

1. If $r_1,\ r_2, \cdots,\ r_k$ are distinct numbers, then

$$y_1 = e^{r_1 x},\ y_2 = e^{r_2 x},\ \cdots,\ y_k = e^{r_k x}$$

are linearly independent functions.

2. For any number a, the functions

$$y_1 = e^{ax},\ y_2 = xe^{ax},\ \cdots,\ y_m = x^{m-1}e^{ax}$$

are linearly independent functions.

3. If $\alpha + i\beta,\ \alpha - i\beta$ are complex conjugates, then

$$y_1 = e^{\alpha x}\cos \beta x,\ y_2 = e^{\alpha x}\sin \beta x,\ y_3 = xe^{ax}\cos bx, \cdots$$

are linearly independent functions.

Examples:

1. Find the general solution of:

$$y''' + 3y'' - 6y' - 8y = 0$$

2. Find the general solution of:

$$y^{(4)} - y''' - 7y'' + y' + 6y = 0$$

1. Find the general solution of:

$$y''' + 3y'' - 6y' - 8y = 0$$

Hint: $r = 2$ is a root of the characteristic equation.

2. Find the general solution of:

$$y^{(4)} - y''' - 7y'' + y' + 6y = 0$$

Hint: $r = 1$, $r = -1$ are roots of the char. poly.

3. Find a fundamental set of solutions of:

$$y^{(4)} + 5y'' - 36y = 0$$

4. Find the general solution of:

$$y^{(4)} + 2y''' + 9y'' - 2y' - 10y = 0$$

Hint: $r = -1 + 3i$ is a root of the char. poly.

5. $y = C_1 e^{2x} + C_2 e^{-4x} + C_3 \cos 3x + C_4 \sin 3x$

is the general solution of a homogeneous equation with constant coefficients. What is the equation?

6. $y = 2e^{-x} - 3\sin 4x + 2x + 5$ is a solution of a homogeneous equation with constant coefficients. What is the equation of least order having this solution?

Answers: **1.** $y = C_1e^{-4x} + C_2e^{-x} + C_3e^{2x}$

2. $y = C_1e^{-x} + C_2e^{x} + C_3e^{3x} + C_4e^{-2x}$

3. $\left\{e^{-2x},\ e^{2x},\ \cos 3x,\ \sin 3x\right\}$

4. $y = C_1e^{-x} + C_2e^{x} + C_3e^{-x}\cos 3x + C_4e^{-x}\sin 3x$

5. $y^{(4)} + 2y''' + y'' + 18y' - 72y = 0$

6. $y^{(5)} + y^{(4)} + 16y''' + 16y'' = 0$

IV. NONHOMOGENEOUOS EQUATIONS (See Sections 3.4, 3.5)

Given the nonhomogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \qquad \text{(N)}$$

The corresponding homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \qquad \text{(H)}$$

is called the **reduced equation** of (N).

THEOREM 1: If $z_1(x)$ and $z_2(x)$ are solutions of (N), then

$$y = z_1(x) - z_2(x)$$

is a solution of the reduced equation (H).

THEOREM 2: Let $y_1(x),\ y_2(x),\ \cdots,\ y_n(x)$ be a fundamental set of solutions of (H) and $z(x)$ be a particular solution of (N). Then

$$y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) + z(x)$$

is the general solution of (N).

V. Finding a particular solution z of (N):

1. **Variation of Parameters** (In theory, we can do this. In practice, difficult.)

2. **Undetermined Coefficients** (This is what we will use.)

Table

A particular solution of $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = f(x)$

If $f(x) =$	try $z(x) =$*
$p(x)e^{rx}$	$z = P(x)e^{rx}$
$p(x)\cos\beta x + q(x)\sin\beta x$	$z = P(x)\cos\beta x + Q(x)\sin\beta x$
$p(x)e^{\alpha x}\cos\beta x + q(x)e^{\alpha x}\sin\beta x$	$z = P(x)e^{\alpha x}\cos\beta x + Q(x)e^{\alpha x}\sin\beta x$

***Note:** If z satisfies the reduced equation, try xz; if xz also satisfies the reduced equation, then try $x^2 z$ ...

7. Find the general solution of

$$y''' + 4y'' - 3y' - 18y = 10e^{2x} + 9x$$

Hint: 2 is a root of the characteristic polynomial

8. Give the form of a particular solution of

$$y^{(4)} + 4y''' + 13y'' + 36y' + 36y = 5e^{-2x} + \sin 2x + 6$$

Hint: -2 is a root of the char. poly.

9. Give the form of a particular solution of

$$y^{(4)} + 2y'' + y = 4\cos x - 2e^{-x} + 5x - 3$$

10. Give the form of the general solution of

$$y^{(4)} - 16y = 2\cos 2x - (3x+5)e^{2x} + 3x + 1$$

11. Give the form of the general solution of

$$y''' - y'' - y' + y = 2xe^{-x} + e^{x} + 5x$$

Hint: 1 is a root of the characteristic polynomial

12. Give the form of the general solution of

$$y''' - y'' - 8y' + 12y = -5e^{-3x} + 4e^{-2x} + xe^{2x} + 3\sin 2x$$

Hint: 2 is a root of the characteristic polynomial

13. Give the form of the general solution of

$$y^{(5)} - 3y^{(4)} + 4y''' - 12y'' = 2xe^{3x} + 5x$$

Hint: 3 is a root of the characteristic polynomial

14. Give the form of a particular solution of

$$y''' - 3y'' + 3y' - y = (2x + 1)e^x + 10$$

Answers

7. $y = C_1e^{2x} + C_2e^{-3x} + C_3xe^{-3x} + \frac{2}{5}xe^{2x} - \frac{1}{2}x + \frac{1}{12}$

8. $z = Ax^2e^{-2x} + Bx\cos 3x + Cx\sin 3x + D$

9. $z = Ax^2\cos x + Bx^2\sin x + Ce^{-x} + Dx + E$

10. $y = C_1e^{2x} + C_2e^{-2x} + C_3\cos 2x + C_4\sin 2x +$

$Ax\cos 2x + Bx\sin 2x + (Cx^2 + Dx)e^{2x} + Ex + F$

11. $y = C_1e^x + C_2xe^x + C_3e^{-x} + (Ax^2 + Bx)e^{-x} +$
$Cx^2e^x + Dx + E$

12. $y = C_1e^{2x}+C_2xe^{2x}+C_3e^{-3x}+Axe^{-3x}+Be^{-2x}+$
$+ (Cx^3 + Dx^2)e^{2x} + E\cos 2x + F\sin 2x$

13. $y = C_1 + C_2x + C_3\cos 2x + C_4\sin 2x + C_5e^{3x} +$
$(Ax^2 + Bx)e^{3x} + (Cx^3 + Dx^2)$

14. $z = (Ax^4 + Bx^3)e^x + C$