CHAPTER 4

THE LAPLACE TRANSFORM

Background: Improper Integrals

Recall: Definite Integral: a, b real numbers, $a \leq b$; f continuous on $[a, b]$

$$\int_a^b f(x)\,dx$$

Improper integrals:

Type I – Infinite interval of integration

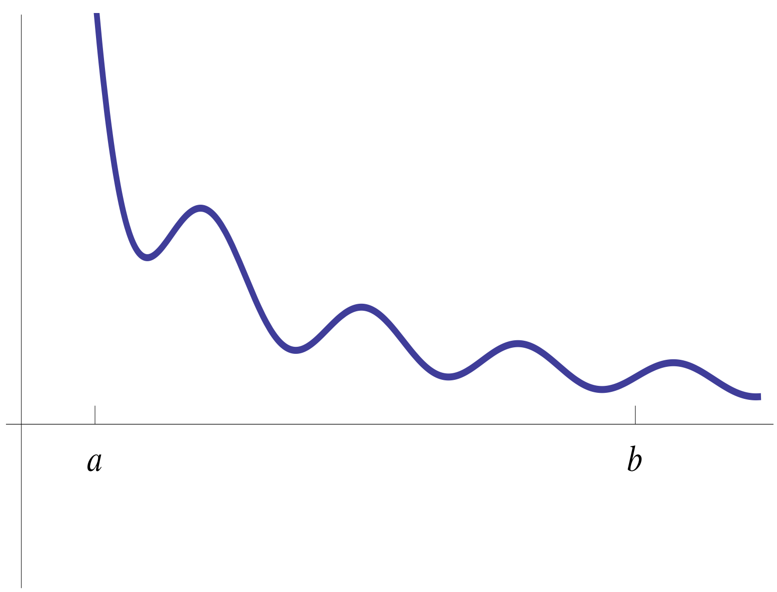
$$\int_a^\infty f(x)\,dx, \quad \int_{-\infty}^b f(x)\,dx, \quad \int_{-\infty}^\infty f(x)\,dx$$

Type II – f undefined at some point $c \in [a,b]$. For example

$$\int_0^2 \frac{1}{(x-1)^2}\,dx$$

Integrals of the form $\int_a^\infty f(x)\,dx$

Def. $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$



$\int_a^{\infty} f(x)\,dx$ **converges** if

$$\lim_{b\to\infty} \int_a^b f(x)\,dx = L$$

exists, in which case

$$\int_a^{\infty} f(x)\,dx = L$$

$\int_a^{\infty} f(x)\,dx$ **diverges** if

$$\lim_{b\to\infty} \int_a^b f(x)\,dx$$

does not exist.

Examples

$$\int_1^\infty \frac{1}{x^2}\,dx$$

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

Let $f(s, x)$ be a function of x and s.

Then

$$\int_a^b f(s, x)\, dx$$

is a function of s. That is,

$$\int_a^b f(s, x)\, dx = F(s)$$

Example: Set $f(x, s) = 3x^2 s + 4xs^2$

$$\int_1^2 \left(3x^2 s + 4xs^2\right)\, dx =$$

I. DEFINITION OF THE LAPLACE TRANSFORM (Text, Section 4.2)

Definition: Let $f = f(x)$ be continuous function on $[0, \infty)$. The **Laplace transform** of f, denoted $\mathcal{L}[f(x)]$, or $F(s)$, is given by

$$\mathcal{L}[f(x)] = F(s) = \int_0^\infty e^{-sx} f(x)\, dx.$$

$$\mathcal{L}[f(x)] = F(s) = \int_0^\infty e^{-sx} f(x)\, dx.$$

The domain of F is the set of real numbers s for which the improper integral converges.

Applications of the Laplace Transform

- A direct method for solving initial-value problems for first and second order linear differential equations with constant coefficients.

- An introduction to transform methods.

Laplace transforms of basic functions

1. $f(x) \equiv c$, constant, on $[0, \infty)$:

$$\mathcal{L}[c] = \int_0^\infty e^{-sx}\, c\, dx = \lim_{b \to \infty} \int_0^b e^{-sx} c\, dx$$

2. $f(x) = e^{\alpha x}$ on $[0, \infty)$:

$$\mathcal{L}[e^{\alpha x}] = \int_0^{\infty} e^{-sx}\, e^{\alpha x}\, dx$$

$$= \lim_{b \to \infty} \int_0^{b} e^{-sx}\, e^{\alpha x}\, dx =$$

3. $f(x) = \cos \beta x$ on $[0, \infty)$:

$$\mathcal{L}[\cos \beta x] = \int_0^\infty e^{-sx} \cos \beta x \, dx$$

$$= \lim_{b \to \infty} \int_0^b e^{-sx} \cos \beta x \, dx$$

Recall, from Calculus II,

$$\int e^{-sx} \cos \beta x \, dx =$$

$$\frac{e^{-sx}(-s \cos \beta x - \beta \sin \beta x)}{s^2 + \beta^2}$$

Therefore

$$\int_0^b e^{-sx} \cos\, \beta x \, dx$$

$$= \frac{e^{-sx}(-s \cos\, \beta x - \beta \sin\, \beta x)}{s^2 + \beta^2}\Bigg]_0^b$$

$$= \frac{e^{-sb}(-s \cos\, \beta b - \beta \sin\, \beta b)}{s^2 + \beta^2} + \frac{s}{s^2 + \beta^2}$$

If $s > 0$, then we have

$$= \frac{1}{e^{sb}} \frac{(-s \cos\, \beta b - \beta \sin\, \beta b)}{s^2 + \beta^2} + \frac{s}{s^2 + \beta^2}$$

Therefore,

$$\mathcal{L}[\cos\, \beta x] = \int_0^\infty e^{-sx} \cos\, \beta x \, dx = \frac{s}{s^2 + \beta^2}$$

$f(x)$	$F(s) = \mathcal{L}[f(x)]$
c	$\dfrac{c}{s}, \quad s > 0$
e^{rx}	$\dfrac{1}{s-r}, \quad s > r$
$\cos \beta x$	$\dfrac{s}{s^2+\beta^2}, \quad s > 0$
$\sin \beta x$	$\dfrac{\beta}{s^2+\beta^2}, \quad s > 0$
$e^{rx} \cos \beta x$	$\dfrac{s-r}{(s-r)^2+\beta^2}, \quad s > r$
$e^{rx} \sin \beta x$	$\dfrac{\beta}{(s-r)^2+\beta^2}, \quad s > r$
$x,$	$\dfrac{1}{s^2}, \quad s > 0$
$x^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$xe^{rx},$	$\dfrac{1}{(s-r)^2}, \quad s > r$
$x^n e^{rx}, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-r)^{n+1}}, \quad s > r$
$x \cos \beta x$	$\dfrac{s^2-\beta^2}{(s^2+\beta^2)^2}, \quad s > 0$
$x \sin \beta x$	$\dfrac{2\beta s}{(s^2+\beta^2)^2}, \quad s > 0$

What functions have Laplace Transforms ?

Definition : f is of exponential order λ if there exists a positive number M and a nonnegative number A such that

$$|f(x)| \leq Me^{\lambda x} \quad \text{on} \quad [A, \infty).$$

Examples:

(a) Bounded functions, e.g., sin x, cos x exponential order 0

(b) Powers of x: $f(x) = x^k$.

exp. order λ for any $\lambda > 0$

(c) Exponential fcns: $f(x) = e^{ax}$.

exp. order λ for any $\lambda \geq a$

(d) "Complex exponentials":

$$f(x) = e^{ax} \cos bx, \;\; e^{ax} \sin bx, \;\; etc.$$

exp. order λ for any $\lambda \geq a$

(e) $f(x) = e^{x^2}$ is **not** of exponential order.

Theorem: Let f be a continuous function on $[0, \infty)$. If f is of exponential order λ, then the Laplace transform $\mathcal{L}[f(x)] = F(s)$ exists for $s > \lambda$.

II. PROPERTIES OF THE LAPLACE TRANSFORM (Text, Section 4.3)

$$\boxed{\mathcal{L}[f(x)] = \int_0^\infty e^{-sx} f(x)\, dx} \quad \text{(definition)}$$

Theorem 1. $\mathcal{L}$ is a linear operator.

That is:

$$\mathcal{L}[f_1(x) + f_2(x)] = \mathcal{L}[f_1(x)] + \mathcal{L}[f_2(x)]$$

$$\mathcal{L}[cf(x)] = c\mathcal{L}[f(x)].$$

Proof:

$$\mathcal{L}[f_1(x) + f_2(x)]$$

$$= \int_0^\infty e^{-sx}[f_1(x) + f_2(x)]\,dx$$

$$= \int_0^\infty \left[e^{-sx} f_1(x) + e^{-sx} f_2(x)\right]\,dx$$

$$= \int_0^\infty e^{-sx} f_1(x)\,dx + \int_0^\infty e^{-sx} f_2(x)\,dx$$

$$= \mathcal{L}[f_1(x)] + \mathcal{L}[f_2(x)]$$

$$\begin{aligned}\mathcal{L}[cf(x)] &= \int_0^\infty e^{-sx}[cf(x)]\,dx \\ &= c\int_0^\infty e^{-sx} f(x)\,dx \\ &= c\,\mathcal{L}[f(x)]\end{aligned}$$

Examples:

1. $f(x) = 3 + 7e^{3x}$

$$\mathcal{L}[f(x)] = \mathcal{L}\left[3 + 7e^{3x}\right] = \mathcal{L}[3] + 7\mathcal{L}[e^{3x}]$$

2. $f(x) = 2x - 5\cos 3x$

$$\mathcal{L}\left[2x - 5\cos 3x\right] = 2\mathcal{L}[x] - 5\mathcal{L}[\cos 3x]$$

II. Laplace transform of derivatives.

Theorem 2. If f is continuously differentiable and of exponential order λ, then $\mathcal{L}[f'(x)]$ exists for $s > \lambda$ and

$$\boxed{\mathcal{L}[f'(x)] = s\mathcal{L}[f(x)] - f(0).}$$

Proof:

Example: Use Theorem 2 to find $\mathcal{L}[\sin\ \beta x]$

$$\sin \beta x = \frac{-1}{\beta}(\cos \beta x)'$$

$$\mathcal{L}[\sin \beta x] = \mathcal{L}\left[\frac{-1}{\beta}(\cos \beta x)'\right]$$

$$= \frac{-1}{\beta}\mathcal{L}[(\cos \beta x)']$$

Corollary 1. If f is twice continuously differentiable with f and f' of exponential order λ, then $\mathcal{L}[f''(x)]$ exists for $s > \lambda$ and

$$\boxed{\mathcal{L}[f''(x)] = s^2\mathcal{L}[f(x)] - sf(0) - f'(0).}$$

Proof: $\mathcal{L}[f''(x)] = \mathcal{L}\left[(f'(x))'\right]$

Corollary 2. In general, if $f,\ f',\ \cdots,\ f^{(n-1)}$ are of exponential order λ, then $\mathcal{L}[f^{(n)}(x)]$ exists for $s > \lambda$ and

$$\mathcal{L}[f^{(n)}(x)] = s^n \mathcal{L}[f(x)] - s^{n-1} f(0) -$$

$$s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

III. $\mathcal{L}[x^n f(x)] = (-1)^n \frac{d^n F}{ds^n}$

If $\mathcal{L}[f(x)] = F(s)$, then

$$\mathcal{L}[xf(x)] = -\frac{dF}{ds},$$

$$\mathcal{L}[x^2 f(x)] = \frac{d^2 F}{ds^2}$$

and, in general,

$$\mathcal{L}[x^n f(x)] = (-1)^n \frac{d^n F}{ds^n}.$$

Examples:

1. $\mathcal{L}[1] = \frac{1}{s}, \quad s > 0$

$$\mathcal{L}[x] = \mathcal{L}[x \cdot 1] = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

$$\mathcal{L}[x^2] = \mathcal{L}[x^2 \cdot 1] = \frac{d^2}{ds^2}\left(\frac{1}{s}\right) = \frac{2}{s^3}$$

In general, $\mathcal{L}[x^n] = \frac{n!}{s^{n+1}}$

2. $\mathcal{L}[e^{-2x}] = \dfrac{1}{s+2}$,

$$\mathcal{L}[xe^{-2x}] = -\frac{d}{ds}\left(\frac{1}{s+2}\right) = \frac{1}{(s+2)^2}$$

$$\mathcal{L}[x^2e^{-2x}] = \frac{d^2}{ds^2}\left(\frac{1}{s+2}\right) = \frac{2}{(s+2)^3}$$

$\cdots$

$$\mathcal{L}[x^ne^{-2x}] = \frac{n!}{(s+2)^{n+1}}$$

In general: $\mathcal{L}[x^ne^{\alpha x}] = \dfrac{n!}{(s-\alpha)^{n+1}}$

IV. "Translation"

If $\mathcal{L}[f(x)] = F(s)$, then

$$\mathcal{L}[e^{rx}f(x)] = F(s-r).$$

Proof:

$$F(s) = \int_0^\infty e^{-sx} f(x)\, dx$$

$$\begin{aligned} F(s-r) &= \int_0^\infty e^{-(s-r)x} f(x)\, dx \\ &= \int_0^\infty e^{-sx+rx} f(x)\, dx \\ &= \int_0^\infty e^{-sx} e^{rx} f(x)\, dx \\ &= \mathcal{L}[e^{rx}f(x)] \end{aligned}$$

Examples:

1. $$\mathcal{L}[\text{cos }\beta x] = F(s) = \frac{s}{s^2 + \beta^2}$$

$$\mathcal{L}[e^{\alpha x}\,\text{cos }\beta x] = F(s-\alpha) = \frac{s-\alpha}{(s-\alpha)^2 + \beta^2}$$

2. $$\mathcal{L}[\text{sin }\beta x] = F(s) = \frac{\beta}{s^2 + \beta^2}$$

$$\mathcal{L}[e^{\alpha x}\,\text{sin }\beta x] = F(s-\alpha) = \frac{\beta}{(s-\alpha)^2 + \beta^2}$$

III. FINDING LAPLACE TRANSFORMS

Examples:

1. Find the Laplace transform of

$$f(x) = 3 + 4e^{3x} - 2\cos 2x.$$

2. Find the Laplace transform of

$$f(x) = 5x^2 - 2e^{-3x}\sin 2x + 5xe^{4x}$$

3. Find the Laplace transform of the solution of the initial-value problem:

$$y' - 2y = 4x; \;\; y(0) = 3.$$

4. Find the Laplace transform of the solution of the initial-value problem:

$$y'' - 2y' + 5y = 2x + e^{-x};$$

$$y(0) = -2, \ y'(0) = 3.$$

Ans:

$Y(s) =$

$$\frac{s^2+2s+2}{s^2(s+1)(s^2-2s+5)} + \frac{7-2s}{s^2-2s+5}$$

IV. INVERSE LAPLACE TRANSFORMS (Text, Section 4.4)

Theorem: If f and g are continuous functions on $[0, \infty)$, and if $\mathcal{L}[f(x)] = \mathcal{L}[g(x)]$, then $f \equiv g$; that is $f(x) = g(x)$ for all $x \in [0, \infty)$. ($\mathcal{L}$ is a one-to-one operator.)

Def. If $F(s)$ is a given transform and if the function f, continuous on $[0,\infty)$, has the property that

$$\mathcal{L}[f(x)] = F(s),$$

then f is called the **inverse Laplace transform of** $F(s)$, and is denoted by

$$f(x) = \mathcal{L}^{-1}[F(s)].$$

The operator $\mathcal{L}^{-1}$ is called the **inverse operator of** $\mathcal{L}$.

$f(x)$	$F(s) = \mathcal{L}[f(x)]$	
c	$\frac{c}{s}$,	$s > 0$
e^{rx}	$\frac{1}{s-r}$,	$s > r$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}$,	$s > 0$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}$,	$s > 0$
$e^{rx} \cos \beta x$	$\frac{s-r}{(s-r)^2+\beta^2}$,	$s > r$
$e^{rx} \sin \beta x$	$\frac{\beta}{(s-r)^2+\beta^2}$,	$s > r$
x,	$\frac{1}{s^2}$,	$s > 0$
$x^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$,	$s > 0$
xe^{rx},	$\frac{1}{(s-r)^2}$,	$s > r$
$x^n e^{rx}, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-r)^{n+1}}$,	$s > r$
$x \cos \beta x$	$\frac{s^2-\beta^2}{(s^2+\beta^2)^2}$,	$s > 0$
$x \sin \beta x$	$\frac{2\beta s}{(s^2+\beta^2)^2}$,	$s > 0$

Examples:

1. If $F(s) = \dfrac{1}{(s+3)^2}$, then

2. If $F(s) = \dfrac{3}{s^2+4}$, then

The operator $\mathcal{L}^{-1}$ is linear; that is

$$\mathcal{L}^{-1}[F(s)+G(s)] = \mathcal{L}^{-1}[F(s)]+\mathcal{L}^{-1}[G(s)]$$

$$\mathcal{L}^{-1}[c\,F(s)] = c\,\mathcal{L}^{-1}[F(s)]$$

Finding inverse Laplace transforms:

Crucial step: Partial fraction decomposition!

Examples:

1. $F(s) = \dfrac{s^2 - 3s - 1}{(s-2)^2(s+4)}$

Find $\mathcal{L}^{-1}[F(s)] = f(x)$.

Answer:

$$f(x) = \frac{1}{4}e^{2x} - \frac{1}{2}xe^{2x} + \frac{3}{4}e^{-4x}$$

2. $F(s) = \dfrac{2s - 1}{s^4 + 4s^2}$

Find $\mathcal{L}^{-1}[F(s)] = f(x)$.

Answer:

$$f(x) = \frac{1}{2} - \frac{1}{4}x - \frac{1}{2}\cos 2x + \frac{1}{8}\sin 2x$$

3. $F(s) = \dfrac{4s+8}{(s-4)(s^2+16)}$

Find $\mathcal{L}^{-1}[F(s)] = f(x)$.

Answer:

$$f(x) = \frac{3}{4}e^{4x} - \frac{3}{4}\cos 4x + \frac{1}{4}\sin 4x$$

4. $F(s) = \dfrac{2s - 1}{(s+1)(s^2 - 4s + 13)}$

Find $\mathcal{L}^{-1}[F(s)] = f(x)$.

Answer:

$$f(x) = -\frac{1}{6}\, e^{-x} + \frac{1}{6}\, e^{2x} \cos 3x + \frac{1}{2}\, e^{2x} \sin 3x$$

5. Given the initial-value problem

$$y' + 2y = 3e^x, \quad y(0) = 4.$$

(a) Find the Laplace transform of the solution.

(b) Find the solution by finding the inverse transform of the result in (a).

(a) $\mathcal{L}[y' + 2y] = \mathcal{L}[3e^x]$

6. Given the initial-value problem

$$y'' - y' - 2y = \sin 2x, \quad y(0) = 1,\ y'(0) = 2.$$

(a) Find the Laplace transform of the solution.

(b) Find the solution by finding the inverse transform of the result in (a).

Answer:

$$y = \frac{13}{12}e^{2x} - \frac{2}{15}e^{-x} + \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x.$$

6. Let $y = y(x)$ be the solution of the initial-value problem:

$$y'' - 2y' - 8y = 0, \quad y(0) = \alpha,\ y'(0) = 4.$$

Find α such that $y = y(x)$ satisfies $\lim_{x \to \infty} y(x) = 0$.