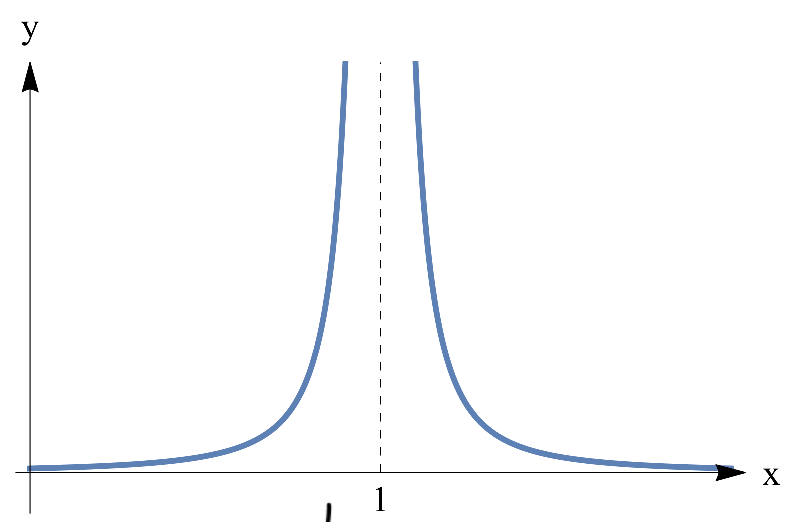
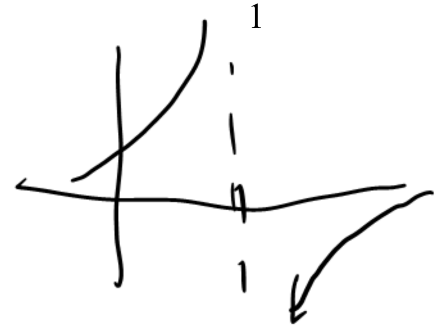
Laplace Transform: Discontinuous Functions (Text, Section 4.5)

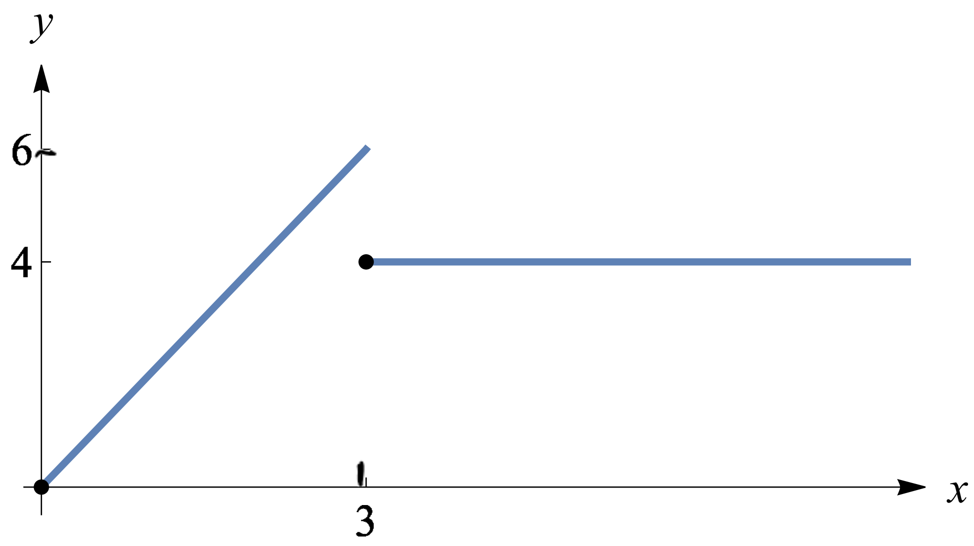
Types of discontinuities: **Infinite**

$$f(x) = \frac{1}{(x-1)^2}$$





♯ Jump $f(x) = \begin{cases} 2x, & 0 \leq x < 3 \\ 4, & 3 \leq x < \infty \end{cases}$

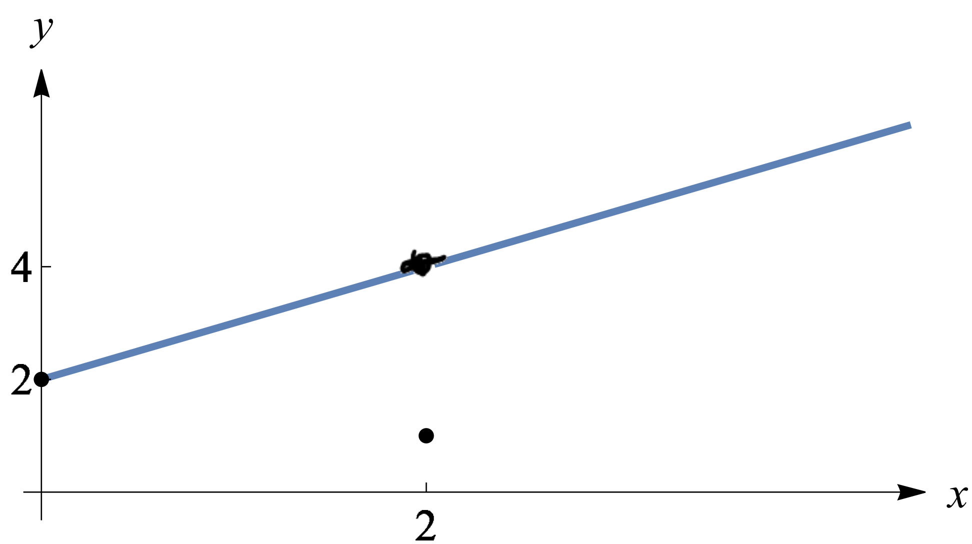


$$\frac{(x-2)(x+2)}{x-2} = x+2$$

$$y = \begin{cases} x+2 & x \neq 2 \\ 1 & x = 2 \end{cases}$$

$$u = \begin{cases} x+2 & x \neq 2 \\ 4 & x = 2 \end{cases}$$

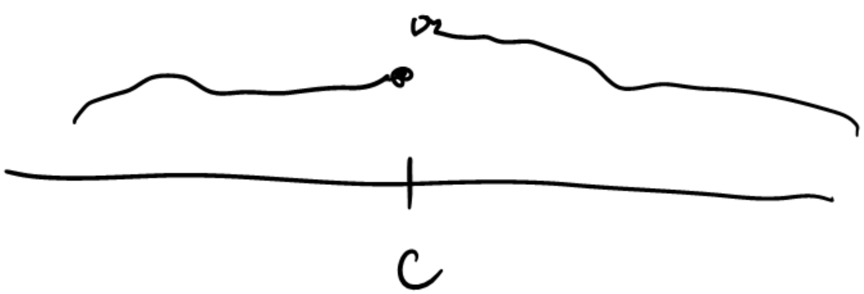
Removable: $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$



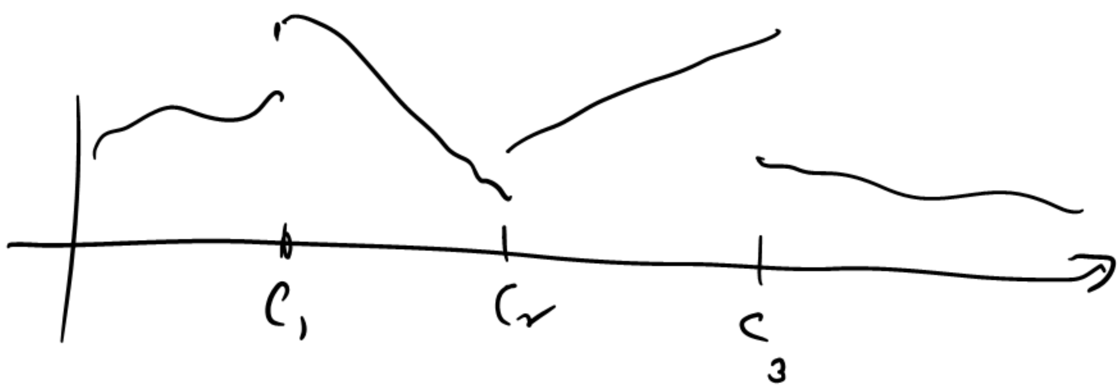
Def. Let the function $f = f(x)$ be defined on an interval I and continuous except at a point $c \in I$. If

$$\lim_{x \to c^-} f(x) \quad \text{and} \quad \lim_{x \to c^+} f(x) \quad \text{exist}$$

but $\displaystyle \lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$, then f is said to have a **jump** (or *finite*) **discontinuity** at c.



Def. A function f defined on an interval I is **piecewise continuous on** I if it is continuous on I except for at most a finite number of points $c_1, c_2, \ldots, c_n$ of I at which it has jump discontinuities.



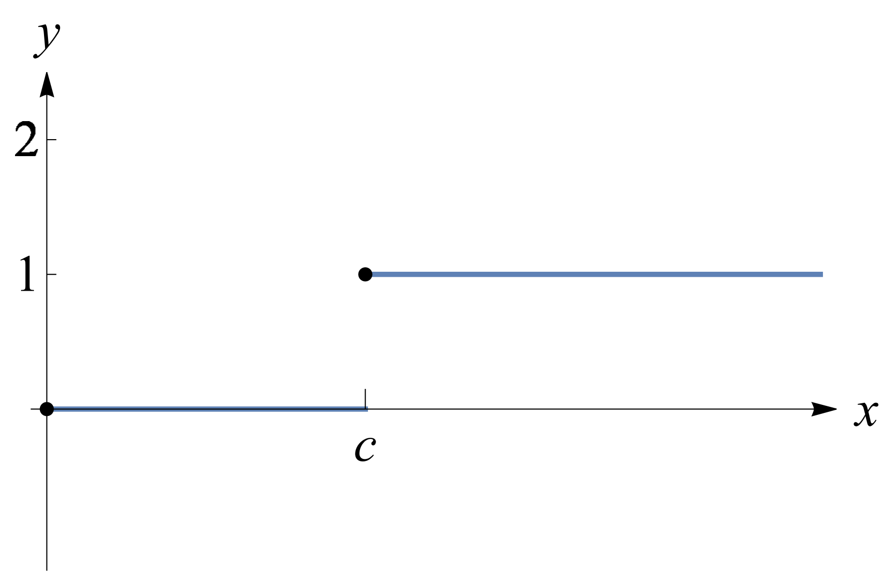
Theorem: If the function f is piecewise continuous on $[0, \infty)$, and of exponential order λ, then the Laplace transform $\mathcal{L}[f(x)]$ exists for $s > \lambda$.

Heaviside fcns

Unit Step Functions: Let $c > 0$. The function

$$u_c(x) = u(x - c) = \begin{cases} 0 & 0 \le x < c \\ 1 & x \ge c \end{cases}$$

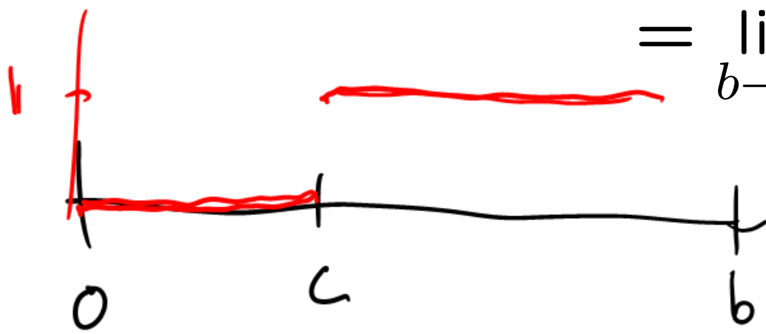
is called a **unit step function**.



Laplace Transform of a Unit Step Function:

$$\mathcal{L}[u(x-c)] = \int_0^\infty e^{-sx} u(x-c)\,dx \qquad \text{def.}$$

$$= \lim_{b\to\infty} \int_0^b e^{-sx} u(x-c)\,dx$$



$$\int_0^b e^{-sx} u(x-c)\,dx = \int_0^c e^{-sx} \overset{0}{u(x-c)}\,dx + \int_c^b e^{-sx} \overset{1}{u(x-c)}\,dx$$

$$= \int_c^b e^{-sx}\,dx = -\frac{e^{-sx}}{s}\Big|_c^b = -\frac{e^{-bs}}{s} + \frac{e^{-cs}}{s}$$

$$= -\frac{1}{s e^{bs}} + \frac{e^{-cs}}{s}$$

as $b \to \infty$, $\frac{1}{s e^{bs}} \to 0$

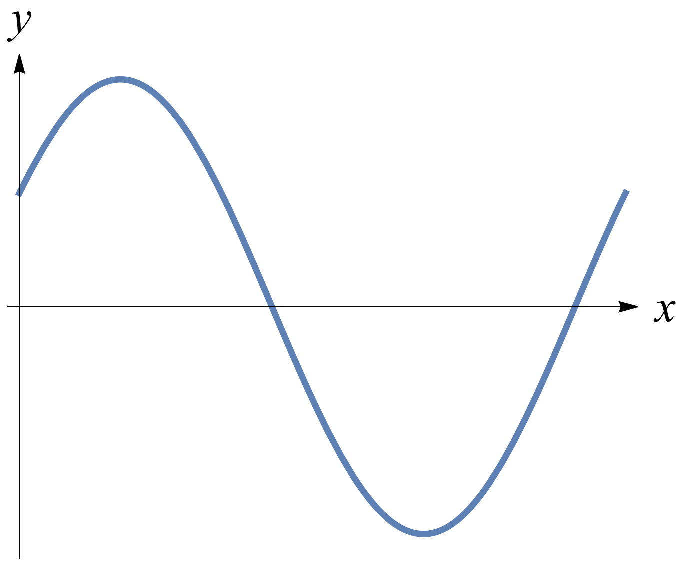
$$\mathcal{L}[u(x-c)] = e^{-cs}\frac{1}{s}, \qquad s > 0.$$

Translation of a Function: if f is defined on $[0, \infty)$ and $c > 0$, then the function

$$f(x-c)u(x-c) = \begin{cases} 0, & 0 \le x < c \\ f(x-c)u(x-c), & x \ge c \end{cases}$$

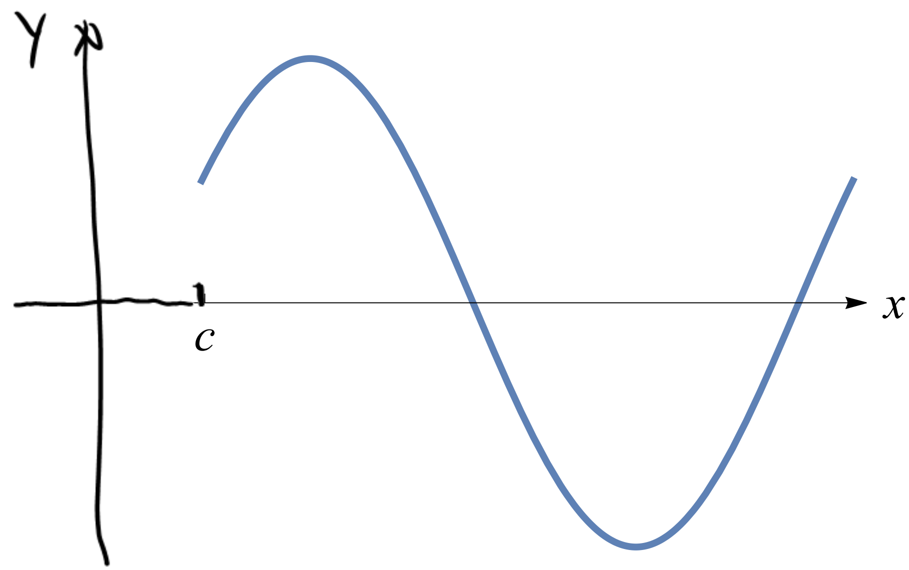
is called the **translation of f** to c.

$f(x)$



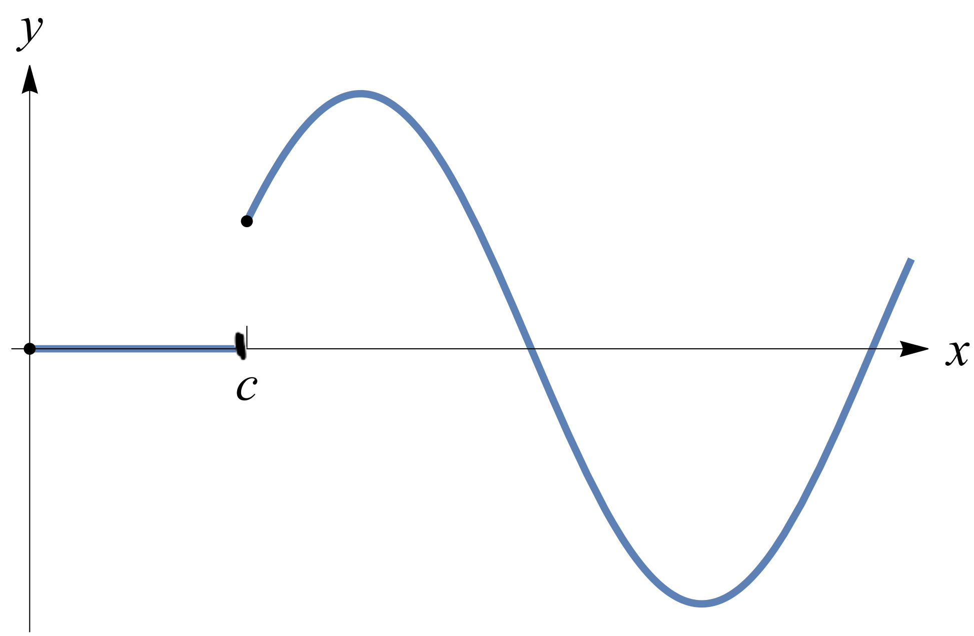
defined on $[0, \infty)$

$f(x - c)$



defined on $[c, \infty)$

$$f(x-c)u(x-c) = \begin{cases} 0 & 0 \le x < c \\ f(x-c), & x \ge c \end{cases}$$



Translations: Express $f(x)$ in terms of $(x-c)$

EXAMPLES:

1. Express $f(x) = 5x + 3$ in terms of $(x-3)$

$f(x) = 5x + 3 = 5((x-3)+3) + 3 = 5(x-3) + 15 + 3$

$= 5(x-3) + 18$

2. Express $f(x) = x^2 - 3x + 7$ in terms of $(x-2)$

$f(x) = [(x-2)+2]^2 - 3[(x-2)+2] + 7$

$= (x-2)^2 + 4(x-2) + 4 - 3(x-2) - 6 + 7$

$= (x-2)^2 + (x-2) + 5$

3. Express $f(x) = \sin 2x$ in terms of $(x - \pi/2)$ ✓

$$\sin 2x = \sin 2\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) = \sin\left[2(x - \pi/2) + \pi\right]$$

$$= \sin 2(x - \pi/2)\overset{-1}{\cos \pi} + \cos 2(x - \pi/2)\overset{0}{\sin \pi}$$

$$= -\sin 2(x - \pi/2)$$

4. Express $f(x) = \cos \pi x$ in terms of $(x - 3)$

$$\cos \pi(x - 3 + 3) = \cos\left[\pi(x-3) + 3\pi\right]$$

$$= \cos \pi(x-3)\overset{-1}{\cos(3\pi)} - \sin \pi(x-3)\overset{0}{\sin 3\pi}$$

$$= -\cos \pi(x-3)$$

Property V. Laplace Transform of a Translated Function:

Suppose that $\mathcal{L}[f(x)] = F(s)$. Then, for any positive number c,

1. $$\boxed{\mathcal{L}[f(x-c)u(x-c)] = e^{-cs}F(s)}$$ Sec 4.5

2. $$\mathcal{L}^{-1}[e^{-cs}F(s)] = f(x-c)u(x-c)$$ Sec 4.6

Proof of (1):

$$\mathcal{L}[f(x-c)u(x-c)] = \int_0^\infty e^{-sx} f(x-c)u(x-c)\,dx \quad \text{def}$$

$$= \lim_{b\to\infty} \int_0^b e^{-sx} f(x-c)u(x-c)\,dx$$

$$\int_0^b e^{-sx} f(x-c)\,u(x-c)\,dx = \int_0^c e^{-sx} f(x-c)\,\overset{0}{\overset{\shortparallel}{u(x-c)}}\,dx + \int_c^b e^{-sx} f(x-c)\,\overset{1}{\overset{\shortparallel}{u(x-c)}}\,dx$$

$$= \int_c^b e^{-sx} f(x-c)\,dx$$

Let $t = x - c$ $\quad$ $t = 0$ when $x = c$ $\quad$ $x = t + c$

$dt = dx$

$$= \int_0^{b-c} e^{-s(t+c)} f(t)\, dt$$

$$= \int_0^{b-c} e^{-st} \cdot e^{-cs} f(t)\, dt$$

$$= e^{-cs} \int_0^{b-c} e^{-st} f(t)\, dt$$

$$\lim_{b\to\infty} e^{-cs} \int_0^{b-c} e^{-st} f(t)\, dt = e^{-cs} \lim_{b\to\infty} \int_0^{b-c} e^{-st} f(t)\, dt$$

$$= e^{-cs} \int_0^{\infty} e^{-st} f(t)\, dt$$

$$= e^{-cs} \mathcal{L}[f]$$

$$= e^{-cs} F(s)$$

Examples

1. Let $f(x) = e^{5x}$; $\mathcal{L}[e^{5x}] = F(s) = \dfrac{1}{s-5}$

Then: $f(x-3) = e^{5(x-3)}$ and

$$\mathcal{L}[f(x-3)u(x-3)] = \mathcal{L}[e^{5(x-3)}u(x-3)] = e^{-3s}\frac{1}{s-5}$$

$$\mathcal{L}\left(f(x-3)\,u(x-3)\right) = e^{-3s}F(s)$$

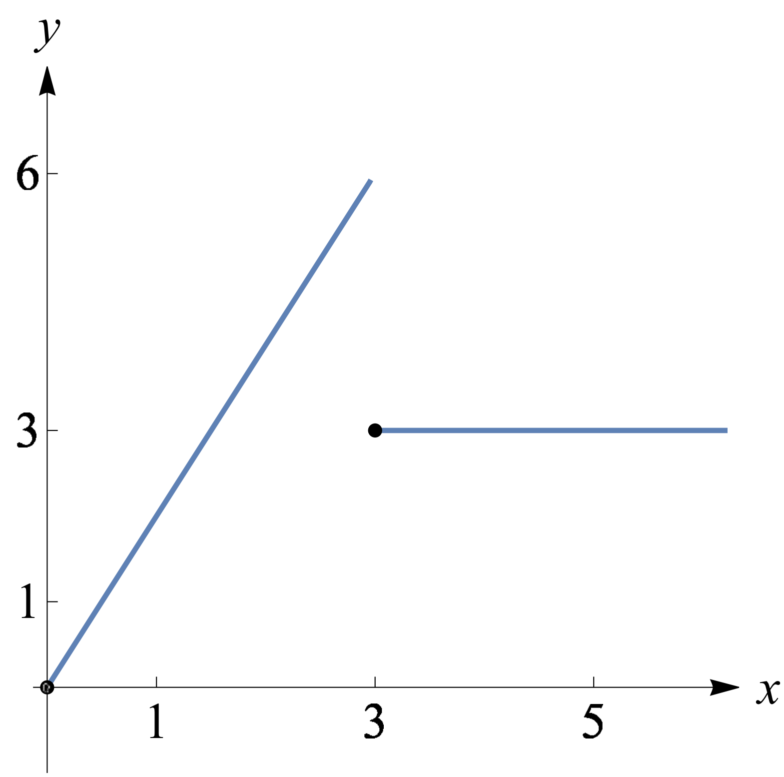
2. Let $f(x) = \cos 4x$; $\mathcal{L}[\cos 4x] = F(s) = \dfrac{s}{s^2+16}$

Then: $f(x-2) = \cos[4(x-2)]$ and

$$\mathcal{L}[\cos 4(x-2)u(x-2)] = e^{-2s}\frac{s}{s^2+16}$$

Examples: Given $f(x)$, find $F(s)$.

1. $$f(x) = \begin{cases} 2x, & 0 \le x < 3 \\ 3, & 3 \le x < \infty \end{cases}$$



Step 1. Re-write f in terms of $u(x-3)$:

$$2x - 2x\,u(x-3) = \begin{cases} f(x) & 0 \le x < 3 \\ 0 & x \ge 3 \end{cases}$$

$$f(x) = 2x - 2x\,u(x-3) + 3\,u(x-3) = \begin{cases} 2x & 0 \le x < 3 \\ 3 & x \ge 3 \end{cases}$$

express this in terms of $x-3$

$$= 2x - 2\left[(x-3)+3\right] u(x-3) + 3\,u(x-3)$$

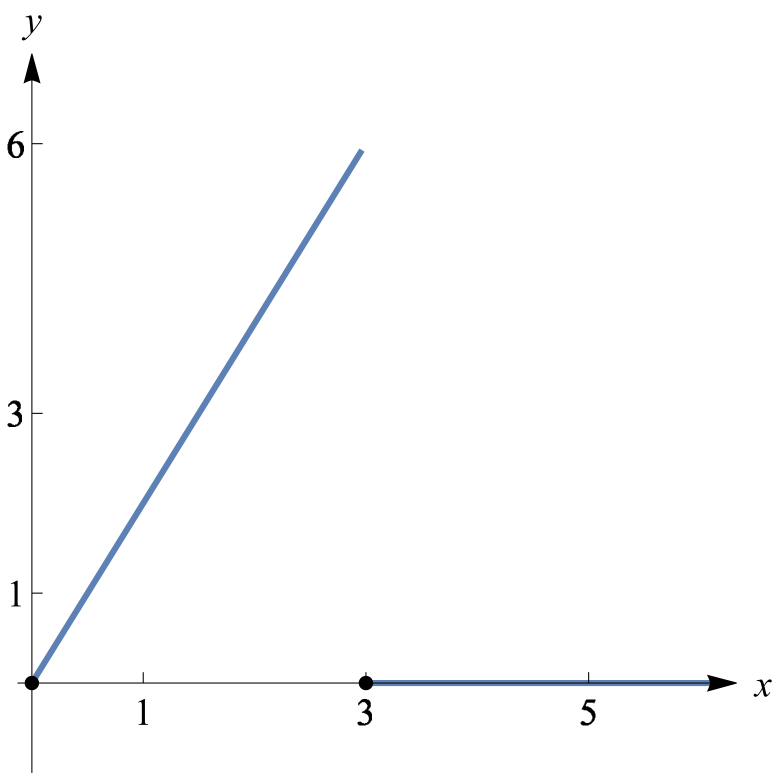
$$= 2x - 2(x-3)\,u(x-3) - 6\,u(x-3) + 3\,u(x-3)$$

$$= 2x - 2(x-3)\,u(x-3) - 3\,u(x-3)$$

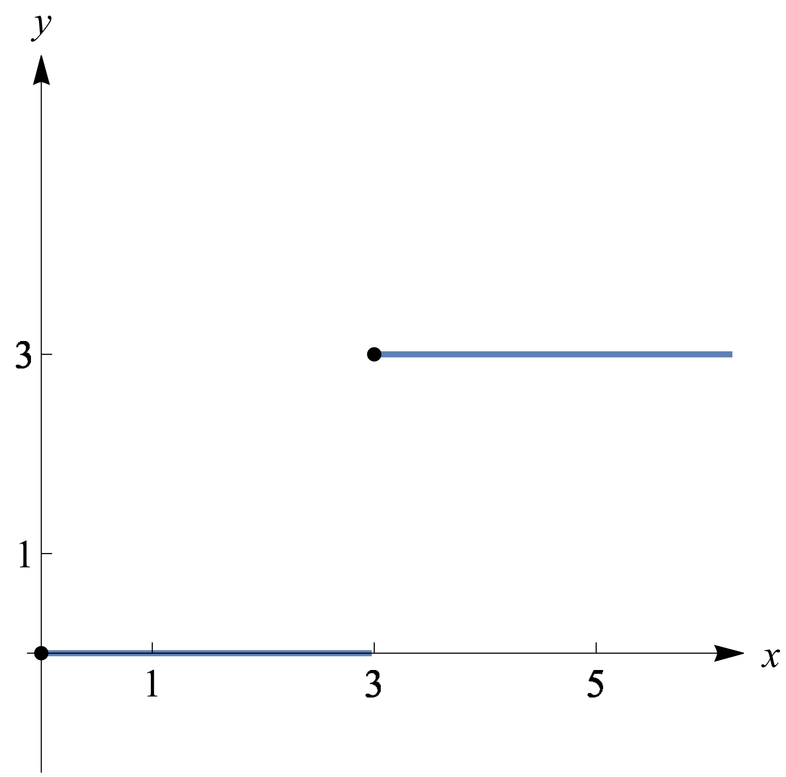
$$f(x) = 2x - 2x\,u(x-3) \overset{-}{\cancel{+}}\, 3u(x-3)$$

Graphs:

$2x - 2x\,u(x-3)$



$3u(x-3)$



Step 2. Write the coefficients of $f(x)$ in terms of $x - 3$ and simplify:

$$f(x) = 2x - 2x\,u(x-3) + 3\,u(x-3)$$

$$= 2x - 2\left[(x-3) + 3\right] u(x-3) + 3\,u(x-3)$$

$$= 2x - 2(x-3)\,u(x-3) - 6\,u(x-3) + 3\,u(x-3)$$

$$= 2x - 2(x-3)\,u(x-3) - 3\,u(x-3)$$

$$f(x) = 2x - 2(x-3)u(x-3) - 3u(x-3)$$

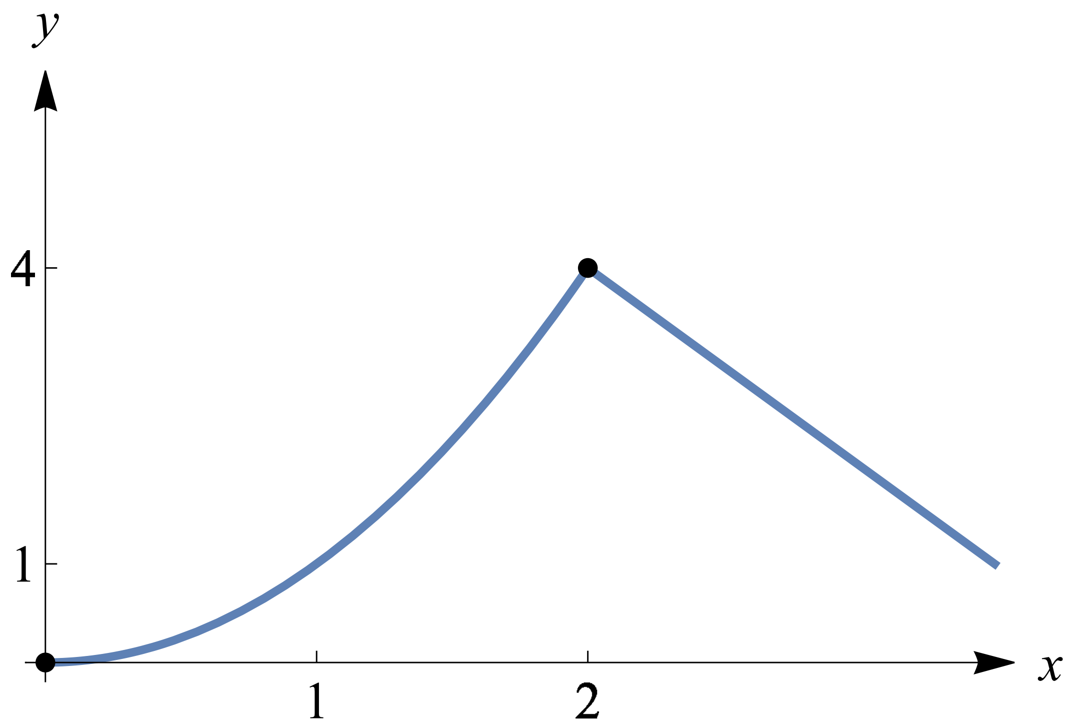
Step 3. Determine $\mathcal{L}[f]$:

$$\mathcal{L}(f(x)) = 2\mathcal{L}(x) - 2\mathcal{L}[(x-3)u(x-3)] - \mathcal{L}[3]\, u(x-3)$$

$$= \frac{2}{s^2} - 2e^{-3s}\frac{1}{s^2} - e^{-3s}\frac{3}{s}$$

$$F(s) = \frac{2}{s^2} - 2e^{-3s}\frac{1}{s^2} - 3e^{-3s}\frac{1}{s}$$

2. $$f(x) = \begin{cases} x^2, & 0 \le x < 2 \\ -2x + 8, & x \ge 2 \end{cases}$$



Step 1. Re-write f in terms of $u(x-2)$:

$$f(x) = x^2 - x^2 u(x-2) + (-2x+8)\,u(x-2)$$

$$= x^2 - x^2 u(x-2) - 2x\,u(x-2) + 8\,u(x-2)$$

$$f(x) = \begin{cases} f_1(x) & 0 \le x < c \\ f_2(x) & x \ge c \end{cases}$$

piecewise form

$$= f_1(x) - f_1(x)\,u(x-c) + f_2(x)\,u(x-c)$$

unit step fcn form

$$f(x) = x^2 - x^2\,u(x-2) + (-2x+8)\,u(x-2)$$

Step 2. Write the coefficients of $f(x)$ in terms of $x-2$ and simplify:

$$f(x) = x^2 - x^2 u(x-2) - 2x\,u(x-2) + 8\,u(x-2)$$

$$= x^2 - [(x-2)+2]^2 u(x-2) - 2[(x-2)+2]\,u(x-2) + 8\,u(x-2)$$

$$= x^2 - [(x-2)^2 + 4(x-2) + 4]\,u(x-2) - 2(x-2)\,u(x-2)$$
$$- 4\,u(x-2) + 8\,u(x-2)$$

$$= x^2 - (x-2)^2 u(x-2) - 6(x-2)\,u(x-2)$$

$$f(x) = x^2 - (x-2)^2 u(x-2) - 6(x-2)u(x-2)$$

Step 3. Determine $\mathcal{L}[f]$:

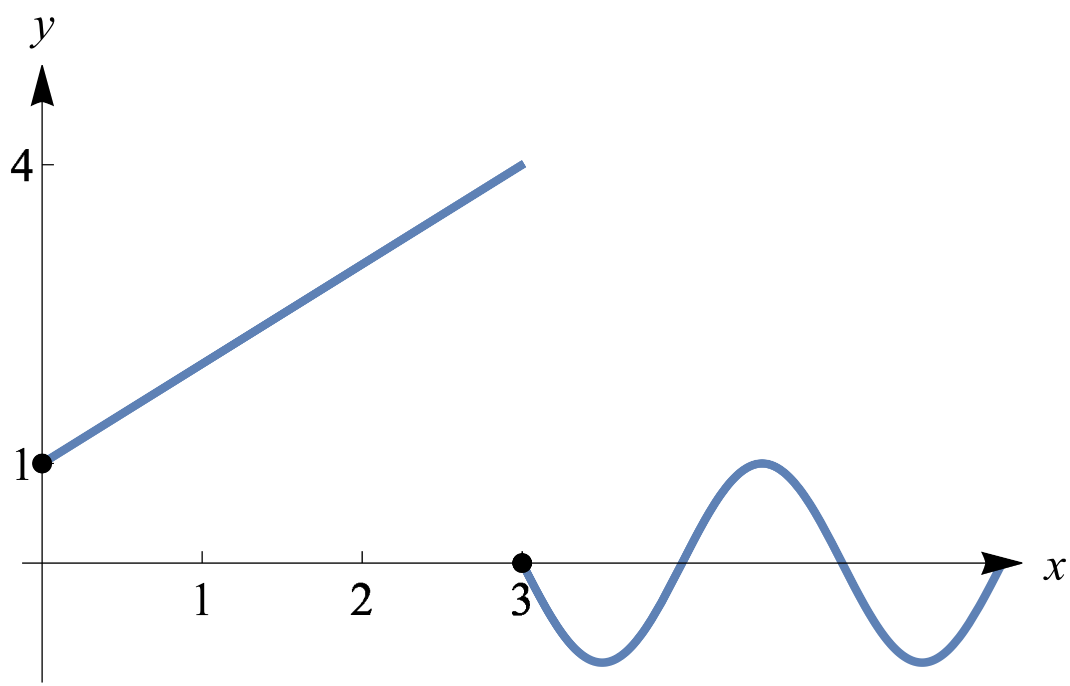
x

$$f(x) = x^2 - (x-2)^2 u(x-2) - 6(x-2)u(x-2)$$

$$\mathcal{L}(f(x)) = \frac{2}{s^3} \; e^{-2s} \frac{2}{s^3} - 6 e^{-2s} \frac{1}{s^2}$$

$$F(s) = \frac{2}{s^3} - e^{-2s}\frac{2}{s^3} - 6\,e^{-2s}\frac{1}{s^2}$$

3. $f(x) = \begin{cases} x + 1, & 0 \leq x < 3 \\ \sin \pi x, & x \geq 3 \end{cases}$



Step 1. Re-write f in terms of $u(x-3)$:

$$\begin{cases} x+1 & 0 \le x < 3 \\ \sin \pi x & x \ge 3 \end{cases}$$

$$f(x) = x+1 - (x+1)\,u(x-3) + \sin \pi x \; u(x-3)$$

$$f(x) = x + 1 - (x+1)u(x-3) + \sin(\pi x)\,u(x-3)$$

Step 2. Write the coefficients of $f(x)$ in terms of $x-3$ and simplify:

$$f(x) = x+1 - [x-3+3+1]\,u(x-3) + \sin\pi(x-3+3)\,u(x-3)$$

$$= x+1 - [(x-3)+4]\,u(x-3) + \sin[\pi(x-3)+3\pi]\,u(x-3)$$

$$= x+1 - (x-3)\,u(x-3) - 4u(x-3) - \sin\pi(x-3)\,u(x-3)$$

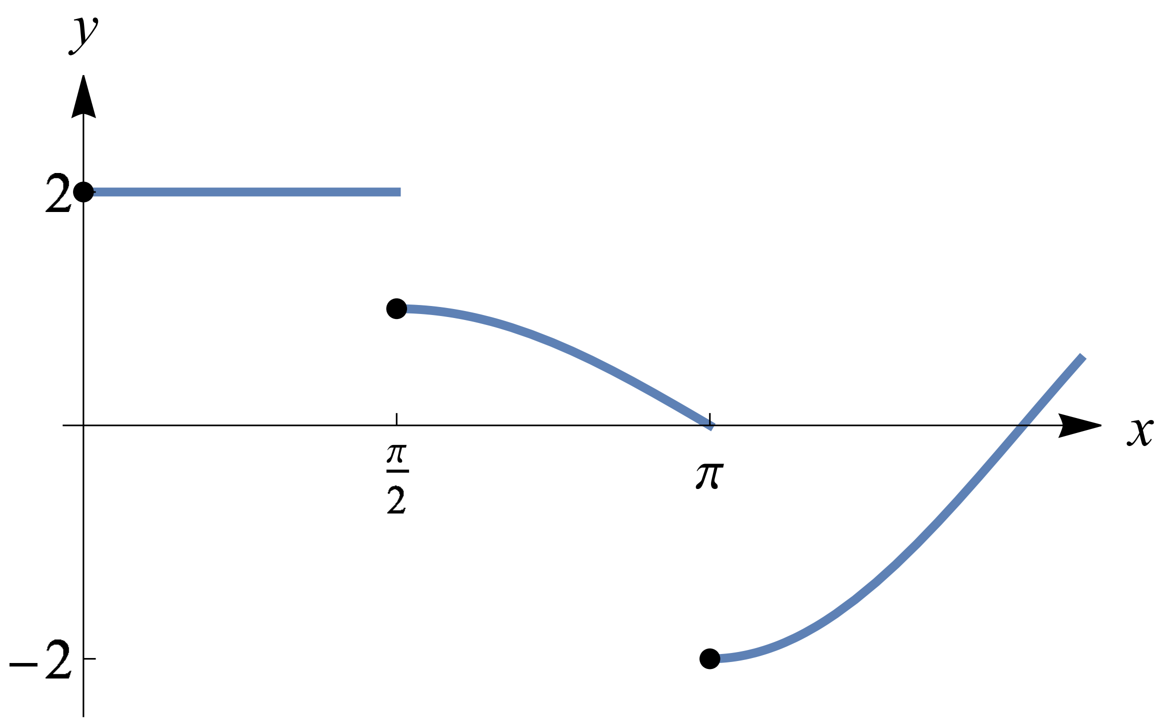
$$f(x) = x+1-4u(x-3)-(x-3)u(x-3)-\sin[\pi(x-3)]u(x-3)$$

Step 3. Determine $\mathcal{L}[f]$:

$$\mathcal{L}(f(x)) = \frac{1}{s^2} + \frac{1}{s} - e^{-3s}\frac{4}{s} - e^{-3s}\frac{1}{s^2} - e^{-3s} \cdot \frac{\pi}{s^2+\pi^2}$$

$$F(s) = \frac{1}{s^2} + \frac{1}{s} - 4e^{-3s}\frac{1}{s} - e^{-3s}\frac{1}{s^2} - e^{-3s}\frac{\pi}{s^2+\pi^2}$$

4. $$f(x) = \begin{cases} 2 & 0 \le x < \pi/2 \\ \sin x & \pi/2 \le x < \pi \\ 2\cos x & x \ge \pi \end{cases}$$



Step 1. Re-write f in terms of $u(x - \pi/2)$ and $u(x - \pi)$:

Step 2. Write coefficients of $f(x)$ in terms of $x-\frac{\pi}{2}$ and $x-\pi$:

(continued)

$$f(x) = 2 - 2u(x - \tfrac{\pi}{2}) + \cos(x - \tfrac{\pi}{2})u(x - \tfrac{\pi}{2}) + \sin(x - \pi)u(x - \pi) -$$

$$2\cos(x - \pi)u(x - \pi)$$

Step 3. Determine $\mathcal{L}[f]$:

$$F(s) = \frac{2}{s} - e^{-\pi s/2}\,\frac{2}{s} + e^{-\pi s/2}\,\frac{s}{s^2+1} + e^{-\pi s}\,\frac{1}{s^2+1} - 2e^{-\pi s}\,\frac{s}{s^2+1}$$

Inverse Transforms & Piecewise Continuous Functions: (Text, Section 4.5)

Recall **Property V**: Suppose that $\mathcal{L}[f(x)] = F(s)$.

Then, for any positive number c,

1. $\mathcal{L}[f(x-c)u(x-c)] = e^{-cs}F(s)$.

2. $\boxed{\mathcal{L}^{-1}[e^{-cs}F(s)] = f(x-c)u(x-c)}$.

Examples

1. $F(s) = \dfrac{1}{s+3}$. Then $f(x) = e^{-3x}$ and

$$\mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s+3}\right] = e^{-3(x-2)}u(x-2)$$

2. $F(s) = \dfrac{2}{s^2+4}$. Then $f(x) = \sin 2x$ and

$$\mathcal{L}^{-1}\left[e^{-\pi s}\frac{2}{s^2+4}\right] = \sin 2(x-\pi)u(x-\pi)$$

3. $F(s) = \dfrac{1}{(s-4)^2}$. Then $f(x) = xe^{4x}$ and

$$\mathcal{L}^{-1}\left[e^{-7s}\frac{1}{(s-4)^2}\right] = (x-7)e^{4(x-7)}u(x-7)$$

Examples: Given $F(s)$, find $f(x)$:

says this is a piecewise fcn jumps at 2

1. $F(s) = \frac{3}{s} - e^{-2s}\frac{2}{s} + e^{-2s}\frac{1}{s-2}$

$f(x) = \begin{cases} 3 & 0 \le x < 2 \\ \_\_\_\_ & \end{cases}$

Answer:

$$f(x) = 3 - 2u(x-2) + e^{2(x-2)}u(x-2)$$

$$= \begin{cases} 3, & 0 \le x < 2 \\ 1 + e^{2(x-2)}, & x \ge 2 \end{cases}$$

2. $F(s) = \frac{2}{s} + 4e^{-3s}\frac{1}{s(s+2)}$

Answer:

$$f(x) = 2 + 2u(x-3) - 2e^{-2(x-3)}u(x-3)$$

$$= \begin{cases} 2, & 0 \leq x < 3 \\ 4 - 2e^{6}\, e^{-2x}, & x \geq 3 \end{cases}$$

3. $F(s) = \dfrac{1 - e^{-\pi s}}{s(s^2 + 4)}$

Answer:

$$f(x) = \frac{1}{4} - \frac{1}{4}\cos 2x - \frac{1}{4}u(x-\pi) + \frac{1}{4}\cos(2[x-\pi])\,u(x-\pi)$$

$$= \frac{1}{4} - \frac{1}{4}\cos 2x - \frac{1}{4}u(x-\pi) + \frac{1}{4}\cos 2x\,u(x-\pi)$$

$$= \begin{cases} \frac{1}{4} - \frac{1}{4}\cos 2x, & 0 \le x < \pi \\ 0, & x \ge \pi \end{cases} .$$

4.

$$F(s) = \frac{4}{s} + \frac{2}{s^2} + 3e^{-2s}\frac{1}{s} - 2e^{-2s}\frac{1}{s^2} - 5e^{-4s}\frac{1}{s^2} + e^{-4s}\frac{1}{s-2}$$

Answer: $$f(x) = \begin{cases} 4 + 2x, & 0 \leq x < 2 \\ 3 + 4x, & 2 \leq x < 4 \\ 23 - x + e^{2(x-4)}, & x \geq 4 \end{cases}$$

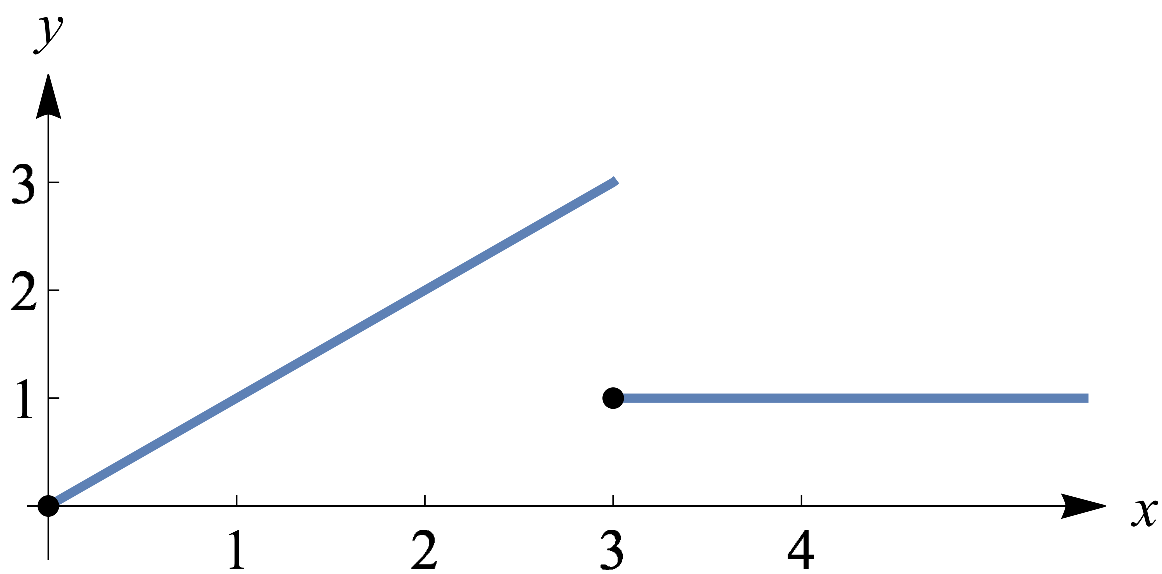
Application to Initial-Value Problems (Text, Section 4.7)

Examples:

1. Use the Laplace transform method to solve the initial-value problem

$$y' + 2y = f(x), \quad y(0) = 1.$$

where $f(x) = \begin{cases} x & 0 \le x < 3 \\ 1 & x \ge 3 \end{cases}$



$$y' + 2y = x - xu(x-3) + 1 \cdot u(x-3)$$

$$= x - [(x-3)+3]u(x-3) + u(x-3)$$

$$= x - (x-3)u(x-3) - 2u(x-3)$$

Laplace transform of the solution:

$$Y(s) = \frac{1}{s^2(s+2)} - \frac{e^{-3s}}{s^2(s+2)} - 2\frac{e^{-3s}}{s(s+2)} + \frac{1}{s+2}$$

$$\frac{1}{s^2(s+2)} = \frac{-1/4}{s} + \frac{1/2}{s^2} + \frac{1/4}{s+2}$$

$$\frac{2}{s(s+2)} = \frac{1}{s} - \frac{1}{s+2}$$

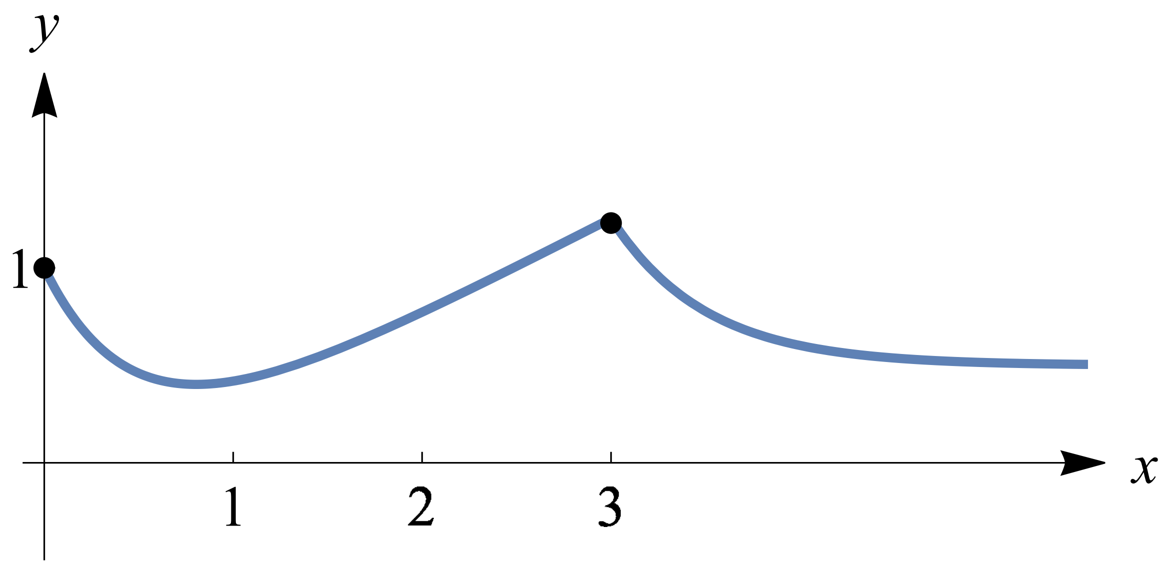
$$Y(s) = \frac{-1/4}{s} + \frac{1/2}{s^2} + \frac{5/4}{s+2} - e^{-3s}\left[\frac{3/4}{s} + \frac{1/2}{s^2} - \frac{3/4}{s+2}\right]$$

The solution:

$$y(x) =$$

$$y = \begin{cases} -\frac{1}{4} + \frac{1}{2}x + \frac{5}{4}e^{-2x}, & 0 \leq x < 3 \\ \frac{1}{2} + \frac{5}{4}e^{-2x} + \frac{3}{4}e^{-2(x-3)}, & x \geq 3. \end{cases}$$

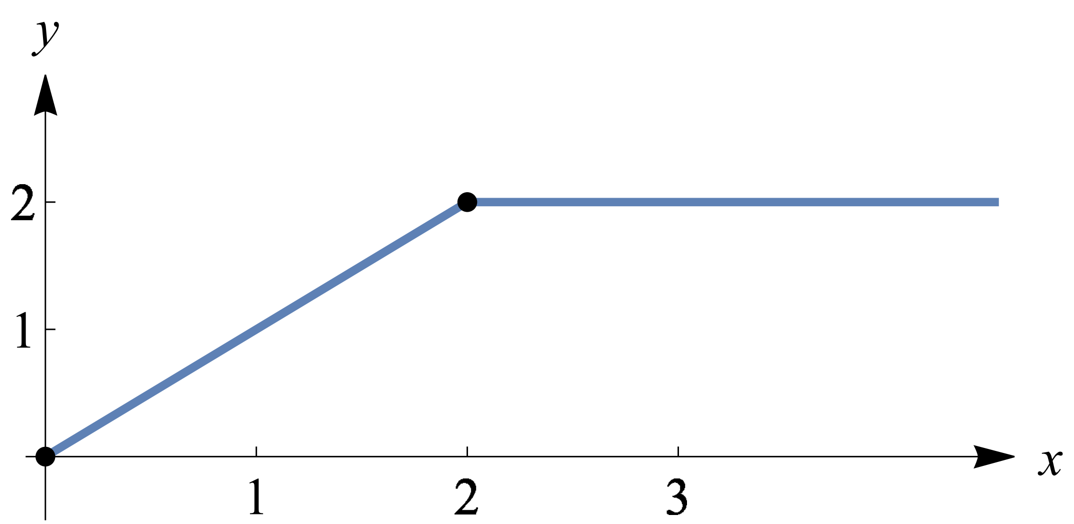
The graph of the solution:



2. Use the Laplace transform method to solve the initial-value problem

$$y' + y = f(x), \quad y(0) = 1.$$

where $f(x) = \begin{cases} x & 0 \leq x < 2 \\ 2 & x \geq 2 \end{cases}$



$$y' + y = x - (x-2)u(x-2)$$

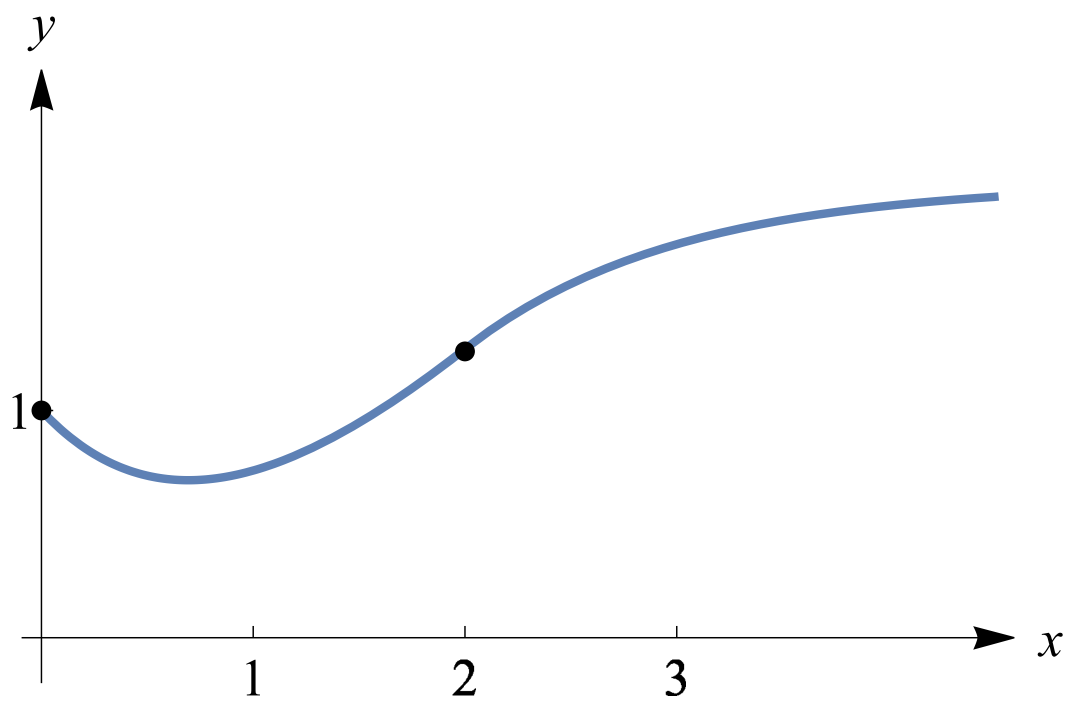
Laplace transform of the solution:

$$Y = \frac{-1}{s} + \frac{1}{s^2} + \frac{2}{s+1} - e^{-2s}\left[\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right]$$

The solution:

$$y(x) = \begin{cases} -1 + x + 2e^{-x} & 0 \le x < 2 \\ 2 + 2e^{-x} - e^{-(x-2)} & x \ge 2 \end{cases}$$

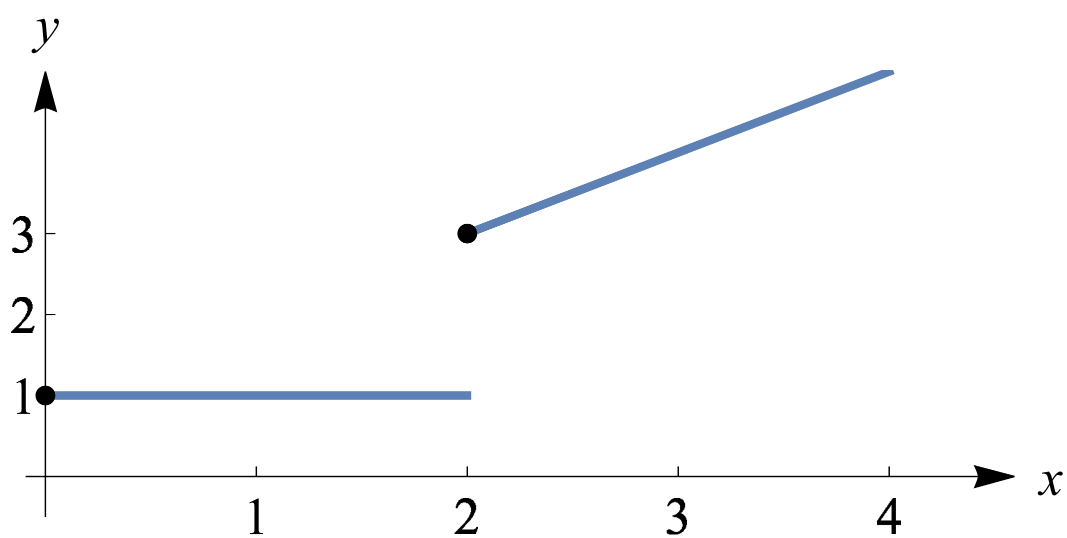
The graph of the solution:



3. Solve the initial-value problem

$$y'' + 2y' + y = f(x), \quad y(0) = y'(0) = 0.$$

where $f(x) = \begin{cases} 1 & 0 \le x < 2 \\ x + 1 & x \ge 2 \end{cases}$



Laplace transform of the solution:

$$Y(s) = \frac{1}{s(s+1)^2} + \frac{2e^{-2s}}{s(s+1)^2} + \frac{e^{-2s}}{s^2(s+1)^2}$$

The solution: $y(x) =$

$$\begin{cases} 1-(x+1)e^{-x}, & 0 \le x < 2 \\ x-1-(x+1)e^{-x}-(x-2)e^{-(x-2)}, & x \ge 2 \end{cases}$$

The graph of the solution: