The Laplace Transform

Let f be continuous function on $[0, \infty)$. *The Laplace transform of* f, denoted by $\mathcal{L}[f(x)]$, or by $F(s)$, is given by

$$\mathcal{L}[f(x)] = F(s) = \int_0^\infty e^{-sx} f(x)\, dx.$$

The domain of F is the set of real numbers s for which the improper integral converges.

f is of *exponential order* λ if there exists a positive number M and a nonnegative number A such that

$$|f(x)| \leq Me^{\lambda x} \quad \text{on} \quad [A, \infty).$$

Examples:

(a) Bounded functions, e.g., sin x, cos x

(b) Powers of x, $f(x) = x^k$.

(c) Exponential functions $f(x) = e^{ax}$.

$f(x) = e^{x^2}$ is **not** of exponential order.

THEOREM: Let f be a continuous function on $[0, \infty)$. If f is of exponential order λ, then the Laplace transform $\mathcal{L}[f(x)] = F(s)$ exists for $s > \lambda$.

Table of Laplace Transforms

$f(x)$	$F(s) = \mathcal{L}[f(x)]$
1	$\frac{1}{s}, \qquad s > 0$
$e^{\alpha x}$	$\frac{1}{s-\alpha}, \qquad s > r$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}, \qquad s > 0$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}, \qquad s > 0$
$e^{\alpha x} \cos \beta x$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}, \qquad s > r$
$e^{\alpha x} \sin \beta x$	$\frac{\beta}{(s-\alpha)^2+\beta^2}, \qquad s > r$
$x^n, \ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \qquad s > 0$
$x^n e^{\alpha x}, \ n = 1, 2, \ldots$	$\frac{n!}{(s-\alpha)^{n+1}}, \qquad s > r$
$x \cos \beta x$	$\frac{s^2-\beta^2}{(s^2+\beta^2)^2}, \qquad s > 0$
$x \sin \beta x$	$\frac{2\beta s}{(s^2+\beta^2)^2}, \qquad s > 0$

Properties of the Laplace Transform

1. $\mathcal{L}$ is a linear operator:

$$\mathcal{L}[f_1(x) + f_2(x)] = \mathcal{L}[f_1(x)] + \mathcal{L}[f_2(x)]$$

$$\mathcal{L}[cf(x)] = c\mathcal{L}[f(x)].$$

2. If f is continuously differentiable and of exponential order λ, then $\mathcal{L}[f'(x)]$ exists for $s > \lambda$ and

$$\mathcal{L}[f'(x)] = s\mathcal{L}[f(x)] - f(0).$$

• If f is twice continuously differentiable with f and f' of exponential order λ, then $\mathcal{L}[f''(x)]$ exists for $s > \lambda$ and

$$\mathcal{L}[f''(x)] = s^2\mathcal{L}[f(x)] - sf(0) - f'(0).$$

• In general, if f, f', $\cdots$, $f^{(n-1)}$ are of exponential order λ, then $\mathcal{L}[f^{(n)}(x)]$ exists for $s > \lambda$ and

$$\mathcal{L}[f^{(n)}(x)] =$$

$$s^n\mathcal{L}[f(x)] - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots y^{(n-1)}(0).$$

3. If $\mathcal{L}[f(x)] = F(s)$, then

$$\mathcal{L}[xf(x)] = -\frac{dF}{ds}, \quad \mathcal{L}[x^2 f(x)] = \frac{d^2F}{ds^2}$$

and, in general,

$$\mathcal{L}[x^n f(x)] = (-1)^n \frac{d^nF}{ds^n}.$$

4. If $\mathcal{L}[f(x)] = F(s)$, then

$$\mathcal{L}[e^{rx} f(x)] = F(s-r).$$

Examples:

1. Find the Laplace transform of

$$f(x) = 3 + 4e^{3x} - 2\cos 2x.$$

Ans: $F(s) = \dfrac{3}{s} + \dfrac{4}{s-3} - \dfrac{2s}{s^2+4}$

2. Find the Laplace transform of the solution of the initial-value problem:

$$y' - 2y = 4x;\ \ y(0) = 3.$$

Ans: $Y(s) = \dfrac{3s^2+4}{s^2(s-2)}$

3. Find the Laplace transform of the solution of the initial-value problem:

$y''-2y'+5y = 2x+e^{-x};\ y(0) = -2,\ y'(0) = 0.$

Ans:

$$Y(s) = \frac{s^2+2s+2}{s^2(s+1)(s^2-2s+5)}+\frac{4-2s}{s^2-2s+5}$$

Inverse Laplace transforms

Theorem If f and g are continuous functions on $[0, \infty)$, and if $\mathcal{L}[f(x)] = \mathcal{L}[g(x)]$, then $f \equiv g$; that is $f(x) = g(x)$ for all $x \in [0, \infty)$. ($\mathcal{L}$ is a one-to-one operator.)

If $F(s)$ is a given transform and if the function f, continuous on $[0, \infty)$, has the property that $\mathcal{L}[f(x)] = F(s)$, then f is called the *inverse Laplace transform of* $F(s)$, and is denoted by

$$f(x) = \mathcal{L}^{-1}[F(s)].$$

The operator $\mathcal{L}^{-1}$ is called the *inverse operator of* $\mathcal{L}$.

The operator $\mathcal{L}^{-1}$ is linear; that is

$$\mathcal{L}^{-1}[F(s) + G(s)] = \mathcal{L}^{-1}[F(s)] + \mathcal{L}^{-1}[G(s)]$$

and

$$\mathcal{L}^{-1}[cF(s)] = c\mathcal{L}^{-1}[F(s)], \quad c \text{ any constant.}$$

Examples:

1. Find the Laplace transform of the solution of the initial-value problem

$$y' + 2y = 3e^{x}, \qquad y(0) = 4,$$

then find the solution of the problem.

(a) $Y = \dfrac{4s-1}{(s-1)(s+2)} = \dfrac{1}{s-1} + \dfrac{3}{s+2}$

(b) $y = e^{x} + 3e^{-2x}$

2. $F(s) = \dfrac{2s-1}{s^4 + 4s^2}$

Find $f(x) = \mathcal{L}^{-1}[F(s)]$.

Ans. $f(x) = \frac{1}{2} - \frac{1}{4}x - \frac{1}{2}\cos 2x + \frac{1}{8}\sin 2x$

3. $F(s) = \dfrac{3s+2}{(s-1)(s^2+2s+5)}$

Find $f(x) = \mathcal{L}^{-1}[F(s)]$.

Ans. $f(x) = \frac{5}{8}e^{x} - \frac{5}{8}e^{-x}\cos 2x + \frac{7}{8}e^{-x}\sin 2x$

4. Find the Laplace transform of the solution of the initial-value problem

$y'' - y' - 2y = \sin 2x, \quad y(0) = 1,\ y'(0) = 2$

then find the solution of the problem.

(a) $Y = \dfrac{2}{(s^2+4)(s+1)(s-2)} + \dfrac{1}{s-2}$

$$= \frac{13/12}{s-2} - \frac{2/15}{s+1} + \frac{1}{20}\,\frac{s}{s^2+4} - \frac{3}{10}\,\frac{2}{s^2+4}.$$

(b) $y = \frac{13}{12}e^{2x} - \frac{2}{15}e^{-x} + \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x.$

Jump Discontinuity: Let the function $f = f(x)$ be defined on an interval I and continuous except at a point $c \in I$. If

$$\lim_{x \to c^-} f(x) \quad \text{and} \quad \lim_{x \to c^-} f(x)$$

exist, but

$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^-} f(x),$$

then f is said to have a *jump* (or *finite*) *discontinuity* at c.

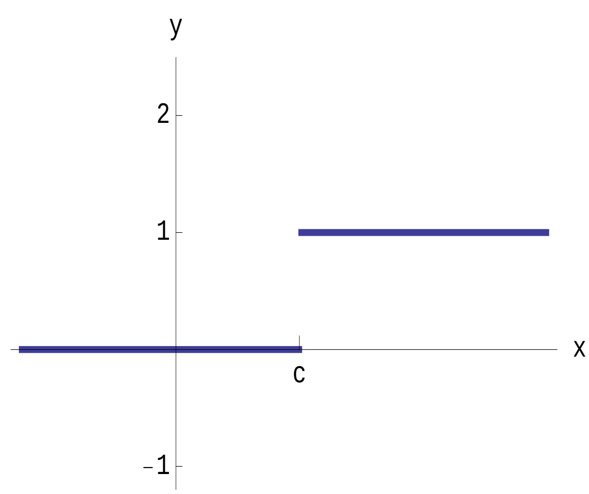
Piecewise Continuous Functions: A function f defined on an interval I is *piecewise continuous on* I if it is continuous on I except for at most a finite number of points $c_1, c_2, \ldots, c_n$ of I at which it has jump discontinuities.

THEOREM: If the function f is piecewise continuous on $[0, \infty)$, and of exponential order λ, then the Laplace transform $\mathcal{L}[f(x)]$ exists for $s > \lambda$.

Unit Step Functions: Let $c > 0$. The function

$$u_c(x) = u(x-c) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$$

is called a *unit step function*.



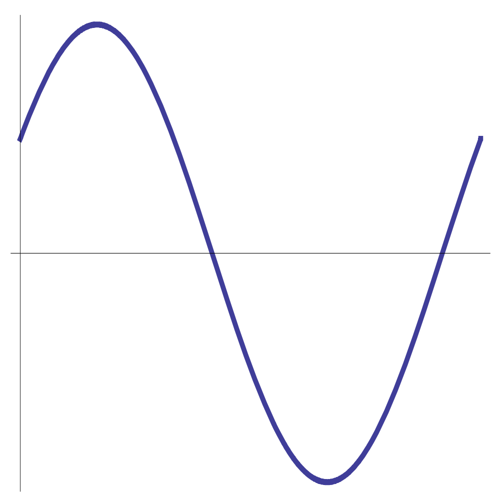
Laplace Transform of a Unit Step Function:

$$\mathcal{L}[u(x-c)] = e^{-cs}\,\frac{1}{s}, \quad s > 0.$$

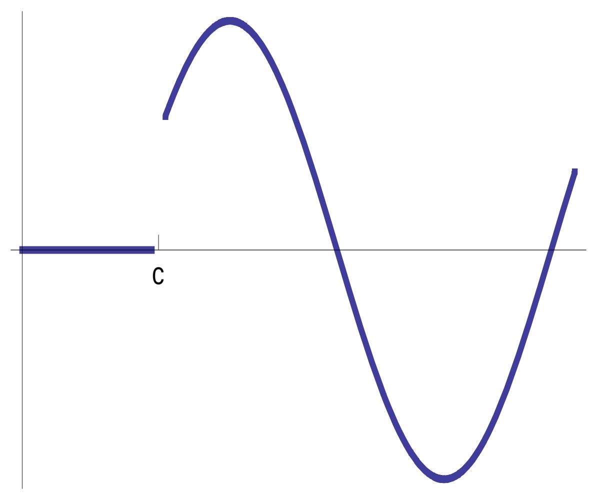
Translation of a Function: if f is defined on $[0, \infty)$ and $c > 0$, then the function

$$f(x-c)u(x-c) = \begin{cases} 0 & x < c \\ f(x-c)u(x-c) & x \geq c \end{cases}$$

is a *translation of* f.



Property 5. Laplace Transform of a Translated Function:



Suppose that $\mathcal{L}[f(x)] = F(s)$. Then, for any positive number c,

1. $\mathcal{L}[f(x-c)u(x-c)] = e^{-cs}F(s)$.

2. $\mathcal{L}^{-1}[e^{-cs}F(s)] = f(x-c)u(x-c)$.

Examples: Find $\mathcal{L}[f(x)]$.

1. $f(x) = \begin{cases} 2x & 0 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$.

Step 1. Express f in terms of $u(x-3)$:

$$f(x) = 3 + 3u(x-3) + 2(x-3)u(x-3)$$

Step 2. Determine $\mathcal{L}[f]$:

$$F(s)] = \frac{3}{s} + 3e^{-3s}\frac{1}{s} + 2e^{-3s}\frac{1}{s^2}$$

2. $f(x) = \begin{cases} x^2 & 0 \le x < 2 \\ 3x & x \ge 2 \end{cases}$.

Step 1. Express f in terms of $u(x-2)$:

$$f(x) = x^2 - (x-2)^2 u(x-2) - (x-2)u(x-2) + 2u(x-2)$$

Step 2. Determine $\mathcal{L}[f]$:

$$F(s)] = \frac{2}{s^3} - e^{-2s}\frac{2}{s^3} - e^{-2s}\frac{1}{s^2} + 2e^{-2s}\frac{1}{s}$$

3. $f(x) = \begin{cases} 1 & 0 \le x < \pi/2 \\ \sin x & \pi/2 \le x \le \pi \\ 2\cos x & x \ge \pi \end{cases}$.

Step 1. Express f in terms of

$$u(x-\pi/2) \text{ and } u(x-\pi):$$

$$f(x) = 1 - u(x - \tfrac{\pi}{2}) + \cos(x - \tfrac{\pi}{2})u(x - \tfrac{\pi}{2}) +$$

$$\sin(x-\pi)u(x-\pi) - 2\cos(x-\pi)u(x-\pi)$$

Step 2. Determine $\mathcal{L}[f]$:

$$F(s) = \frac{1}{s} - e^{\pi s/2}\,\frac{1}{s} + e^{-\pi s/2}\,\frac{s}{s^2+1} +$$

$$+ e^{-\pi s}\,\frac{1}{s^2+1} - 2e^{-\pi s}\,\frac{s}{s^2+1}$$

Examples: Given $F(s)$ find $f(x)$:

1. $F(s) = \frac{3}{s} - 2e^{-2s}\frac{1}{s} + e^{-2s}\frac{1}{s-2}$

$$f(x) = 3 - 2u(x-2) + e^{2(x-2)}u(x-2)$$

$$= \begin{cases} 3 & 0 \le x < 2 \\ 1 + e^{2(x-2)} & x \ge 2 \end{cases} .$$

2. $F(s) = \frac{2}{s} + 4e^{-3s}\frac{1}{s(s+2)}$

$$f(x) = 2 + 2u(x-3) - 2e^{-2(x-3)}u(x-3)$$

$$= \begin{cases} 2 & 0 \le x < 3 \\ 4 - 2e^{6}\,e^{-2x} & x \ge 3 \end{cases} .$$

3. $F(s) = \dfrac{1 - e^{-\pi s}}{s(s^2 + 4)}$

$$f(x) = \tfrac{1}{4} - \tfrac{1}{4}\cos 2x - \tfrac{1}{4}u(x - \pi) +$$

$$\tfrac{1}{4}\cos(2[x - \pi])u(x - \pi)$$

$$= \tfrac{1}{4} - \tfrac{1}{4}\cos 2x - \tfrac{1}{4}u(x - \pi) + \tfrac{1}{4}\cos 2x(u(x - \pi)$$

$$= \begin{cases} \tfrac{1}{4} - \tfrac{1}{4}\cos 2x & 0 \le x < \pi \\ 0 & x \ge \pi \end{cases} .$$

Example: Use the Laplace transform method to solve the initial-value problem

$$y' + 3y = f(x), \quad y(0) = 1.$$

where $f(x) = \begin{cases} \sin x & 0 \le x < \pi \\ 0 & x \ge \pi \end{cases}$.

$$Y(s) = \frac{1}{(s^2+1)(s+3)} + \frac{e^{-\pi s}}{(s^2+1)(s+3)} +$$

$$\frac{1}{s+3}$$

$$y = \begin{cases} \frac{11}{10}e^{-3x} + \frac{3}{10}\sin x - \frac{1}{10}\cos x & 0 \le x < \pi \\ \frac{11}{10}e^{-3x} + \frac{1}{10}e^{-3(x-\pi)} & x \ge \pi \end{cases}.$$

Example: Use the Laplace transform method to solve the initial-value problem

$$y'' + 2y' + y = f(x), \qquad y(0) = y'(0) = 0.$$

where $f(x) = \begin{cases} 1 & 0 \le x < 2 \\ x - 1 & x \ge 2 \end{cases}$.

$$Y(s) = \frac{1}{s(s+1)^2} + \frac{e^{-2s}}{s^2(s+1)^2}$$

$$y = \begin{cases} 1 - e^{-x} - xe^{-x} \\ -3 - e^{-x} - xe^{-x} + x + xe^{-(x-2)} - e^{-(x-2)} \end{cases}$$