Chapter 5. Linear Algebra

Sections 5.1 – 5.4

A **linear (algebraic) equation in** n **unknowns**, $x_1, x_2, \ldots, x_n$, is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers.

Linear equations in one unknown

x:

$$ax = b.$$

Exactly one of following holds:

(1) there is precisely one solution (a unique solution)

$$x = a^{-1}b = \frac{b}{a}, \quad a \neq 0,$$

(2) there are no solutions

$$0x = b, \quad b \neq 0$$

(3) there are infinitely many solutions

$$0x = 0.$$

Linear equations in two unknowns x, y and geometry

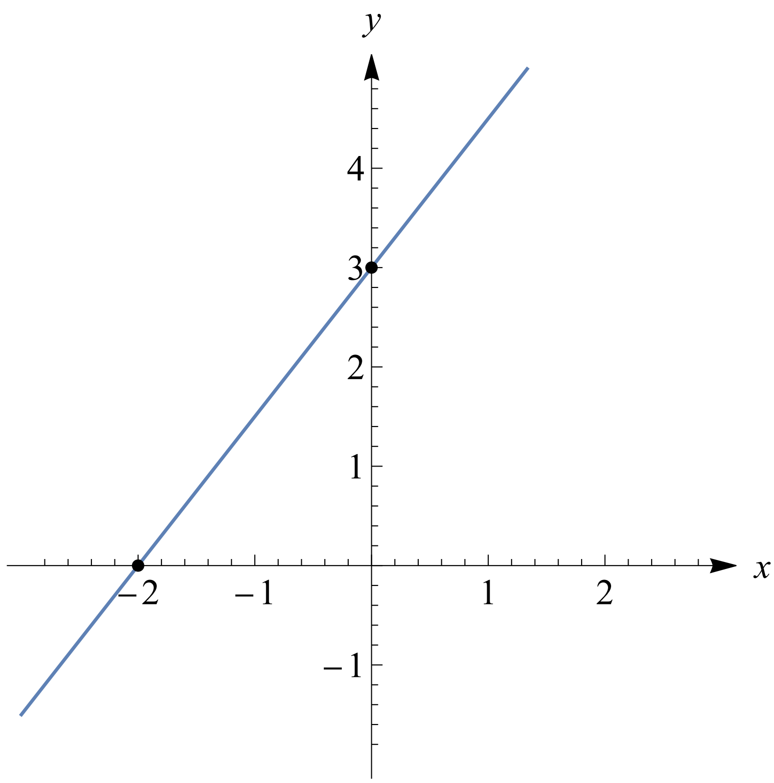
$$ax + by = \alpha.$$

A solution of the equation is an ordered pair of numbers (x_0, y_0).

Assuming a, b, not both 0, then the set of all ordered pairs that satisfy the equation is a straight line (in the x, y-plane). The equation has infinitely many solutions — the set of points that lie on the line.

Example:

$-3x + 2y = 6$



A system of two linear equations in two unknowns:

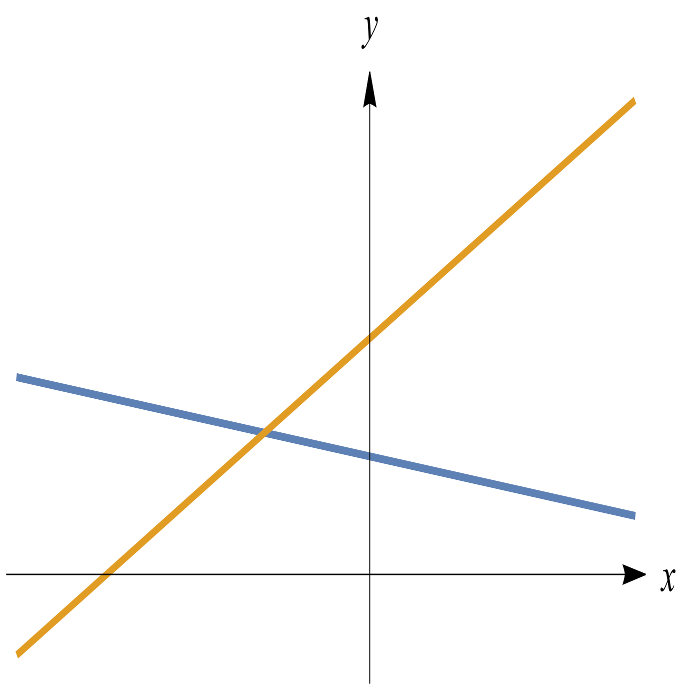
$$ax + by = \alpha$$

$$cx + dy = \beta$$

Find ordered pairs (x, y) that satisfy both equations simultaneously.

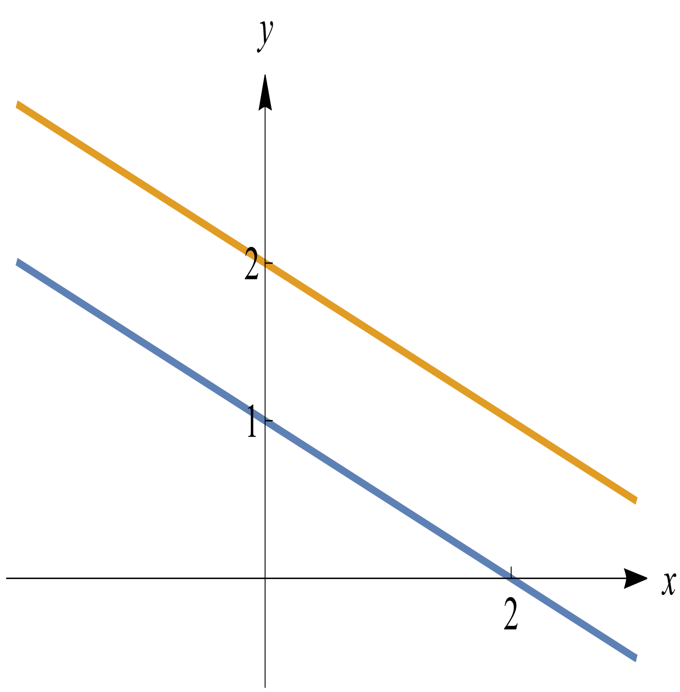
Two lines in the plane either

(a) have a unique point of intersection (the lines have different slopes), and the system has a unique solution



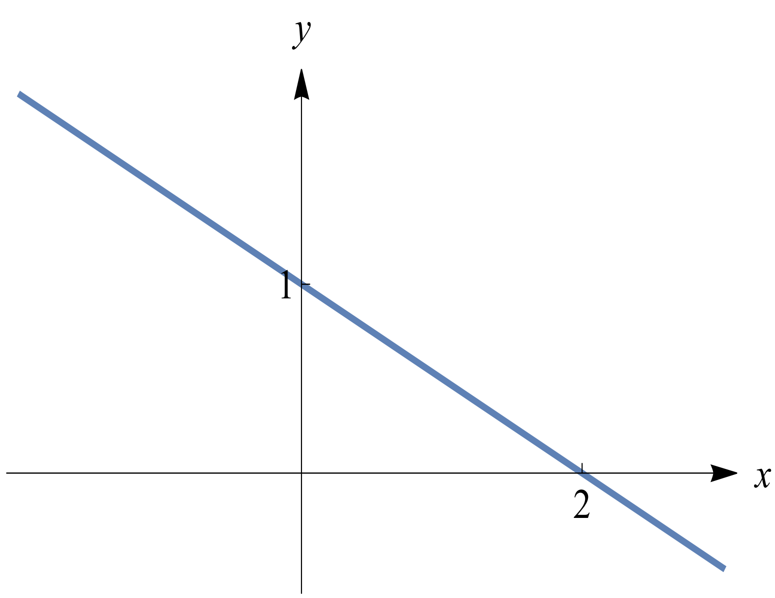
(b) are parallel (the lines have the same slope but, for example, different y-intercepts)

The system has NO solutions, there is no point that lies on <u>both</u> lines



(c) coincide (same slope, same y-intercept), and the system has infinitely many solutions.

Example:



$$x + 2y = 2$$

$$2x + 4y = 4$$

Two equations in two unknowns: There is either

(a) **a unique solution**,

(b) **no solution**,

or

(c) **infinitely many solutions**.

A system of three linear equations in two unknowns:

$$ax + by = \alpha$$

$$cx + dy = \beta$$

$$ex + fy = \gamma$$

Find ordered pairs (x, y) that satisfy the three equations simultaneously.

There is either a

(a) **unique solution**,

(b) **no solution**, (this is usually what happens)

or

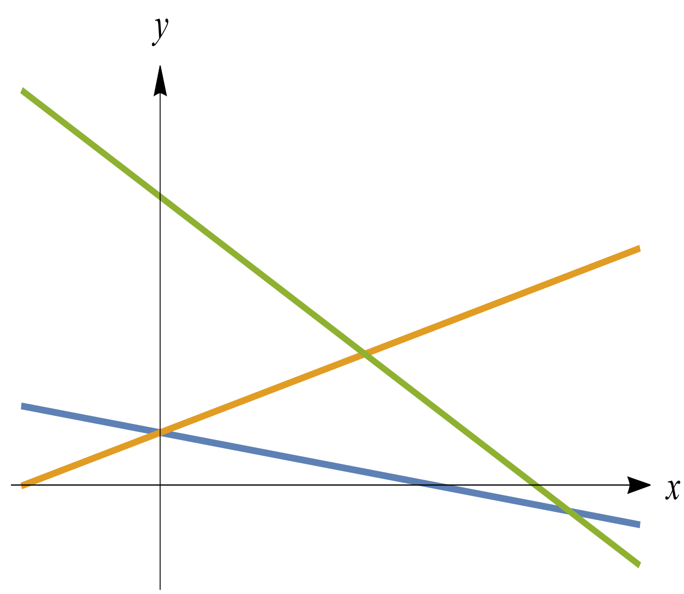
(c) **infinitely many solutions**.

Example:

$$x + y = 2$$

$$-2x + y = 2$$

$$4x + y = 11$$

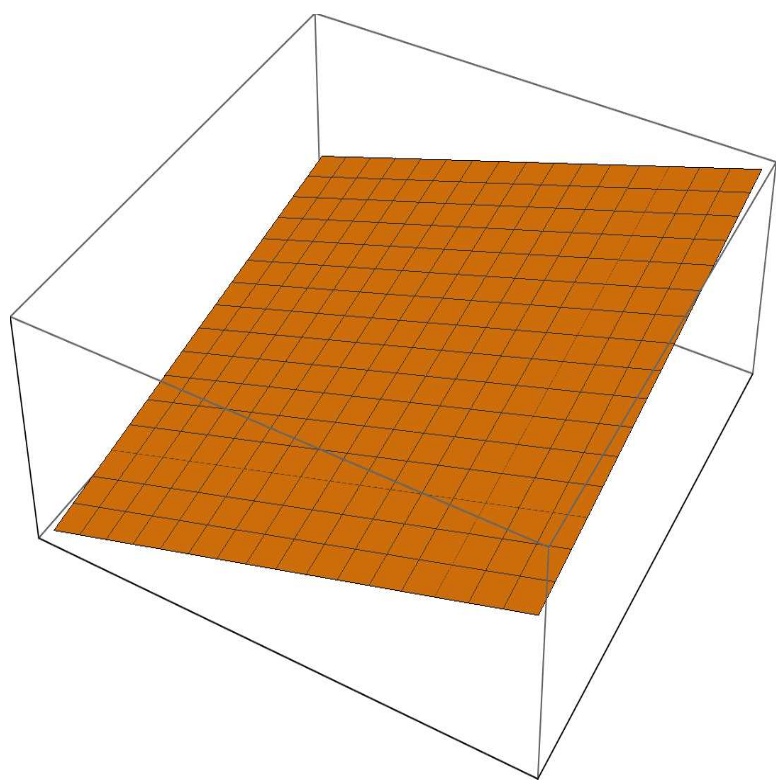


Linear equations in three unknowns $x,\ y,\ z$ and geometry

$$ax + by + cz = \alpha.$$

A solution of the equation is an ordered triple of numbers (x_0, y_0, z_0).

Assuming a, b, c, not all 0, then the set of all ordered triples that satisfy the equation is a plane (in 3-space).

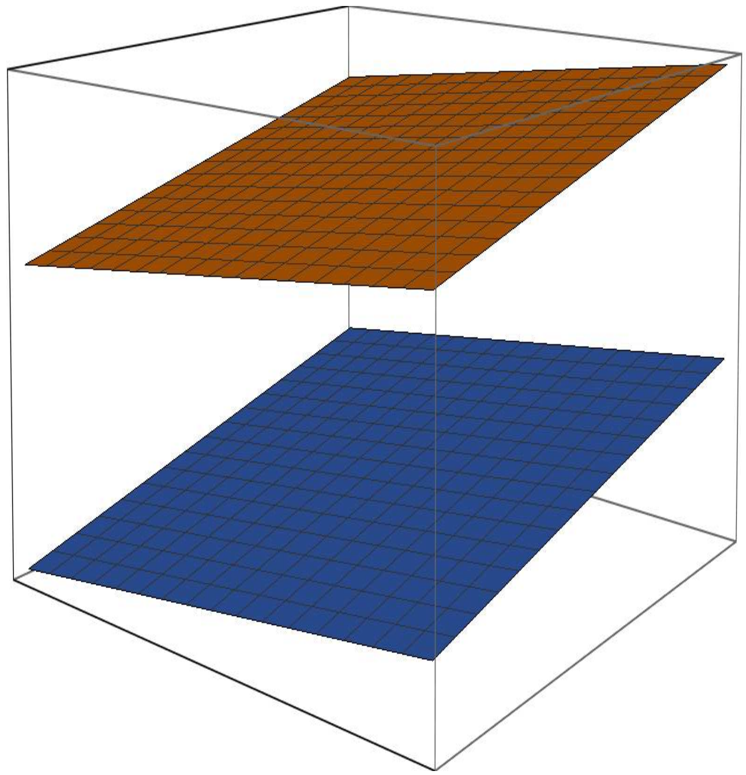


A system of two linear equations in three unknowns

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

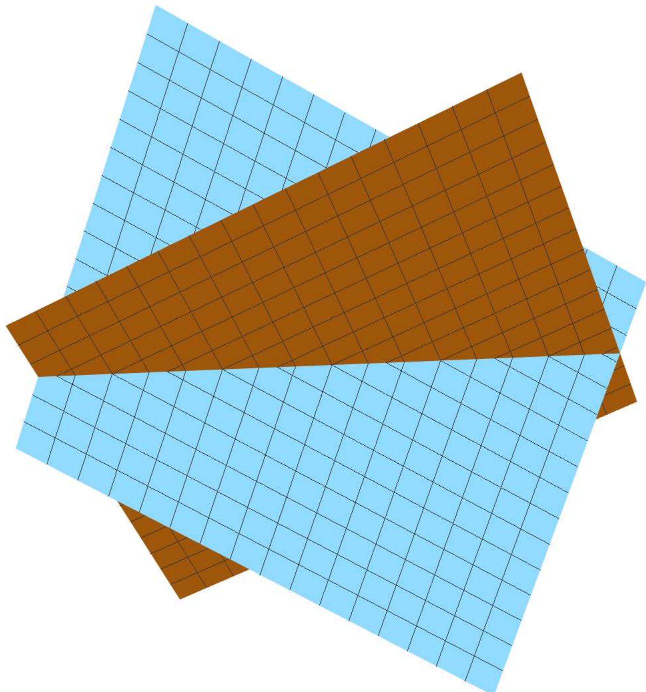
- Either the two planes are parallel (the system has no solutions),



• they coincide (infinitely many solutions, a whole plane of solutions),

- they intersect in a straight line

(again, infinitely many solutions.)



A system of three linear equations in three unknowns.

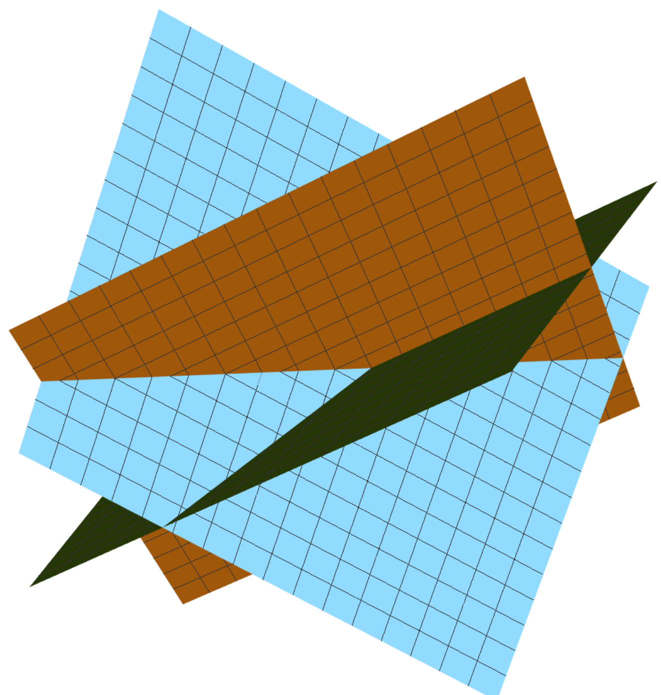
$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The system represents three planes in 3-space.

(a) The system has a unique solution; the three planes have a unique point of intersection;



(b) The system has infinitely many solutions; the three planes intersect in a line, or the three planes intersect in a plane.

(c) The system has no solution; there is no point the lies on all three planes.

Solution methods for systems of Linear Algebraic Equations

Example 1: Solve the system

$$x + 3y = -5$$

$$2x - y = 4$$

Equivalent system

$$
\begin{aligned}
x + 3y &= -5 \\
y &= -2
\end{aligned}
$$

Solution set:

$$x = 1, \quad y = -2$$

Definition: Two systems of linear equations S_1 and S_2 are **equivalent** if they have exactly the same solution set.

Example 2: Solve the system

$$x - 2y + 4z = 12$$

$$2x - y + 5z = 18$$

$$-x + 3y - 3z = -8$$

Example 2 con't

Equivalent system

$$
\begin{aligned}
x - 2y + 4z &= 12 \\
y - z &= -2 \\
z &= 3
\end{aligned}
$$

Solution set:

$$x = 2, \quad y = 1, \quad z = 3$$

The Elementary Operations

The operations that produce equivalent systems are called **elementary operations**.

1. Multiply/divide an equation by a nonzero number.

2. Interchange two equations.

3. Multiply an equation by a number and add it to another equation.

Example 3: Solve the system

$$3x - 4y - z = 3$$

$$2x - 3y + z = 1$$

$$x - 2y + 3z = 2$$

Example 3 continued

Equivalent system

$$
\begin{aligned}
x - 2y + 3z &= 2 \\
y - 5z &= -3 \\
0z &= 1
\end{aligned}
$$

The system has no solution.

Example 4: Solve the system

$$x_1 - 2x_2 + x_3 - x_4 = -2$$

$$-2x_1 + 5x_2 - x_3 + 4x_4 = 1$$

$$3x_1 - 7x_2 + 2x_3 + x_4 = 9$$

Example 4 continued

Equivalent system

$$x_1 - 2x_2 + x_3 - x_4 = -2$$

$$x_2 + x_3 + 2x_4 = -3$$

$$x_4 = 2$$

Solution set:

$$x_1 = -14 - 3a,$$

$$x_2 = -7 - a,$$

$$x_3 = a,$$

$$x_4 = 2, \qquad a \text{ any real number.}$$

Solve the system (same as Ex. 2)

$$x - 2y + 4z = 12$$

$$2x - y + 5z = 18$$

$$-x + 3y - 3z = -8$$

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 3 & -3 & -6 \\ 0 & 1 & 1 & 4 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Corresponding (equivalent) system

of equations:

$$\begin{aligned} x - 2y + 4z &= 12 \\ y - z &= -2 \\ z &= 3 \end{aligned}$$

Solution set:

$$x = 2,\ y = 1,\ z = 3$$

Matrix, Augmented Matrix, Matrix of Coefficients

A **matrix** is a rectangular array of numbers. A matrix with m rows and n columns is an $m \times n$ matrix.

Matrix representation of a system of linear equations

$$
\begin{array}{ccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n = b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n = b_2 \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n = b_m
\end{array}
$$

Augmented matrix and matrix of coefficients:

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Matrix of coefficients:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{32} & \cdots & a_{mn} \end{pmatrix}$$

Elementary row operations:

1. Interchange row i and row j

$$R_i \leftrightarrow R_j.$$

2. Multiply row i by a nonzero number k

$$kR_i \to R_i.$$

3. Multiply row i by a number k and add the result to row j

$$kR_i + R_j \to R_j.$$

Examples

5. Solve the system (same as Ex. 3)

$$3x - 4y - z = 3$$

$$2x - 3y + z = 1$$

$$x - 2y + 3z = 2$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & -4 & -1 & 3 \\ 2 & -3 & 1 & 1 \\ 1 & -2 & 3 & 2 \end{array}\right).$$

Row reduce

$$\left(\begin{array}{ccc|c} 3 & -4 & -1 & 3 \\ 2 & -3 & 1 & 1 \\ 1 & -2 & 3 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 2 & -3 & 1 & 1 \\ 3 & -4 & -1 & 3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 2 & -10 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Corresponding system of equations:

$$x - 2y + 3z = 2$$

$$0x + y - 5z = -3$$

$$0x + 0y + 0z = 1$$

That is

$$
\begin{aligned}
x - 2y + 3z &= 2 \\
y - 5z &= -3 \\
0z &= 1
\end{aligned}
$$

Solution set: no solution.

6. Solve the system

$$x + y - 3z = 1$$

$$2x + y - 4z = 0$$

$$-3x + 2y - z = 7$$

Augmented matrix:

$$\left(\begin{array}{rrr|r} 1 & 1 & -3 & 1 \\ 2 & 1 & -4 & 0 \\ -3 & 2 & -1 & 7 \end{array}\right)$$

Row reduce

$$\left(\begin{array}{rrr|r} 1 & 1 & -3 & 1 \\ 2 & 1 & -4 & 0 \\ -3 & 2 & -1 & 7 \end{array}\right)$$

Equivalent system:

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Corresponding system of equations:

$$x + y - 3z = 1$$

$$0x + y - 2z = 2$$

$$0x + 0y + 0z = 0$$

or

$$x + y - 3z = 1$$

$$y - 2z = 2$$

$$0z = 0$$

or

$$x + y - 3z = 1$$

$$y - 2z = 2$$

This system has infinitely many solutions given by:

$$x = -1 + a,$$

$$y = 2 + 2a,$$

$$z = a, \qquad a \text{ any real number.}$$

7. Solve the system (same as Ex. 4)

$$x_1 - 2x_2 + x_3 - x_4 = -2$$

$$-2x_1 + 5x_2 - x_3 + 4x_4 = 1$$

$$3x_1 - 7x_2 + 2x_3 + x_4 = 9$$

Augmented matrix:

$$\left(\begin{array}{rrrr|r} 1 & -2 & 1 & -1 & -2 \\ -2 & 5 & -1 & 4 & 1 \\ 3 & -7 & 2 & 1 & 9 \end{array}\right)$$

Row Reduce $\left(\begin{array}{rrrr|r} 1 & -2 & 1 & -1 & -2 \\ -2 & 5 & -1 & 4 & 1 \\ 3 & -7 & 2 & 1 & 9 \end{array}\right) \longrightarrow$

$$\left(\begin{array}{rrrr|r} 1 & -2 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 & -3 \\ 0 & -1 & -1 & 4 & 15 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{rrrr|r} 1 & -2 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 & -3 \\ 0 & 0 & 0 & 6 & 12 \end{array}\right) \longrightarrow \left(\begin{array}{rrrr|r} 1 & -2 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

Equivalent system

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= -2 \\ x_2 + x_3 + 2x_4 &= -3 \\ x_4 &= 2 \end{aligned}$$

Solution set:

$$x_1 = -14 - 3a,$$

$$x_2 = -7 - a,$$

$$x_3 = a,$$

$$x_4 = 2, \qquad a \text{ any real number.}$$

Row echelon form:

1. Rows consisting entirely of zeros are at the bottom of the matrix.

2. The first nonzero entry in a nonzero row is a 1. It is called the *leading* 1.

3. If row i and row $i+1$ are nonzero rows, then the leading 1 in row $i+1$ is to the right of the leading 1 in row i.

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \qquad \text{(Example 4)}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \text{(Example 5)}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{(Example 6)}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \qquad \text{(Example 7)}$$

NOTE:

1. All the entries below a leading 1 are zero.

2. The number of leading 1's is less than or equal to the number of rows.

3. The number of leading 1's is less than or equal to the number of columns.

Solution method for systems of linear equations:

1. Write the augmented matrix $(A|b)$ for the system.

2. Use elementary row operations to transform the augmented matrix to row echelon form.

3. Write the system of equations corresponding to the row echelon form.

4. Back substitute to find the solution set.

This method is called **Gaussian elimination with back substitution.**

Consistent/Inconsistent systems:

A system of linear equations is **consistent** if it has at least one solution.

That is, a system is consistent if it has either a unique solution or infinitely many solutions.

A system that has no solutions is **inconsistent**.

Consistent systems:

A consistent system is said to be **independent** if it has a unique solution.

A system with infinitely many solutions is called **dependent**.

8. Solve the system of equations

$$2x_1 + 5x_2 - 5x_3 - 7x_4 = 8$$

$$x_1 + 2x_2 - 3x_3 - 4x_4 = 2$$

$$-3x_1 - 6x_2 + 11x_3 + 16x_4 = 0$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} 2 & 5 & -5 & -7 & 8 \\ 1 & 2 & -3 & -4 & 2 \\ -3 & -6 & 11 & 16 & 0 \end{array}\right).$$

Transform to row echelon form:

$$\left(\begin{array}{rrrr|l} 2 & 5 & -5 & -7 & 8 \\ 1 & 2 & -3 & -4 & 2 \\ -3 & -6 & 11 & 16 & 0 \end{array}\right).$$

Equivalent system:

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & -4 & 2 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right).$$

Corresponding system of equations:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 - 4x_4 &= 2 \\ x_2 + x_3 + x_4 &= 4 \\ x_3 + 2x_4 &= 3 \end{aligned}$$

Solution set:

$$x_1 = 9 - 4a,$$

$$x_2 = 1 + a,$$

$$x_3 = 3 - 2a,$$

$$x_4 = a, \qquad a \text{ any real number.}$$

9. Solve the system of equations

$$x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 = 2$$

$$3x_1 - 9x_2 + 7x_3 - x_4 + 3x_5 = 7$$

$$2x_1 - 6x_2 + 7x_3 + 4x_4 - 5x_5 = 7$$

Augmented matrix:

$$\left(\begin{array}{ccccr|c} 1 & -3 & 2 & -1 & 2 & 2 \\ 3 & -9 & 7 & -1 & 3 & 7 \\ 2 & -6 & 7 & 4 & -5 & 7 \end{array}\right)$$

Transform to row echelon form:

$$\left(\begin{array}{ccccc|c} 1 & -3 & 2 & -1 & 2 & 2 \\ 3 & -9 & 7 & -1 & 3 & 7 \\ 2 & -6 & 7 & 4 & -5 & 7 \end{array}\right)$$

Equivalent system:

$$\left(\begin{array}{ccccc|c} 1 & -3 & 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Corresponding system of equations:

$$x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 = 2$$

$$0x_1 + 0x_2 + x_3 + 2x_4 - 3x_5 = 1$$

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 0.$$

which is

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 &= 2 \\ x_3 + 2x_4 - 3x_5 &= 1 \end{aligned}$$

Solution set:

$$x_1 = 3a + 5b - 8c,$$

$$x_2 = a,$$

$$x_3 = 1 - 2b + 3c,$$

$$x_4 = b,$$

$$x_5 = c,$$

a, b, c arbitrary real numbers

10. For what value(s) of k, if any, does the system

$$x + y - z = 1$$
$$2x + 3y + kz = 3$$
$$x + ky + 3z = 2$$

have:

(a) a unique solution?

(b) infinitely many solutions?

(c) no solution?

$$\left(\begin{array}{ccr|c} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & (k+3)(k-2) & k-2 \end{array}\right)$$

(a) Unique solution: $k \neq 2, -3$.

(b) Infinitely many solns: $k = 2$.

(c) No solution: $k = -3$.

11. For what value(s) of k, if any, does the system

$$\begin{aligned} x + 2y + 3z &= 4 \\ y + 5z &= 9 \\ 2x + 3y + (k^2 - 8)z &= k + 2 \end{aligned}$$

have:

(a) a unique solution?

(b) infinitely many solutions?

(c) no solution?

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 9 \\ 2 & 3 & k^2-8 & k+2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 9 \\ 0 & 0 & k^2-9 & k+3 \end{array}\right)$$

(a) Unique solution: $k \neq -3, 3$.

(b) Infinitely many solns: $k = -3$.

(c) No solution: $k = 3$.

If an $m \times n$ matrix A is reduced to row echelon form, then the number of non-zero rows in its row echelon form is called the **rank** of A.

Equivalently, the rank of a matrix is the number of leading 1's in its row echelon form.

The rank of a matrix is less than or equal to the number of rows. (Obvious)

Consistent/Inconsistent Systems

Theorem: A system of linear equations is consistent if and only if the rank of the coefficient matrix **equals** the rank of the augmented matrix.

If the rank of the augmented matrix is **greater than** the rank of the coefficient matrix, then the system has no solutions.

5.4. Reduced Row Echelon Form

Examples

1. Solve the system (See Example 5, pg. 41)

$$x - 2y + 4z = 12$$

$$2x - y + 5z = 18$$

$$-x + 3y - 3z = -8$$

Augmented matrix:

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{array}\right)$$

Row reduce to:

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Corresponding (equivalent) system

of equations

$$
\begin{aligned}
x - 2y + 4z &= 12 \\
y - z &= -2 \\
z &= 3
\end{aligned}
$$

Back substitute to get:

$$x = 2, \quad y = 1, \quad z = 3.$$

Or, continue row operations:

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \rightarrow$$

Corresponding system of equations

$$\begin{aligned} x &= 2 \\ y &= 1 \\ z &= 3 \end{aligned}$$

2. Solve the system (c.f. Example 8)

$$2x_1 + 5x_2 - 5x_3 - 7x_4 = 8$$

$$x_1 + 2x_2 - 3x_3 - 4x_4 = 2$$

$$-3x_1 - 6x_2 + 11x_3 + 16x_4 = 0$$

Augmented matrix:

$$\left(\begin{array}{rrrr|r} 2 & 5 & -5 & -7 & 8 \\ 1 & 2 & -3 & -4 & 2 \\ -3 & -6 & 11 & 16 & 0 \end{array}\right) \to$$

Row echelon form:

$$\left(\begin{array}{rrrr|r} 1 & 2 & -3 & -4 & 2 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right).$$

Corresponding system of equations:

$$x_1 + 2x_2 - 3x_3 - 4x_4 = 2$$

$$x_2 + x_3 + x_4 = 4$$

$$x_3 + 2x_4 = 3$$

Solution set:

$$x_1 = 9 - 4a,$$

$$x_2 = 1 + a,$$

$$x_3 = 3 - 2a,$$

$$x_4 = a, \qquad a \text{ any real number.}$$

Alternative solution: Continue the row operations

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & -4 & 2 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right) \rightarrow$$

Reduced row echelon form

$$\left(\begin{array}{rrrr|r} 1 & 0 & 0 & 4 & 9 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right)$$

Corresponding system of equations:

$$x_1 + 4x_4 = 9$$

$$x_2 - x_4 = 1$$

$$x_3 + 2x_4 = 3$$

$x_3 = 3-2x_4$, $x_2 = 1+x_4$, $x_1 = 9-4x_4$,

x_4 any real number.

3. Solve the system of equations

$$2x_1 + 3x_2 - 5x_3 - 2x_4 = 2$$

$$-2x_1 - 4x_2 + 4x_3 - 3x_4 = 6$$

$$x_1 + 2x_2 - 2x_3 + 3x_4 = 0$$

$$\left(\begin{array}{rrrr|r} 2 & 3 & -5 & -2 & 2 \\ -2 & -4 & 4 & -3 & 6 \\ 1 & 2 & -2 & 3 & 0 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{rrrr|r} 1 & 2 & -2 & 3 & 0 \\ -2 & -4 & 4 & -3 & 6 \\ 2 & 3 & -5 & -2 & 2 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{rrrr|r} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & -1 & -1 & -8 & 2 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{rrrr|r} 1 & 2 & -2 & 3 & 0 \\ 0 & 1 & 1 & 8 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \text{row echelon form}$$

$$\left(\begin{array}{rrrr|r} 1 & 2 & -2 & 3 & 0 \\ 0 & 1 & 1 & 8 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \longrightarrow$$

$$\left(\begin{array}{rrrr|r} 1 & 2 & -2 & 0 & -6 \\ 0 & 1 & 1 & 0 & -18 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \longrightarrow$$

$$\left(\begin{array}{rrrr|r} 1 & 0 & -4 & 0 & 30 \\ 0 & 1 & 1 & 0 & -18 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$x_4 = 2,\ x_2 = -18 - x_3,\ x_1 = 30 + 4x_3$

Reduced Row Echelon Form

1. Rows consisting entirely of zeros are at the bottom of the matrix.

2. The first nonzero entry in a nonzero row is a 1.

3. The leading 1 in row $i+1$ is to the right of the leading 1 in row i.

4. The leading 1 is the only nonzero entry in its column.

Examples

Example 1

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \qquad \left(\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Example 2

$$\left(\begin{array}{rrrr|r} 1 & 2 & -3 & -4 & 2 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right) \qquad \left(\begin{array}{rrrr|r} 1 & 0 & 0 & 4 & 9 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right)$$

Example 3

$$\left(\begin{array}{rrrr|r} 1 & 2 & -2 & 3 & 0 \\ 0 & 1 & 1 & 8 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \qquad \left(\begin{array}{rrrr|r} 1 & 0 & -4 & 0 & 30 \\ 0 & 1 & 1 & 0 & -18 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

Homogeneous Systems

The system of linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

is **homogeneous** if

$$b_1 = b_2 = \cdots = b_m = 0,$$

otherwise, the system is **nonhomogeneous**.

C.f. Linear differential equations.

A homogeneous system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = 0 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = 0 \end{array}$$

ALWAYS has at least one solution, namely

$$x_1 = x_2 = \cdots = x_n = 0,$$

called the **trivial solution**. (Sound familiar?) **Homogeneous systems are ALWAYS CONSISTENT.**

3. Solve the homogeneous system

$$x - 2y + 2z = 0$$

$$4x - 7y + 3z = 0$$

$$2x - y + 2z = 0$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 4 & -7 & 3 & 0 \\ 2 & -1 & 2 & 0 \end{array}\right)$$

Row echelon form:

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Corresponding system of equations:

$$
\begin{aligned}
x - 2y + 2z &= 0 \\
y - 5z &= 0 \\
z &= 0.
\end{aligned}
$$

This system has the unique solution

$$
\begin{aligned}
x &= 0, \\
y &= 0, \\
z &= 0.
\end{aligned}
$$

The trivial solution is the only solution.

4. Solve the homogeneous system

$$3x - 2y + z = 0$$

$$x + 4y + 2z = 0$$

$$7x + 4z = 0$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & -2 & 1 & 0 \\ 1 & 4 & 2 & 0 \\ 7 & 0 & 4 & 0 \end{array}\right)$$

Row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 5/14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Corresponding system of equations:

$$\begin{aligned} x + 4y + 2z &= 0 \\ y + \frac{5}{14}z &= 0 \end{aligned}$$

This system has infinitely many solutions:

$$x = -\frac{2}{7}a, \quad y = -\frac{5}{14}a, \quad z = a,$$

a any real number.