Section 5.5. Matrices and Vectors

A **matrix** is a rectangular array of objects arranged in rows and columns. The objects are called the **entries**.

A matrix with m rows and n columns is called an $m \times n$ matrix. $m \times n$ is called the **size** of the matrix, and the numbers m and n are its **dimensions**.

A matrix in which the number of rows equals the number of columns, $m = n$, is called a **square matrix of order** n.

If A is an $m \times n$ matrix, then a_{ij} denotes the element in the i^{th} row and j^{th} column of A:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}.$$

The notation $A = (a_{ij})$ also represents this display.

Special Cases: Vectors

A $1 \times n$ matrix

$$\mathbf{v} = (a_1 \; a_2 \; \ldots \; a_n)$$

also written as

$$\mathbf{v} = (a_1, \; a_2, \; \ldots, \; a_n)$$

is called an **row vector**.

An $m \times 1$ matrix

$$\mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

is called a **column vector**.

The entries of a row or column vector are called the **components** of the vector.

"Physics" & "Geometry/Mathematics"

Physics: A quantity that has magnitude and direction.

Examples: velocity, acceleration, force

Mathematics: A directed line segment.

Examples:

Arithmetic of Matrices

Let $A = (a_{ij})$ be an $m \times n$ matrix and let $B = (b_{ij})$ be a $p \times q$ matrix.

- **Equality:** $A = B$ if and only if

1. $m = p$ and $n = q$;

2. $a_{ij} = b_{ij}$ for all i and j.

That is, $A = B$ if and only if A and B are identical.

Example:

$$\begin{pmatrix} a & b & 3 \\ 2 & c & 0 \end{pmatrix} = \begin{pmatrix} 7 & -1 & x \\ 2 & 4 & 0 \end{pmatrix}$$

if and only if

$$a = 7,\ b = -1,\ c = 4,\ x = 3.$$

Matrix Operations

I. Addition: Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices.

$A+B$ is the $m\times n$ matrix $C = (c_{ij})$ where

$c_{ij} = a_{ij} + b_{ij}$ for all i and j.

That is,

$$A + B = (a_{ij} + b_{ij}).$$

Addition of matrices is not defined for matrices of different sizes.

Examples:

(a)

$$\begin{pmatrix} 2 & 4 & -3 \\ 2 & 5 & 0 \end{pmatrix} + \begin{pmatrix} -4 & 0 & 6 \\ -1 & 2 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 2+(-4) & 4+0 & -3+6 \\ 2+(-1) & 5+2 & 0+0 \end{pmatrix} = \begin{pmatrix} -2 & 4 & 3 \\ 1 & 7 & 0 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 2 & 4 & -3 \\ 2 & 5 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 5 & -3 \\ 0 & 6 \end{pmatrix} \quad \text{is not defined.}$$

PROPERTIES: Let A, B, and C be matrices of the same size. Then:

1. $A+B = B+A$ (Commutative)

2. $(A+B)+C = A+(B+C)$ (Associative)

A matrix with all entries equal to 0 is called a **zero matrix**. E.g.,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The symbol $\mathbf{O}$ will be used to denote the zero matrix of arbitrary size.

3. $A + \mathbf{O} = \mathbf{O} + A = A$.

The zero matrix is the **additive identity**.

The **negative of a matrix** A, denoted by $-A$ is the matrix whose entries are the negatives of the entries of A. $-A$ is also called the **additive inverse of** A.

Example:

$$A = \begin{pmatrix} 1 & 7 & -2 \\ 2 & 0 & 6 \\ -4 & -1 & 5 \end{pmatrix}$$

$$-A = \begin{pmatrix} -1 & -7 & 2 \\ -2 & 0 & -6 \\ 4 & 1 & -5 \end{pmatrix}$$

4. $A + (-A) = \mathbf{O}$.

Subtraction: Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. Then

$$A - B = A + (-B).$$

Example:

$$\begin{pmatrix} 2 & 4 & -3 \\ 2 & 5 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 0 & -6 \\ 1 & -2 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & 4 & -3 \\ 2 & 5 & 0 \end{pmatrix} + \begin{pmatrix} -4 & 0 & 6 \\ -1 & 2 & 0 \end{pmatrix} =$$

PROPERTIES: Let A, B, and C be matrices of the same size. Then:

1. $A+B=B+A$ (commutative)

2. $(A+B)+C = A+(B+C)$ (associative)

3. $A+\mathbf{O}=\mathbf{O}+A=A$ (additive identity)

4. $A+(-A) = (-A)+A = \mathbf{O}$ (additive inverse)

II. Multiplication of a Matrix by a Number

The **product of a number** k **and a matrix** A, denoted kA, is given by

$$kA = (ka_{ij}).$$

This product is called **multiplication by a scalar**.

Examples:

$$2(\,1,\;\;3\,)$$

$$-3\begin{pmatrix} 2 & -1 & 4 \\ 1 & 5 & -2 \\ 4 & 0 & 3 \end{pmatrix} =$$

PROPERTIES: Let A, B be $m \times n$ matrices and let α, β be real numbers. Then

1. $1\,A = A$

2. $0\,A = \mathbf{O}$

3. $\alpha(A + B) = \alpha\,A + \alpha\,B$

4. $(\alpha + \beta)A = \alpha\,A + \beta\,A$

(3 and 4 are called **distributive** laws)

III. Matrix Multiplication

1. The Product of a Row Vector and a Column Vector: The product of a $1 \times n$ row vector and an $n \times 1$ column vector, called the **dot product**, is the **number** given by

$$(a_1,\ a_2,\ a_3,\ \ldots,\ a_n) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

$$= a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n.$$

Also called **scalar product** (because the result is a number (scalar)), and **inner product**.

The product of a row vector and a column vector (of the same dimension and in that order!) is a **number**.

The product of a row vector and a column vector of different dimensions is not defined.

Examples:

$$(3,\ -2,\ 5)\begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}$$

$$(-2,\ 3,\ -1,\ 4)\begin{pmatrix} 2 \\ 4 \\ -3 \\ -5 \end{pmatrix}$$

2. The Product of Two Matrices: Let $A = (a_{ij})$ be an $m \times p$ matrix and let $B = (b_{ij})$ be a $p \times n$ matrix.

The matrix product of A and B (**in that order**), denoted AB, is the $m \times n$ matrix $C = (c_{ij})$, where c_{ij} is the product of the i^{th} row of A and the j^{th} column of B.

$$AB = C = \left(c_{ij}\right)$$

Let A and B be given matrices. The product AB, in that order, is defined if and only if the number of columns of A equals the number of rows of B.

If the product AB is defined, then the size of the product is: (no. of rows of A)×(no. of columns of B):

$$\underset{m\times p}{A}\ \underset{p\times n}{B} = \underset{m\times n}{C}$$

Examples:

1. $A = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & -2 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & -2 \end{pmatrix} =$$

$$BA = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix} =$$

NOTE: $AB \neq BA$ matrix multiplication is NOT COMMUTATIVE.

2. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 2 \\ -3 & 0 & 5 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ -3 & 0 & 5 \end{pmatrix} =$$

$BA =$?

$\underset{2\times 3}{B} \ \underset{2\times 2}{A}$ $\quad BA$ does not exist.

3. $A = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$$

Compare with real numbers: $\alpha\beta = 0$ implies $\alpha = 0$, or $\beta = 0$, or $\alpha = \beta = 0$

Remember: Matrix multiplication is not commutative;

$$AB \neq BA$$

in general.

However, there do exist occasions when $AB = BA$.

PROPERTIES: Let A, B, and C be matrices.

1. $AB \neq BA$ in general; NOT COMMUTATIVE.

2. $(AB)C = A(BC)$ matrix multiplication is associative.

3. Multiplicative Identity (the analog of the number 1). ??

Let A be a square matrix of order n. The entries $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ form the **main diagonal** of A.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

(multiplicative) Identity Matrices:

For each positive integer $n > 1$, let I_n denote the square matrix of order n whose entries on the main diagonal are all 1, and all other entries are 0. The matrix I_n is called the $n \times n$ **identity matrix**.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and so on.

3. If A is an $m \times n$ matrix, then

$$I_m A = A \quad \text{and} \quad AI_n = A.$$

If A is a square matrix of order n, then

$$AI_n = I_n A = A,$$

4. **(multiplicative) Inverse** ?????

Recall, for real numbers: If $a \neq 0$

$$a\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)a = 1$$

or $\quad a \cdot a^{-1} = a^{-1} \cdot a = 1$

$\dfrac{1}{a}$ or a^{-1} is the *multiplicative inverse* of a

0 does not have a multiplicative inverse!!

For matrices: Given a matrix A. A matrix B is a *multiplicative inverse* of A if

$AB = BA = I$ (the identity matrix)

Problems:

- AB and BA might not both exist.

- If AB and BA both exist, they might have different size.

- If AB and BA both exist and have

the same size, $AB \neq BA$, in general.

- If AB and BA both exist and have the same size, then A and B must be square.

Distributive Laws:

1. $A(B + C) = AB + AC$. This is called the *left distributive law*.

2. $(A + B)C = AC + BC$. This is called the *right distributive law*.

3. $k(AB) = (kA)B = A(kB)$

Other ways to look at systems of linear equations.

A system of m linear equations in n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_3 + \cdots + a_{3n}x_n = b_3$$

....

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Because of the way we defined multiplication, the system can be written as:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}$$

or in the vector-matrix form

$$A\mathbf{x} = \mathbf{b} \quad \text{c.f.} \quad ax = b. \tag{1}$$

Solution: $\mathbf{x} = A^{-1}\mathbf{b}$?????

Section 5.6. Square matrices

1. Inverse

Let A be an $n \times n$ matrix. An $n \times n$ matrix B with the property that

$$AB = BA = I_n$$

is called the **multiplicative inverse of A** or, more simply, **the inverse of A**.

Uniqueness: If A has an inverse, then it is unique. That is, there is one and only one matrix B such that

$$AB = BA = I.$$

B is denoted by A^{-1}.

Procedure for finding the inverse:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

We want $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

1) Solve $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \end{array}\right) \rightarrow$$

2) Now solve $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} y \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 1 \end{array}\right) \rightarrow$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1/2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1/2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

Solve the two systems simultaneously:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \longrightarrow$$

Examples:

1. $$A = \begin{pmatrix} 2 & -8 \\ -1 & 6 \end{pmatrix}$$

$$\left(\begin{array}{rr|rr} 2 & -8 & 1 & 0 \\ -1 & 6 & 0 & 1 \end{array}\right) \longrightarrow$$

2. $$B = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

$$\left(\begin{array}{rr|rr} 2 & -1 & 1 & 0 \\ -4 & 2 & 0 & 1 \end{array}\right)$$

B does not have an inverse.

NOTE: Not every nonzero $n \times n$ matrix A has an inverse!

3. $C = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$

Form the augmented matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right) \rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|rrr} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right)$$

$$C^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$$

4. $B = \begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix}$

Form the augmented matrix

$$\left(\begin{array}{rrr|rrr} 1 & 3 & -4 & 1 & 0 & 0 \\ 1 & 5 & -1 & 0 & 1 & 0 \\ 3 & 13 & -6 & 0 & 0 & 1 \end{array}\right)$$

and row reduce

$$\left(\begin{array}{rrr|rrr} 1 & 3 & -4 & 1 & 0 & 0 \\ 1 & 5 & -1 & 0 & 1 & 0 \\ 3 & 13 & -6 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 4 & 6 & -3 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{rrr|rrr} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right)$$

B does not have an inverse.

Finding the inverse of A. Let A be an $n \times n$ matrix.

a. Form the augmented matrix $(A|I_n)$.

b. Reduce $(A|I_n)$ to reduced row echelon form. If the reduced row echelon form is

$$(I_n|B), \quad \text{then} \quad B = A^{-1}$$

If the reduced row echelon form is **not** $(I_n|B)$, then A does not have an inverse.

That is, if the reduced row echelon form of A is not the identity, then A does not have an inverse.

Application: Solve the system

$$\begin{array}{rrrcr} x & & +2z & = & 3 \\ 2x & -y & +3z & = & 5 \\ 4x & +y & +8z & = & -2 \end{array}$$

The system written in matrix form:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$$

$C\mathbf{x} = \mathbf{b}$; solution: $\mathbf{x} = C^{-1}\mathbf{b}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} -29 \\ -14 \\ 15 \end{pmatrix}$$

Note 1: The $n \times n$ system of equations

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution if and only if the matrix of coefficients, A, has an inverse.

2. Determinants

Find the solution set:

$$ax + by = \alpha$$

$$cx + dy = \beta$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad A\mathbf{x} = \mathbf{b}$$

$$(ad - bc)x = (\alpha d - \beta b)$$

$$(ad - bc)y = (a\beta - c\alpha)$$

The number $ad - bc$ is called the determinant of A, denoted $\det A$.

NOTE: The determinant $ad - bc$ **determines** whether or not the 2×2 system has a unique solution:

- $ad - bc \neq 0$ implies unique solution,
- $ad - bc = 0$ implies **either** infinitely many solutions or no solutions.

NOTE: The determinant does not completely "determine."

A. Calculations

1. $1 \times 1 \quad A = (a)$

$$\det A = a$$

2. $2 \times 2 \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

2. $3 \times 3 \quad A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

$$\det A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

Re-write as:

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

This is called the expansion across the first row.

Example: $A = \begin{pmatrix} 2 & -3 & 3 \\ 0 & 4 & 2 \\ -2 & 1 & 3 \end{pmatrix}$

$$\begin{vmatrix} 2 & -3 & 3 \\ 0 & 4 & 2 \\ -2 & 1 & 3 \end{vmatrix} =$$

$$2\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 2 \\ -2 & 3 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ -2 & 1 \end{vmatrix} =$$

$$2(12-2)-(-3)(0+4)+3(0+8) =$$

$$2(10) - (-3)(4) + 3(8) = 56$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1) =$$

Re-write as:

$$-a_2(b_1c_3-b_3c_1)+b_2(a_1c_3-a_3c_1)-c_2(a_1b_3-a_3b_1) =$$

$$-a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

Expansion down the second column.

Example: $A = \begin{pmatrix} 2 & -3 & 3 \\ 0 & 4 & 2 \\ -2 & 1 & 3 \end{pmatrix}$

$$\begin{vmatrix} 2 & -3 & 3 \\ 0 & 4 & 2 \\ -2 & 1 & 3 \end{vmatrix} =$$

$$-(-3)\begin{vmatrix} 0 & 2 \\ -2 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} =$$

$$3(4) + 4(12) - 4 = 56$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$$

$$= a_3b_1c_2-a_3b_2c_1-b_3a_1c_2+b_3a_2c_1+c_3a_1b_2-c_3a_2b_1$$

$$= a_3(b_1c_2-b_2c_1)-b_3(a_1c_2-a_2c_1)+c_3(a_1b_2-a_2b_1)$$

$$= a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Expansion down the third column

and so on. You can expand across any row, or down any column.

BUT: Associated with each position is an algebraic sign:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

For example, across the second row:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$= -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

3. 4×4 determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

Sign chart

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

An application of determinants

Recall: Find the solution set:

$$ax + by = \alpha$$

$$cx + dy = \beta$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad A\mathbf{x} = \mathbf{b}$$

$$(ad - bc)x = \alpha d - b\beta$$

$$(ad - bc)y = a\beta - \alpha c$$

$$(\det A)\, x = \alpha d - b\beta = \begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix}$$

$$(\det A)\, y = a\beta - \alpha c = \begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix}$$

$$(\det A)\, x = \alpha d - \beta b = \begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix}$$

$$(\det A)\, y = a\beta - c\alpha = \begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix}$$

If $\det A \neq 0$, then

$$x = \frac{\begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix}}{\det A}$$

$$y = \frac{\begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix}}{\det A}$$

If $\det A = 0$, then the system either has infinitely many solutions or no solutions.

Cramer's Rule & systems of equations.

Given a system of n linear equations in n unknowns: (a "square" system)

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

$$x_i = \frac{\det A_i}{\det A}, \qquad \text{(provided} \quad \det A \neq 0)$$

where A_i is the matrix A with the i^{th} column replaced by the vector $\mathbf{b}$.

Note:

- The $n \times n$ system of equations $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\det A \neq 0$.

- If $\det A = 0$, then the system either has infinitely many solutions or no solutions.

Definitions:

A is **nonsingular** if $\det A \neq 0$;

A is **singular** if $\det A = 0$.

Examples:

1. Given the system

$$\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & 3 \\ 2x & - & y & + & z & = & 5 \\ -4x & + & y & - & 2z & = & -2 \end{array}$$

Does Cramer's rule apply?

$$\det A = \begin{vmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ -4 & 1 & -2 \end{vmatrix} =$$

$$\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & 3 \\ 2x & - & y & + & z & = & 5 \\ -4x & + & y & - & 2z & = & -2 \end{array}$$

Find y.

$$y = \frac{\begin{vmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ -4 & -2 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ -4 & 1 & -2 \end{vmatrix}} =$$

2. Given the system

$$\begin{array}{rcrcrcr} -2x & + & 7y & + & 6z & = & -1 \\ 5x & + & y & - & 2z & = & 7 \\ 3x & + & 8y & + & 4z & = & -1 \end{array}$$

Does Cramer's rule apply?

$$\det A = \begin{vmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{vmatrix} = 0$$

Cramer's Rule does not apply

Does the system have infinitely many solutions or no solutions??

Compare with $ax = b$ when $a = 0$.

B. Properties of determinants:

Let A be an $n \times n$ matrix.

1. If A has a row or column of zeros, then $\det A = 0$

Example:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \\ 4 & 0 & 8 \end{pmatrix}$$

Expand down second column:

$$-0\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 0$$

2. If A is a diagonal matrix,

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix},$$

$$\det A = a_1 \begin{vmatrix} b_2 & 0 \\ 0 & c_3 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 0 & c_3 \end{vmatrix} + 0 \begin{vmatrix} 0 & b_2 \\ 0 & 0 \end{vmatrix}$$

$$= a_1 \cdot b_2 \cdot c_3.$$

In particular, $\det I_n = 1$

For example, I_3:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

3. If A is a triangular matrix,

e.g.,

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & b_2 & b_3 & b_4 \\ 0 & 0 & c_3 & c_4 \\ 0 & 0 & 0 & d_4 \end{pmatrix} \quad \text{(upper triangular)}$$

Expand down first column!!

$$\det A = a_1 \cdot b_2 \cdot c_3 \cdot d_4$$

4. If B is obtained from by interchanging any two rows (columns), then

$$\det B = -\det A.$$

NOTE: If A has two identical rows (or columns), then

$\det A = 0$.

5. Multiply a row (column) of A by a nonzero number k to obtain a matrix B. Then

$$\det B = k \det A.$$

6. Multiply a row (column) of A by a number k and add it to another row (column) to obtain a matrix B. Then

$$\det B = \det A.$$

7. Let A and B be $n \times n$ matrices. Then

$$\det [AB] = \det A \det B.$$

Example:

Calculate $\begin{vmatrix} 3 & 3 & 1 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 2 \\ 2 & 10 & 3 & 2 \end{vmatrix}$

$$= -6\begin{vmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 60 \end{vmatrix} = -360$$

Equivalences: The following statements are equivalent:

1. The system of equations: $A\mathbf{x} = \mathbf{b}$ has a unique solution.

2. The reduced row echelon form of A is I_n.

3. The rank of A is n.

4. A has an inverse.

5. $\det A \neq 0$.