Summary: Solving Systems of Equations

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n = b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n = b_m \end{array}$$

$$A\mathbf{x} = \mathbf{b}$$

1. If A is not square (i.e., more unknowns then equations or more equations than unknowns), then augmented matrix $\longrightarrow$ row echelon or reduced row echelon form.

2. If A is square, then

$(A|b) \longrightarrow$ row echelon form & back substitute

$(A|b) \longrightarrow$ reduced row echelon form

find A^{-1}, if possible, then $\mathbf{x} = A^{-1}\mathbf{b}$

Cramer's rule: $x_i = \dfrac{\det A_i}{\det A}$, provided $\det A \neq 0$

Note: If A^{-1} does not exist, or if $\det A = 0$, then the system either has infinitely many solutions or no solutions.

BUT, Special Case:

If the system is homogeneous, then it has infinitely many solutions; "no solutions" is not an option for homogeneous systems,

Section 5.7. Vector Spaces

$\mathbb{R}^2 = \{(a, b) : a, b \in \mathbb{R}\}$ – "the plane"

$\mathbb{R}^3 = \{(a, b, c) : a, b, c \in \mathbb{R}\}$ – "3-space"

$\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4)\}$ – "4-space"

$\mathbb{R}^n = \{(x_1, x_2, x_3, \ldots, x_n)\}$ ordered

n-tuples of real numbers

For any two vectors $\mathbf{u} = (a_1,\ a_2,\ \ldots, a_n)$ and $\mathbf{v} = (b_1,\ b_2,\ \ldots, b_n)$ in $\mathbb{R}^n$, we have

$$\mathbf{u} + \mathbf{v} = (a_1,\ a_2,\ \ldots, a_n) + (b_1,\ b_2,\ \ldots, b_n)$$

$$= (a_1 + b_1,\ a_2 + b_2,\ \ldots,\ a_n + b_n)$$

and for any real number λ,

$$\lambda\,\mathbf{v} = \lambda\,(a_1,\ a_2,\ \ldots, a_n)$$

$$= (\lambda a_1, \lambda a_2,\ \ldots, \lambda a_n).$$

Clearly, the sum of two vectors in $\mathbb{R}^n$ is another vector in $\mathbb{R}^n$ and a scalar multiple of a vector in $\mathbb{R}^n$ is a vector in $\mathbb{R}^n$.

Properties:

Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$.

Addition:

1. $\mathbf{u} + \mathbf{v} \in \mathbb{R}^n$ **closed**

2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ **commutative**

3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

associative

4. **Zero vector:** $\mathbf{0} = (0,\ 0,\ 0, \ldots,\ 0)$,

$\mathbf{v}+\mathbf{0} = \mathbf{0}+\mathbf{v} = \mathbf{v}$ (additive identity).

5. **Additive Inverse:** For each vector $\mathbf{v} \in \mathbb{R}^n$, there is a unique vector $-\mathbf{v}$ such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{v} - \mathbf{v} = \mathbf{0}$$

$-\mathbf{v}$ is the additive inverse (or negative) of $\mathbf{v}$.

Multiplication by a scalar:

If α and β are numbers, and $\mathbf{u}$ and $\mathbf{v}$ are vectors, then:

1. $\alpha\,\mathbf{v} \in \mathbb{R}^n$ (closed)
2. $1\mathbf{v} = \mathbf{v}$ (1 multiplicative identity)
3. $\alpha(\beta\,\mathbf{v}) = (\alpha\beta)\,\mathbf{v}$ (associative property)
4. $(\alpha + \beta)\mathbf{v} = \alpha\,\mathbf{v} + \beta\,\mathbf{v}$ (distributive property)

5. $\alpha(\mathbf{u}+\mathbf{v}) = \alpha\,\mathbf{u} + \alpha\,\mathbf{v}$ (distributive property)

Any non-empty set V on which there are defined two operations, addition (+) and multiplication by a scalar, which satisfy the properties 1 - 5 for addition and 1 - 5 for multiplication by a scalar is called a **vector space**.

Examples:

1. $\mathbb{R}^n$

2. $\mathcal{C}(0,1) = \{f : (0,1) \to \mathbb{R} \,|\, f$ is continuous$\}$

3. The set $\mathcal{S}$ of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

Section 5.8. Linear Dependence/Independence in $\mathbb{R}^n$

Let V be a vector space and let

$$\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3, \cdots,\ \mathbf{v}_k\}$$

be a set of vectors in V. Let

$$\{c_1,\ c_2,\ c_3, \cdots,\ c_k\}$$

be real numbers. Then

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k$$

is a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

$\mathbb{R}^2 = \{(a, b) : a, b \in \mathbb{R}\}$ – "the plane"

$\mathbb{R}^3 = \{(a, b, c) : a, b, c \in \mathbb{R}\}$ – "3-space"

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ be a set of vectors in $\mathbb{R}^n$.

The set $\mathcal{S}$ is **linearly dependent** if there exist k numbers c_1, c_2, $\cdots, c_k$ **NOT ALL ZERO** such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

($c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ is a **linear combination** of $\mathbf{v}_1$, $\mathbf{v}_2, \ldots,$ $\mathbf{v}_k$)

$\mathcal{S}$ is **linearly independent** if it is not linearly dependent. That is, $\mathcal{S}$ is linearly independent if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

implies $c_1 = c_2 = \cdots = c_k = 0$.

The set $\mathcal{S}$ is **linearly dependent** if there exist k numbers $c_1, c_2, \cdots, c_k$ NOT ALL ZERO such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Another way to say this:

The set $\mathcal{S}$ is **linearly dependent** if one of the vectors can be written as a linear combination of the other vectors.

The set $\mathcal{S}$ is **linearly dependent** if there exist k numbers $c_1, c_2, \cdots, c_k$ NOT ALL ZERO such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{O}.$$

NOTE: If there is one such set

$$\{c_1,\ c_2,\ c_3,\ \ldots,\ c_k\},$$

then there are **infinitely many** such sets.

1. Two vectors $\mathbf{v}_1, \mathbf{v}_2$.

Linearly dependent iff one vector is a constant multiple of the other.

Examples:

$$\mathbf{v}_1 = (1, -2, 4), \quad \mathbf{v}_2 = (-\frac{1}{2}, 1, -2)$$

linearly dependent: $\mathbf{v}_1 = -2\mathbf{v}_2$

$$\mathbf{v}_1 = (\,2,\ -4,\ 5\,), \quad \mathbf{v}_2 = (\,0,\ 0,\ 0\,)$$

linearly dependent: $\mathbf{v}_2 = 0\mathbf{v}_1$

$$\mathbf{v}_1 = (\,5,\ -2,\ 0\,), \quad \mathbf{v}_2 = (\,-3,\ 1,\ 9\,)$$

linearly independent

2. Given the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^2$:

$$\mathbf{v}_1 = (1,\ -1),\ \mathbf{v}_2 = (-2,\ 3)$$

$$\mathbf{v}_3 = (3,\ -5)$$

1. $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ Dependent or independent??

Solution 1. Independent: $\mathbf{v}_1$ is not a constant multiple of $\mathbf{v}_2$; and $\mathbf{v}_2$ is not a constant multiple of $\mathbf{v}_1$.

Solution 2. Suppose they are dependent. Then there are two numbers c_1, c_2, not both 0, such that

$$c_1(1,-1)+c_2(-2,3)=\mathbf{O}.$$

That is

$$c_1 - 2c_2 = 0$$

$$-c_1 + 3c_2 = 0$$

$\mathbf{v}_1 = (1,\ -1),\ \mathbf{v}_2 = (-2,\ 3)$

$$\mathbf{v}_3 = (3,\ -5)$$

Dependent or independent??

Does there exist three numbers, c_1, c_2, c_3, not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (0,0)$$

$c_1(1,\ -1) + c_2(2,\ -3) + c_3(3,\ -5)$

I.e, does the system of equations

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ -c_1 - 3c_2 - 5c_3 &= 0 \end{aligned}$$

have nontrivial solutions?

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & 3 & -5 & 0 \end{array} \right)$$

3. Four vectors: $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4$ in $\mathbb{R}^3$

$$\mathbf{v}_1 = (\,1,\ -1,\ 2\,), \quad \mathbf{v}_2 = (\,2,\ -3,\ 0\,),$$

$$\mathbf{v}_3 = (\,-1,\ -2,\ 2\,), \quad \mathbf{v}_4 = (\,0,\ 4,\ -3\,)$$

Are there 4 real numbers c_1, c_2, c_3, c_4, NOT ALL ZERO, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = (0,0,0)?$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ -1 & -3 & -2 & 4 & 0 \\ 2 & 0 & 2 & -3 & 0 \end{array}\right)$$

- **A homogeneous system with more unknowns than equations ALWAYS has infinitely many nontrivial solutions.**

- **Let $\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_k$ be a set of k vectors in $\mathbb{R}^n$. If $k > n$, then the set of vectors is (automatically) linearly dependent.**

In $\mathbb{R}^3$:

$\mathbf{v}_1 = (1, -1, 2), \quad \mathbf{v}_2 = (2, -3, 0),$

$\mathbf{v}_3 = (-1, -2, 2), \quad \mathbf{v}_4 = (0, 4, -3)$

Dependent or independent??

(a) $\{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \mathbf{v}_4\}$

(b) $\{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3\}$

(c) $\{\mathbf{v}_1, \ \mathbf{v}_2\}$

(b) $\mathbf{v}_1 = (\,1,\ -1,\ 2\,)$, $\quad \mathbf{v}_2 = (\,2,\ -3,\ 0\,)$,

$\mathbf{v}_3 = (\,-1,\ -2,\ 2\,)$

Does $\quad c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$

have non-trivial solutions? That is,

does

$$\begin{aligned} c_1 + 2c_2 - c_3 &= 0 \\ -c_1 - 3c_2 - 2c_3 &= 0 \\ 2c_1 + 2c_3 &= 0 \end{aligned}$$

have non-trivial solutions?

Augmented matrix and row reduce:

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ -1 & -3 & -2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right)$$

NOTE: It's enough to row reduce:

$$\begin{pmatrix} 1 & 2 & -1 \\ -1 & -3 & -2 \\ 2 & 0 & 2 \end{pmatrix}$$

since the numbers to the right of the bar will always be 0.

Or, calculate the determinant

$$\begin{vmatrix} 1 & 2 & -1 \\ -1 & -3 & -2 \\ 2 & 0 & 2 \end{vmatrix}$$

- $\det \neq 0$ implies unique solution

$$c_1 = c_2 = c_3 = 0$$

and **independent**,

- $\det = 0$ implies infinitely many solutions and **dependent**

$$\begin{vmatrix} 1 & 2 & -1 \\ -1 & -3 & -2 \\ 2 & 0 & 2 \end{vmatrix} =$$

Note: When testing a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ for independence/dependence:

1. If you use the definition

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

Then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ appear as columns.

2. If you use the determinant, then you can write the vectors either as rows or columns.

From the previous example:

$$\begin{vmatrix} 1 & 2 & -1 \\ -1 & -3 & -2 \\ 2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -3 & 0 \\ -1 & -2 & 2 \end{vmatrix}$$

This follows from the fact that to evaluate a determinant you can expand across any row or down any column.

4. $\mathbf{v}_1 = (\,a,\ 1,\ -1\,),\quad \mathbf{v}_2 = (\,-1,\ 2a,\ 3\,),$

$\mathbf{v}_3 = (\,-2,\ a,\ 2\,),\quad \mathbf{v}_4 = (\,3a,\ -2,\ a\,)$

For what values of a are the vectors linearly dependent?

5. $\mathbf{v}_1 = (\,a,\ 1,\ -1\,), \quad \mathbf{v}_2 = (\,-1,\ 2a,\ 3\,),$

$\mathbf{v}_3 = (\,-2,\ a,\ 2\,)$

For what values of a are the vectors

$\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ linearly dependent?

$$
\begin{aligned}
a\,c_1 - c_2 - 2c_3 &= 0 \\
c_1 + 2a\,c_2 + a\,c_3 &= 0 \\
-c_1 + 3c_2 + 2c_3 &= 0
\end{aligned}
$$

Augmented matrix and row reduce:

$$\left(\begin{array}{rrr|r} a & -1 & -2 & 0 \\ 1 & 2a & a & 0 \\ -1 & 3 & 2 & 0 \end{array}\right)$$

Or row reduce:

$$\begin{pmatrix} a & -1 & -2 \\ 1 & 2a & a \\ -1 & 3 & 2 \end{pmatrix}$$

Or calculate the determinant

$$\begin{vmatrix} a & -1 & -2 \\ 1 & 2a & a \\ -1 & 3 & 2 \end{vmatrix}$$

5. $\mathbf{v}_1 = (1, -1, 2, 1), \quad \mathbf{v}_2 = (3, 2, 0, -1)$

$\mathbf{v}_3 = (-1, -4, 4, 3), \quad \mathbf{v}_4 = (2, 3, -2, -2)$

a. $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4\}$

dependent or independent?

b. If dependent, what is the maximum number of independent vectors?

Row reduce

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ -1 & -4 & 4 & 3 \\ 2 & 3 & -2 & -2 \end{pmatrix}$$

Or, calculate the determinant.

$$\begin{vmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ -1 & -4 & 4 & 3 \\ 2 & 3 & -2 & -2 \end{vmatrix}$$

$v_1 = (1, -1, 2, 1),\ V_2 = (3, 2, 0, -1),$

$v_3 = (-1, -4, 4, 3),\ v_4 = (2, 3, -2, -2)$

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ -1 & 4 & 4 & 3 \\ 2 & 3 & -2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -6 & -4 \\ 0 & -5 & 6 & 4 \\ 0 & 5 & -6 & -4 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -6 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Tests for independence/dependence

Let $\mathcal{S} = \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ be a set of vectors in $\mathbb{R}^n$.

Case 1: $k > n$: $\mathcal{S}$ is linearly dependent.

Case 2: $k = n$ **:**

1. Solve the system of equations

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

If a unique solution:

$$c_1 = c_2 = \cdots = c_n = 0,$$

the vectors are **independent**.

If infinitely many solutions:

The vectors are **dependent**.

OR 2. Form the $n \times n$ matrix A whose rows are $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ and row reduce A:

if the reduced matrix has n nonzero rows, i.e., if the rank of A is n, then **independent**;

if the reduced matrix has one or more zero rows, then **dependent**.

OR 3. Calculate $\det A$:

If $\det A \neq 0$,

the vectors are **independent**.

If $\det A = 0$,

the vectors are **dependent**.

Note: If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is a linearly independent set of vectors in $\mathbb{R}^n$, then each vector in $\mathbb{R}^n$ has a unique representation as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.

Case 3: $k < n$: Form the $k \times n$ matrix A whose rows are $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$

1. Row reduce A:

   if the reduced matrix has k nonzero rows –independent;

   one or more zero rows – dependent.

Equivalently, solve the system of equations

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

If unique solution: $c_1 = c_2 = \cdots = c_n = 0$, then **independent**; If infinitely many solutions, then **dependent**.

Linear Independence/Dependence & Row Operations

Example. $\mathbf{v}_1 = (1, -2, 3)$,

$\mathbf{v}_2 = (2, -3, 1)$, $\mathbf{v}_3 = (3, -4, -1)$

Dependent or independent?

$$\begin{pmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \\ 3 & -4 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -5 \\ 0 & 2 & -10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

That is, R_3 is a linear combination of R_1 and R_2; the vectors are linearly dependent; the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{O}$$

has infinitely many non-zero solutions.

The General Result: Given a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$. Form the matrix V with $\mathbf{v}_1, \mathbf{v}_2 \ldots$ as rows and row reduce.

If you get a row of 0's, the vectors are linearly dependent and at least one of the vectors is a linear combination of the other vectors.

(THIS IS WHY A SYSTEM WITH INFINITELY MANY SOLUTIONS IS CALLED **DEPENDENT**. See Chapter 5, Part 1, pg 55.)

If you get no rows of 0's, the vectors are linearly independent

Given an $m \times n$ matrix A. $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation!

$A[\mathbf{v}+\mathbf{w}] = A\mathbf{v}+A\mathbf{w}$ and $A[\alpha\,\mathbf{v}] = \alpha\,A\mathbf{v}$

Example:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \end{pmatrix}$$

is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Section 5.9. Eigenvalues/Eigenvectors

Example: Set

$$A = \begin{pmatrix} 1 & -3 & 1 \\ -1 & 1 & 1 \\ 3 & -3 & -1 \end{pmatrix}.$$

A is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

$$\begin{pmatrix} 1 & -3 & 1 \\ -1 & 1 & 1 \\ 3 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 1 \\ -1 & 1 & 1 \\ 3 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -2 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 1 \\ -1 & 1 & 1 \\ 3 & -3 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ -4 \\ 6 \end{pmatrix} = -2 \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 1 \\ -1 & 1 & 1 \\ 3 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Let A be an $n \times n$ matrix.

A number λ is an **eigenvalue** of A if there is a **non-zero vector** $\mathbf{v}$ such that

$$A\,\mathbf{v} = \lambda\,\mathbf{v}$$

How to find eigenvalues: Suppose λ is a number and $\mathbf{v}$ is a non-zero vector such that $A\mathbf{v} = \lambda\,\mathbf{v}$. Then

To find the eigenvalues of A, find the values of λ that satisfy

$$\det(A - \lambda I) = 0.$$

Example: Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix}$$

$$= (2-\lambda)(4-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = 0$$

Eigenvalues: $\lambda_1 = 5, \quad \lambda_2 = 1$

Let $A = \begin{pmatrix} 1 & -3 & 1 \\ -1 & 1 & 1 \\ 3 & -3 & -1 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -3 & 1 \\ -1 & 1-\lambda & 1 \\ 3 & -3 & -1-\lambda \end{vmatrix}$$

$$= -\lambda^3 + \lambda^2 + 4\lambda - 4$$

$\det(A - \lambda I) = 0$ implies

$$-\lambda^3 + \lambda^2 + 4\lambda - 4 = 0$$

or $\quad \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$

Terminology:

- $\det(A - \lambda I)$ is a polynomial of degree n, called the **characteristic polynomial** of A.

- The zeros of the characteristic polynomial are the eigenvalues of A

- The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

• A **non-zero vector** $\mathbf{v}$ that satisfies

$$A\mathbf{v} = \lambda\mathbf{v}$$

is called an **eigenvector** corresponding to the eigenvalue λ.

Note: Eigenvectors are NOT unique; an eigenvalue has infinitely many eigenvectors.

Examples: 1. $A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$

Characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-1-\lambda) - 4$$

$$= \lambda^2 - \lambda - 6$$

Characteristic equation:

$$\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$$

Eigenvalues: $\lambda_1 = 3, \quad \lambda_2 = -2$

Eigenvectors:

$$(A - \lambda I) = \begin{pmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix}$$

$\lambda_1 = 3$: Solve

$$(A - 3 I)\mathbf{x} = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\lambda_2 = -2$: Solve

$$(A - (-2)\, I)\mathbf{x} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

2. $A = \begin{pmatrix} 4 & -1 \\ 4 & 0 \end{pmatrix}$

Characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 \\ 4 & -\lambda \end{vmatrix}$$

$$= \lambda^2 - 4\lambda + 4$$

Characteristic equation:

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$$

Eigenvalues: $\lambda_1 = \lambda_2 = 2$

Eigenvectors:

$$(A - \lambda I) = \begin{pmatrix} 4 - \lambda & -1 \\ 4 & -\lambda \end{pmatrix}$$

$\lambda_1 = \lambda_2 = 2$

Solve:

$$(A - 2I)\mathbf{x} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3. $A = \begin{pmatrix} 5 & 3 \\ -6 & -1 \end{pmatrix}$

Characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 3 \\ -6 & -1-\lambda \end{vmatrix}$$

$$= \lambda^2 - 4\lambda + 13$$

Characteristic equation:

$$\lambda^2 - 4\lambda + 13 = 0$$

Eigenvalues:

$$\lambda_1 = 2 + 3i, \quad \lambda_2 = 2 - 3i$$

Eigenvectors: $(A - \lambda I) = \begin{pmatrix} 5-\lambda & 3 \\ -6 & -1-\lambda \end{pmatrix}$

$\lambda_1 = 2 + 3i$: Solve

$(A - (2+3i)\, I)\mathbf{x} = \begin{pmatrix} 3-3i & 3 \\ -6 & -3-3i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

NOTE: If $a + b\,i$ is an eigenvalue of A with eigenvector $\alpha + \beta\,i$, then $a - b\,i$ is also an eigenvalue of A and $\alpha - \beta\,i$ is a corresponding eigenvector.

4. $A = \begin{pmatrix} 4 & -3 & 5 \\ -1 & 2 & -1 \\ -1 & 3 & -2 \end{pmatrix}$

Characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -3 & 5 \\ -1 & 2-\lambda & -1 \\ -1 & 3 & -2-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 4\lambda^2 - \lambda - 6$$

Characteristic equation:

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = (\lambda - 3)(\lambda - 2)(\lambda + 1) = 0.$$

Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 =$

-1

Eigenvectors:

$$(A - \lambda I) = \begin{pmatrix} 4-\lambda & -3 & 5 \\ -1 & 2-\lambda & -1 \\ -1 & 3 & -2-\lambda \end{pmatrix}$$

$\lambda_1 = 3$:

$$(A - \lambda I) = \begin{pmatrix} 4-\lambda & -3 & 5 \\ -1 & 2-\lambda & -1 \\ -1 & 3 & -2-\lambda \end{pmatrix}$$

$\lambda_2 = 2$

$$(A - \lambda I) = \begin{pmatrix} 4-\lambda & -3 & 5 \\ -1 & 2-\lambda & -1 \\ -1 & 3 & -2-\lambda \end{pmatrix}$$

$\lambda_3 = -1$

5. $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$

Characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 11\lambda^2 - 39\lambda + 45$$

Characteristic equation:

$$\lambda^3 - 11\lambda^2 + 39\lambda - 45 = (\lambda - 3)^2(\lambda - 5) = 0.$$

Eigenvalues: $\lambda_1 = 5, \lambda_2 = \lambda_3 = 3$

Eigenvectors:

$$(A - \lambda\, I) = \begin{pmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

$\lambda_1 = 5$

$$(A - \lambda I) = \begin{pmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

$$\lambda_2 = \lambda_3 = 3$$

6. $A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$

Characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 12\lambda + 16$$

Characteristic equation:

$$\lambda^3 - 12\lambda - 16 = (\lambda - 4)(\lambda + 2)^2 = 0.$$

Eigenvalues: $\lambda_1 = 4, \ \lambda_2 = \lambda_3 = -2$

Eigenvectors:

$$(A - \lambda I) = \begin{pmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{pmatrix}$$

$\lambda_1 = 4$:

$$(A - \lambda I) = \begin{pmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{pmatrix}$$

$\lambda_2 = \lambda_3 = -2$:

Some Facts

THEOREM If

$$\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$$

are eigenvectors of a matrix A corresponding to distinct eigenvalues

$$\lambda_1,\ \lambda_2, \ldots,\ \lambda_k,$$

then $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ are linearly independent.

THEOREM Let A be a (real) $n \times n$ matrix. If the complex number $\lambda = a + bi$ is an eigenvalue of A with corresponding (complex) eigenvector $\mathbf{u} + i\mathbf{v}$, then $\lambda = a - bi$, the *conjugate* of $a + bi$, is also an eigenvalue of A and $\mathbf{u} - i\mathbf{v}$ is a corresponding eigenvector.

THEOREM Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

$$\det A = (-1)^n \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ \cdots \ \lambda_n.$$

That is, $\det A$ is the ± 1 times the product of the eigenvalues of A.

(The λ's are not necessarily distinct, the multiplicity of an eigenvalue may be greater than 1, and they are not necessarily real.)

Equivalences: (A an $n \times n$ matrix)

1. $A\mathbf{x} = \mathbf{b}$ has a unique solution.

2. The reduced row echelon form of A is I_n.

3. The rank of A is n.

4. A has an inverse.

5. $\det A \neq 0$.

6. 0 is not an eigenvalue of A

The following are equivalent:

1. $\det A = 0$.

2. A does not have an inverse.

3. The rank of A is less than n.

4. The reduced row echelon form of A is not I_n.

5. The system $A\mathbf{x} = \mathbf{b}$ does not have a unique solution.

6. 0 is an eigenvalue of A.