Chapter 6: Systems of Linear Differential Equations - Part 1 (See Section 3.1)

Let $a_{11}(t),\ a_{12}(t),\ \ldots,\ a_{nn}(t),$

$$b_1(t),\ b_2(t),\ \ldots,\ b_n(t)$$

be continuous functions on the interval I.

The system of n first-order linear differential equations

$$
\begin{aligned}
x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\
x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\
\vdots & \qquad\qquad\qquad\qquad \vdots \\
x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t)
\end{aligned}
$$

is called a **first-order linear differential system**.

The system is **homogeneous** if

$b_1(t) \equiv b_2(t) \equiv \cdots \equiv b_n(t) \equiv 0$ on I.

It is **nonhomogeneous** if the functions $b_i(t)$ are not all identically zero on I.

Set

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

The system can be written in the

vector-matrix form

$$\mathbf{x}' = A(t)\,\mathbf{x} + \mathbf{b}(t). \qquad \text{(S)}$$

The matrix $A(t)$ is called the **matrix of coefficients** or the **coefficient matrix**.

The vector $\mathbf{b}(t)$ is called the non-homogeneous term, or "forcing function."

A **solution** of the linear differential system (S) is a differentiable vector function

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

that satisfies (S) on the interval I.

Example 1:

$$x_1' = x_1 + 2x_2 - 5e^{2t}$$

$$x_2' = 3x_1 + 2x_2 + 3e^{2t}$$

Vector/matrix form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix}$$

or

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix}$$

Show that

$$\mathbf{x}(t) = \begin{pmatrix} -e^{2t} \\ 2\,e^{2t} \end{pmatrix} \quad \text{is a solution of}$$

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix}$$

Example 2:

$$x_1' = 3x_1 - x_2 - x_3$$

$$x_2' = -2x_1 + 3x_2 + 2x_3$$

$$x_3' = 4x_1 - x_2 - 2x_3$$

Vector/matrix form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{pmatrix} \mathbf{x}$$

Show that

$$\mathbf{x}(t) = \begin{pmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{pmatrix}$$

is a solution.

In fact, as we shall see

$\mathbf{x}(t) =$

$$C_1 \begin{pmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix} + C_3 \begin{pmatrix} e^{-t} \\ -3e^{-t} \\ 7e^{-t} \end{pmatrix}$$

is a solution for any numbers $C_1,\ C_2, C_3$, and this is the general solution of the system.

THEOREM. The initial-value problem

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{b}(t), \;\; \mathbf{x}(t_0) = \mathbf{c}$$

has a unique solution $\mathbf{x} = \mathbf{x}(t)$.

II. Homogeneous Systems: General Theory (See Section 3.2)

$$\begin{aligned}
x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n(t)\\
x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n(t)\\
\vdots & \qquad\qquad\qquad \vdots\\
x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n(t)
\end{aligned}$$

$$\mathbf{x}' = A(t)\mathbf{x}. \qquad \text{(H)}$$

Note: The zero vector $\mathbf{z}(t) \equiv \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a solution of (H). This solution is called the **trivial solution**.

THEOREM 1. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of (H), then $\mathbf{u} = \mathbf{x}_1 + \mathbf{x}_2$ is also a solution of (H); the sum of any two solutions of (H) is a solution of (H).

THEOREM 2. If $\mathbf{x}$ is a solution of (H) and α is any real number, then $\mathbf{u} = \alpha\mathbf{x}$ is also a solution of (H); any constant multiple of a solution of (H) is a solution of (H).

In general,

THEOREM. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are solutions of (H), and if $C_1, C_2, \ldots, C_k$ are real numbers, then

$$C_1\mathbf{x}_1 + C_2\mathbf{x}_2 + \cdots + C_k\mathbf{x}_k$$

is a solution of (H); any linear combination of solutions of (H) is also a solution of (H).

Linear Dependence/Independence of vectors – in general Let

$$\mathbf{v}_1(t) = \begin{pmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{pmatrix}, \ \mathbf{v}_2(t) = \begin{pmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2} \end{pmatrix},$$

$$\ldots, \ \mathbf{v}_k(t) = \begin{pmatrix} v_{1k} \\ v_{2k} \\ \vdots \\ v_{nk} \end{pmatrix}$$

be vector functions defined on some interval I.

The vectors are **linearly dependent** on I if there exist k real numbers $c_1,\ c_2,\ \ldots,\ c_k,$ not all zero, such that

$$c_1\mathbf{v}_1(t)+c_2\mathbf{v}_2(t)+\cdots+c_k\mathbf{v}_k(t) \equiv 0 \quad \text{on} \quad I.$$

Otherwise the vectors are **linearly independent** on I.

THEOREM. Let

$$\mathbf{v}_1(t),\ \mathbf{v}_2(t),\ \ldots,\ \mathbf{v}_k(t)$$

be k, k-component vector functions defined on an interval I. If the vectors are **linearly dependent**, then the determinant

$$W(t) = \begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kk} \end{vmatrix} \equiv 0 \quad \text{on} \quad I.$$

That is, the determinant is 0 **for all** $t \in I$.

Equivalently,

THEOREM. Let

$$\mathbf{v}_1(t),\ \mathbf{v}_2(t),\ \ldots,\ \mathbf{v}_k(t)$$

be k, k-component vector functions defined on an interval I. The vectors are **linearly independent** if the determinant

$$W(t) = \begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kk} \end{vmatrix} \neq 0$$

for at least one $t \in I$.

The determinant

$$W(t) = \begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kk} \end{vmatrix}$$

is called the **Wronskian** of the vector functions $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

SPECIAL CASE: Solutions of (H)

THEOREM. Let $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_n$ be n solutions system of n equations (H). Exactly one of the following holds:

1. $W(\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n)(t) \equiv 0$ on I and the solutions are linearly dependent.

2. $W(\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n)(t) \neq 0$ for all $t \in I$ and the solutions are linearly independent.

THEOREM. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent solutions of (H). Let $\mathbf{u}$ be *any* solution of (H). Then there exists a unique set of constants $C_1, C_2, \ldots, C_n$ such that

$$\mathbf{u} = C_1\mathbf{x}_1 + C_2\mathbf{x}_2 + \cdots + C_n\mathbf{x}_n.$$

That is, every solution of (H) can be written as a unique linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$.

A set of n linearly independent solutions of (H)

$$\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n$$

is called a **fundamental set of solutions**. A fundamental set is also called a **solution basis** for (H).

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be a fundamental set of solutions of (H). Then

$$\mathbf{x} = C_1\mathbf{x}_1 + C_2\mathbf{x}_2 + \cdots + C_n\mathbf{x}_n,$$

$C_1, C_2, \ldots, C_n$ arbitrary constants, is the **general solution** of (H).

Example: $\mathbf{x}_1 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$ are solutions of

$$\mathbf{x}' = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}\mathbf{x} \quad \text{(Verify)}$$

$$W(\mathbf{x}_1.\mathbf{x}_2) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & e^{3t} \end{vmatrix} = -e^{5t} \neq 0$$

Therefore, $\left\{ \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}, \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix} \right\}$ is a fundamental set of solutions and

$$\mathbf{x}(t) = C_1 \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$$

is the general solution of the system.

III. An nth order linear equation can be converted into a system of n first order linear equations

Consider the second order equation

$$y'' + p(t)y' + q(t)y = 0$$

Solve for y''

$$y'' = -q(t)y - p(t)y'$$

Introduce new dependent variables

x_1, x_2, as follows:

$$x_1 = y$$

$$x_2 = x_1' \ (= y')$$

Vector-matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note that this system is just a very special case of the "general" homogeneous system of two, first-order differential equations:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or

$$\mathbf{x}' = A(t)\mathbf{x}$$

Example 1: $y'' - 5y' + 6y = 0$

Characteristic equation:

Fundamental set:

General solution:

In system form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Corresponding solutions of the system are:

Solution of equation: y

Corresponding solution of system:

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

That is: $y \rightarrow \mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$

$$y_1 = e^{2t} \longrightarrow \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$y_2 = e^{3t} \longrightarrow \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The Wronskian

$$W(\mathbf{x}_1, \mathbf{x}_2) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t} \neq 0$$

and so $\mathbf{x}_1 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}$

is a fundamental set of solutions of the system.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The general solution of the system

is

$$\mathbf{x}(t) = C_1 \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}$$

and $\begin{pmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{pmatrix}$ is the fundamental matrix.

Example 2: $y'' - \frac{5}{t}y' + \frac{8}{t^2}y = 0$

Look for solutions of the form $y = t^r$

$y_1 = t^2, \; y_2 = t^4$ are independent solutions

In system form, the equation is:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -8/t^2 & 5/t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Corresponding solutions of system:

$$y \to \mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$y_1 = t^2 \longrightarrow \begin{pmatrix} t^2 \\ 2t \end{pmatrix} = \mathbf{x}_1,$$

$$y_2 = t^4 \longrightarrow \begin{pmatrix} t^4 \\ 4t^3 \end{pmatrix} = \mathbf{x}_2$$

$$\mathbf{x}_1 = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} t^4 \\ 4t^3 \end{pmatrix}$$

is a fundamental set of solutions of

the system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -8/t^2 & 5/t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The matrix

$$\mathbf{X}(t) = \begin{pmatrix} t^2 & t^4 \\ 2t & 4t^3 \end{pmatrix}$$

is a fundamental matrix.

Consider the third-order equation

$$y''' + p(t)y'' + q(t)y' + r(t)y = 0$$

or

$$y''' = -r(t)y - q(t)y' - p(t)y''.$$

Introduce new dependent variables

$x_1,\ x_2,\ x_3,$ as follows:

$$x_1 = y$$

$$x_2 = x_1' \ (= y')$$

$$x_3 = x_2' \ (= y'')$$

Then

$$y''' = x_3' = -r(t)x_1 - q(t)x_2 - p(t)x_3$$

The third-order equation can be written equivalently as the system of three first-order equations:

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = -r(t)x_1 - q(t)x_2 - p(t)x_3$$

That is

$$x_1' = 0x_1 + 1\,x_2 + 0x_3$$

$$x_2' = 0x_1 + 0x_2 + 1\,x_3$$

$$x_3' = -r(t)x_1 - q(t)x_2 - p(t)x_3$$

Vector-matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r(t) & -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Note that this system is just a very special case of the “general” system of three, first-order differential equations:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or in vector-matrix form:

$$\mathbf{x}' = A(t)\mathbf{x}$$

Example 3:

$$y''' - 3y'' - 4y' + 12y = 0.$$

which can be written

$$y''' = -12y + 4y' + 3y''.$$

Set

$$x_1 = y$$

$$x_2 = x_1' \ (= y')$$

$$x_3 = x_2' \ (= y'')$$

Then

$$x_3' = y''' = -12x_1 + 4x_2 + 3x_3$$

and equivalent system:

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = -12x_1 + 4x_2 + 3x_3$$

which is

$$x_1' = 0x_1 + 1\,x_2 + 0x_3$$

$$x_2' = 0x_1 + 0x_2 + 1\,x_3$$

$$x_3' = -12x_1 + 4x_2 + 3x_3$$

Vector-matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or

$$\mathbf{x}' = A\mathbf{x}$$

$$y''' - 3y'' - 4y' + 12y = 0$$

Characteristic equation:

$$r^3 - 3r^2 - 4r + 12 = (r-3)(r-2)(r+2)$$

Fundamental set:

$$\left\{e^{3t},\ e^{2t},\ e^{-2t}\right\}$$

General solution:

$$y = C_1 e^{3t} + C_2 e^{2t} + C_3 e^{-2t}$$

System:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

If y is a solution of the equation,

then

$$\mathbf{x} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

is the corresponding solution of the

system.

$$y \longrightarrow \mathbf{x} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

Equation:

$$y''' - 3y'' - 4y' + 12y = 0$$

Fundamental set:

$$\left\{y_1 = e^{3t}, \quad y_2 = e^{2t}, \quad y_3 = e^{-2t}\right\}$$

Equivalent differential system:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Solutions:

$$y_1 = e^{3t} \longrightarrow \mathbf{x}_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

$$y_2 = e^{2t} \longrightarrow \mathbf{x}_2 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$$y_3 = e^{-2t} \longrightarrow \mathbf{x}_3 = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$$W(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{vmatrix} e^{3t} & e^{2t} & e^{-2t} \\ 3e^{3t} & 2e^{2t} & -2e^{-2t} \\ 9e^{3t} & 4e^{2t} & 4e^{-2t} \end{vmatrix}$$

$$= -20e^{3t} \neq 0$$

Therefore,

$$\mathbf{x}_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

$$\mathbf{x}_3 = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \end{pmatrix}$$

is a fundamental set of solutions of

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and

$$\mathbf{x}(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + C_3 e^{-2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

is the general solution.

$$\mathbf{X} = \begin{pmatrix} e^{3t} & e^{2t} & e^{-2t} \\ 3e^{3t} & 2e^{2t} & -2e^{-2t} \\ 9e^{3t} & 4e^{2t} & 4e^{-2t} \end{pmatrix}$$

is the fundamental matrix.

IV. Homogeneous Systems with Constant Coefficients

(See Section 3.3)

$$x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$-\quad -\quad -\quad -$$

$$x_n' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

where $a_{11}, a_{12}, \ldots, a_{nn}$ are constants.

The system in vector-matrix form is

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

or

$$\mathbf{x}' = A\mathbf{x}.$$

Solutions of $\mathbf{x}' = A\mathbf{x}$

Example 1: (See Example 1, pg. 27)

$$\mathbf{x}_1 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is a solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}\mathbf{x}$$

How is the number 2 and the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ related to the matrix $\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$?

$$\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} =$$

THAT IS:

2 is an eigenvalue of A and

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is a corresponding eigenvector.

You can verify that 3 is an eigenvalue of A with eigenvector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

NOTE:

$$y'' - 5y' + 6y = 0$$

Characteristic equation

$$r^2 - 5r + 6 = (r-2)(r-3) = 0 \quad (*)$$

Characteristic roots: $r_1 = 2, \; r_2 = 3$

Fundamental set:

$$\left\{ y_1 = e^{2t}, \quad y_2 = e^{3t} \right\}$$

Vector-matrix system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$$

Characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -6 & 5 - \lambda \end{vmatrix} =$$

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$$

Eigenvalues: $\lambda_1 = 2, \ \lambda_2 = 3$

Fund set: $\mathbf{x}_1 = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{x}_2 = e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Example 2: (See Example 3, pg. 38)

$$\mathbf{x}_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = e^{3t}\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

is a solution of

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

How is the number 3 and the vector $\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$ related to the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} ?$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} =$$

THAT IS:

3 is an eigenvalue of A and

$$\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

is a corresponding eigenvector.

$$y''' - 3y'' - 4y' + 12y = 0$$

Characteristic equation:

$$r^3 - 3r^2 - 4r + 12 = (r-3)(r-2)(r+2) = 0$$

Characteristic roots:

$$r_1 = 3,\ r_2 = 2,\ r_3 = -2$$

Fundamental set:

$$\left\{y_1 = e^{3t}, \quad y_2 = e^{2t}, \quad y_3 = e^{-2t}\right\}$$

Vector-matrix form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -12 & 4 & 3-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 3\lambda^2 + 4\lambda - 12 = 0$$

or

$$\lambda^3 - 3\lambda^2 - 4\lambda + 12 = (\lambda - 3)(\lambda - 2)(\lambda + 2) = 0$$

Eigenvalues:

$$\lambda_1 = 3,\ \lambda_2 = 2,\ \lambda_3 = -2$$

Eigenvectors:

$$y_1 = e^{3t} \longrightarrow \mathbf{x}_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

$$y_2 = e^{2t} \longrightarrow \mathbf{x}_2 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$$y_3 = e^{-2t} \longrightarrow \mathbf{x}_3 = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

Fundamental set:

$$\left\{ e^{3t}\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, e^{2t}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, e^{-2t}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \right\}$$

and general solution

$$\mathbf{x} = C_1 e^{3t}\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + C_3 e^{-2t}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

In general, given the homogeneous system with constant coefficients

$$\mathbf{x}' = A\mathbf{x}.$$

THEOREM 1. If λ is an eigenvalue of A and $\mathbf{v}$ is a corresponding eigenvector, then

$$\mathbf{x} = e^{\lambda t}\mathbf{v}$$

is a solution.

Proof:

Let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{v}$.

Set $\mathbf{x} = e^{\lambda t}\mathbf{v}$

THEOREM 2. If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$, then

$$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1,\ \mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2,\ \cdots, \mathbf{x}_k = e^{\lambda_k t}\mathbf{v}_k$$

are linearly independent solutions of

$$\mathbf{x}' = A\mathbf{x}.$$

Corollary. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$, then

$$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1,\ \mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2,\ \cdots, \mathbf{x}_n = e^{\lambda_n t}\mathbf{v}_n$$

is a fundamental set of solutions of

$$\mathbf{x}' = A\mathbf{x} \quad \text{and}$$

$$\mathbf{x}(t) = C_1\mathbf{x}_1 + C_2\mathbf{x}_2 + \ \cdots \ + C_n\mathbf{x}_n$$

is the general solution.

Example 1: Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \mathbf{x}.$$

Step 1. Find the eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix}$$

$$= \lambda^2 - \lambda - 6.$$

Characteristic equation:

$$\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0.$$

Eigenvalues: $\lambda_1 = 3, \ \lambda_2 = -2.$

Step 2. Find the eigenvectors:

$$A - \lambda I = \begin{pmatrix} 2 - \lambda & 2 \\ 2 & -1 - \lambda \end{pmatrix}$$

$\lambda_1 = 3$:

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix}$$

$$\lambda_2 = -2$$

$$\lambda_1 = 3,\ \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix};\ \lambda_2 = -2,\ \mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Solutions:

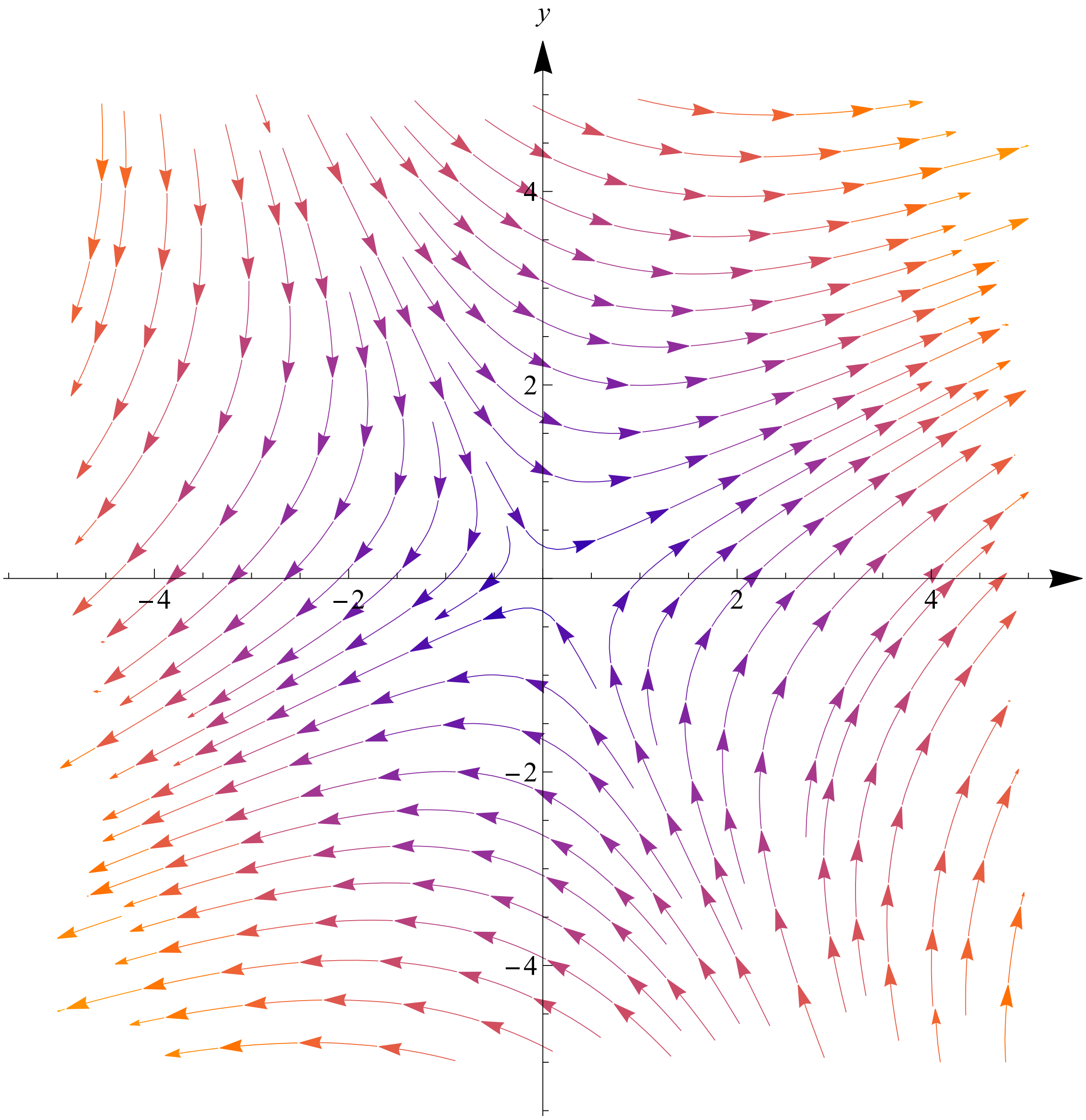
Fundamental set of solution vectors:

$$\left\{ \mathbf{x}_1 = e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$

General solution of the system:

$$\mathbf{x} = C_1 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Graphs



Example 2: Solve $\mathbf{x}' = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{pmatrix} \mathbf{x}$.

(See the example on pg. 8)

Step 1. Find the eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 4\lambda^2 - \lambda - 6.$$

Characteristic equation:

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = (\lambda - 3)(\lambda - 2)(\lambda + 1) = 0.$$

Eigenvalues:

$$\lambda_1 = 3, \quad \lambda_2 = 2, \quad \lambda_3 = -1.$$

Step 2. Find the eigenvectors:

$$A - \lambda I = \begin{pmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{pmatrix}$$

$\lambda_1 = 3$:

$\lambda_2 = 2$:

$\lambda_3 = -1$:

$$\lambda_1 = 3: \ \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

$$\lambda_2 = 2: \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$\lambda_3 = -1: \ \mathbf{v}_3 = \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}.$$

Solutions.

Fundamental set of solutions:

$$\mathbf{x}_1 = e^{3t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$\mathbf{x}_3 = e^{-t}\begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}.$$

The general solution of the system:

$$\mathbf{x} = C_1 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}.$$

Example 3: Find the solution of the initial-value problem

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}.$$

(See Example 2.)

General solution:

$$\mathbf{x} = C_1 e^{3t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 e^{t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{-t}\begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}.$$

To find the solution satisfying the

initial condition, set $t = 0$ and solve

$$C_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -3 \\ 1 & 1 & 7 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}.$$

Augmented matrix:

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ -1 & 0 & -3 & -3 \\ 1 & 1 & 7 & 1 \end{array}\right)$$

Solution:

$$C_1 = 3,\ C_2 = -2,\ C_3 = 0.$$

The solution of the initial-value problem is:

$$\mathbf{x} = 3e^{3t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - 2e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Example 4: Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{pmatrix} \mathbf{x}$$

Step 1. Find the eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{vmatrix}$$

$$= -\lambda^3 - 2\lambda^2 + 23\lambda + 60.$$

Characteristic equation:

$$\lambda^3 + 2\lambda^2 - 23\lambda - 60 = (\lambda+3)(\lambda+4)(\lambda-5) = 0.$$

Eigenvalues:

$$\lambda_1 = -3, \quad \lambda_2 = -4, \quad \lambda_3 = 5.$$

Step 2. Find the eigenvectors:

$$A - \lambda I = \begin{pmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{pmatrix}$$

$\lambda_1 = -3$:

$\lambda_2 = -4$:

$\lambda_3 = 5$:

Solutions:

$$\lambda_1 = -3: \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$\lambda_2 = -4: \quad \mathbf{v}_2 = \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix},$$

$$\lambda_3 = 5: \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}.$$

Fundamental set of solutions:

$$\mathbf{x}_1 = e^{-3t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = e^{-4t}\begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix},$$

$$\mathbf{x}_3 = e^{5t}\begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}.$$

The general solution of the system:

$$\mathbf{x} = C_1 e^{-3t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{-4t}\begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix} + C_3 e^{5t}\begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}$$