V. HOMOGENEOUS SYSTEMS WITH CONSTANT COEFFICIENTS, TWO DIFFICULTIES:

1. A has complex eigenvalues and complex eigenvectors.

2. A has an eigenvalue of multiplicity greater than 1.

1. Complex Eigenvalues/Vectors

Example 1: Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix} \mathbf{x}.$$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -2 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5.$$

The eigenvalues are:

$$\lambda_1 = -1 + 2i, \;\; \lambda_2 = -1 - 2i.$$

$$A - \lambda I = \begin{pmatrix} -3 - \lambda & -2 \\ 4 & 1 - \lambda \end{pmatrix}$$

For $\lambda_1 = -1 + 2i$: Solve

$$\left(\begin{array}{cc|c} -2 - 2i & -2 & 0 \\ 4 & 2 - 2i & 0 \end{array}\right) \longrightarrow$$

The solution set is:

$$x_2 = -(1+i)x_1, \quad x_1 \quad \text{arbitrary}$$

Set $x_1 = 1$. Then, for $\lambda_1 = -1+2i$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1-i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

and, for $\lambda_2 = -1-2i$:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

$\lambda_1 = -1+2i$

$$A - (-1+2i)I = \begin{pmatrix} -2-2 & -2 \\ 4 & 2-2i \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -2-2i & -2 & 0 \\ 4 & 2-2i & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -\frac{1}{2}-\frac{1}{2}i & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2}-\frac{1}{2}i & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & \frac{1}{2}-\frac{1}{2}i & 0 \\ -\frac{1}{2}-\frac{1}{2}i & -\frac{1}{2} & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & \frac{1}{2}-\frac{1}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 = \left(-\frac{1}{2}+\frac{1}{2}i\right)x_2$, x_2 arb.

Set $x_2 = 2$: $\vec{v}_1 = \begin{pmatrix} -1+i \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solutions:

$$u_1(t) = e^{(-1+2i)t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

$$u_2(t) = e^{(-1-2i)t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} - i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

Complex-valued! We want real-valued

solutions (see Section 3.3 Case III).

From 3.3,

$e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula)

$e^{-i\theta} = \cos\theta - i\sin\theta$

So $e^{(-1+2i)t} = e^{-t}e^{2it}$

$= e^{-t}[\cos 2t + i\sin 2t]$

$e^{(-1-2i)t} = e^{-t}e^{-2it}$

$= e^{-t}[\cos 2t - i\sin 2t]$

Solutions

$$\mathbf{u}_1 = e^{\lambda_1 t}\mathbf{v}_1 =$$

$$= e^{(-1+2i)t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] =$$

$$e^{-t}(\cos 2t + i \sin 2t)\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

$$= e^{-t}\left[\cos 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] +$$

$$i\, e^{-t}\left[\cos 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right].$$

$$\mathbf{u}_2 = e^{\lambda_2 t}\mathbf{v}_2$$

$$= e^{(-1-2i)t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} - i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] =$$

$$e^{-t}(\cos 2t - i \sin 2t)\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} - i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

$$= e^{-t}\left[\cos 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] -$$

$$i\, e^{-t}\left[\cos 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right].$$

Fundamental set:

$$\mathbf{x}_1 = \frac{\mathbf{u}_1 + \mathbf{u}_2}{2}$$

$$\vec{x}_1 = e^{-t}\left[\cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

$$\mathbf{x}_2 = \frac{\mathbf{u}_1 - \mathbf{u}_2}{2i}$$

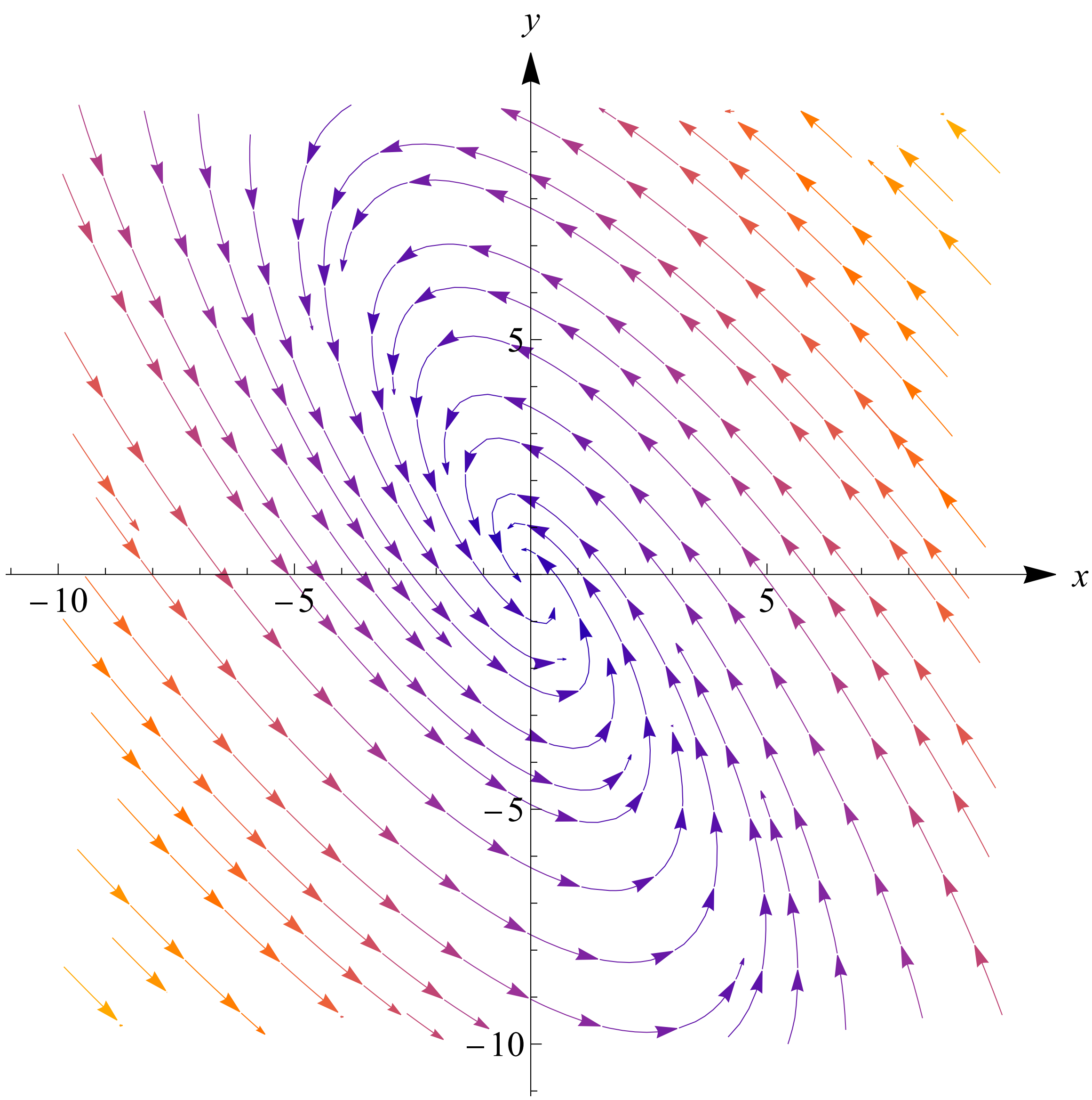
$$\vec{x}_2 = e^{-t}\left[\cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

General solution:

$$\mathbf{x} = C_1\, e^{-t}\left[\cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] +$$

$$C_2\, e^{-t}\left[\cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

Graphs



Example 2: Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix} \mathbf{x}.$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -5 \\ 2 & 3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 4\lambda + 13.$$

Characteristic equation:

$$\lambda^2 - 4\lambda + 13 = 0$$

Eigenvalues:

$$\lambda_1 = 2 + 3i, \;\; \lambda_2 = 2 - 3i.$$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -5 \\ 2 & 3-\lambda \end{pmatrix}$$

For $\lambda_1 = 2 + 3i$: Solve

$$\left(\begin{array}{cc|c} -1-3i & -5 & 0 \\ 2 & 1-3i & 0 \end{array}\right) \longrightarrow$$

Eigenvectors:

$$\lambda_1 = 2 + 3i, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

$$\lambda_2 = 2 - 3i, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - i \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Fundamental set:

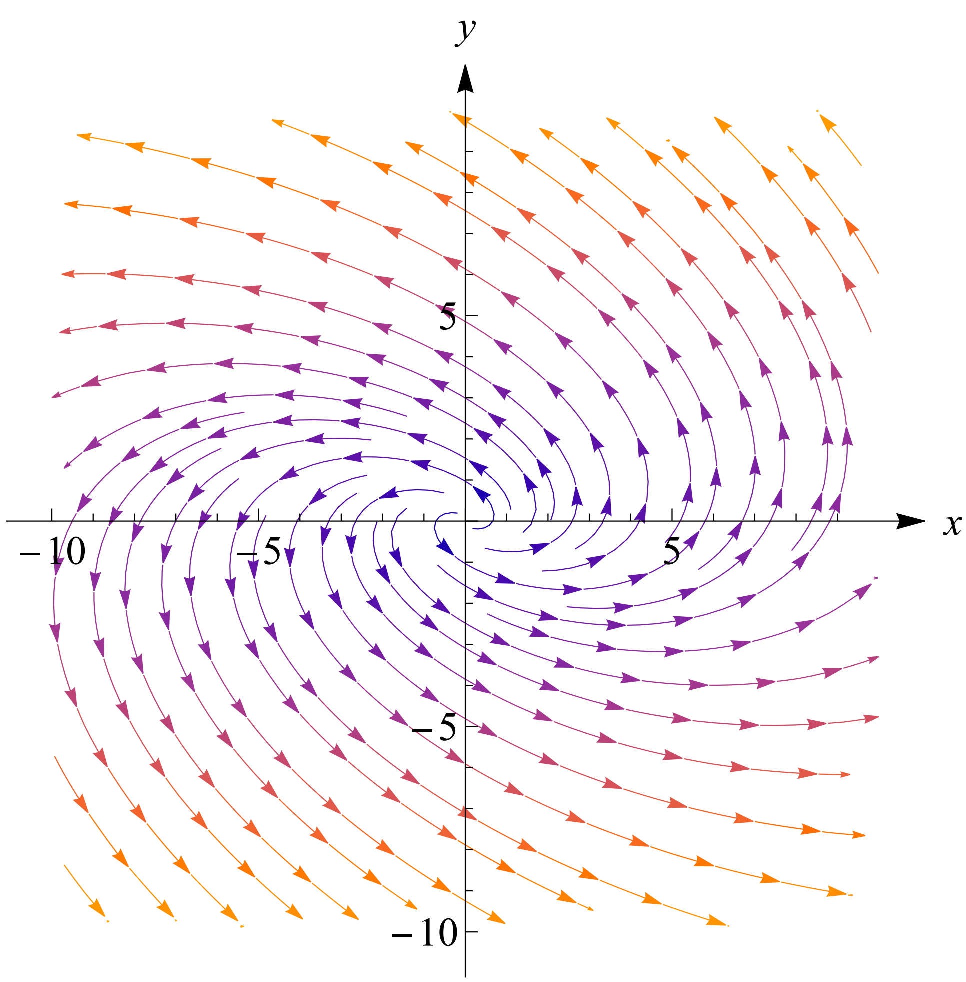
$$\mathbf{x}_1 = e^{2t} \left[\cos 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right]$$

$$\mathbf{x}_2 = e^{2t} \left[\cos 3t \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right]$$

General solution:

$$\mathbf{x} = C_1 e^{2t}\left[\cos 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} 3 \\ 0 \end{pmatrix}\right] +$$

$$C_2 e^{2t}\left[\cos 3t \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right]$$



Summary: $\mathbf{x}' = A\mathbf{x}$, A $n \times n$ const.

$a+ib$, $a-ib$ complex eigenvalues.

$\vec{\alpha} + i\,\vec{\beta}$, $\vec{\alpha} - i\,\vec{\beta}$ corresponding eigenvectors.

Independent (complex-valued) solutions:

$$\mathbf{u}_1 = e^{(a+ib)t}\left(\vec{\alpha} + i\,\vec{\beta}\right)$$

$$\mathbf{u}_2 = e^{(a-ib)t}\left(\vec{\alpha} - i\,\vec{\beta}\right)$$

Corresponding real-valued solutions, fundamental set:

$$\mathbf{x}_1 = e^{at}\left[\cos bt\ \vec{\alpha} - \sin bt\ \vec{\beta}\right]$$

$$\mathbf{x}_2 = e^{at}\left[\cos bt\ \vec{\beta} + \sin bt\ \vec{\alpha}\right]$$

General solution:

$$\mathbf{x} = C_1\, e^{at}\left[\cos bt\ \vec{\alpha} - \sin bt\ \vec{\beta}\right] +$$

$$C_2 e^{at}\left[\cos bt\ \vec{\beta} + \sin bt\ \vec{\alpha}\right]$$

Example 3: Determine a fundamental set of solution vectors of

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 & -1 \\ 3 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \mathbf{x}.$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -4 & -1 \\ 3 & 2-\lambda & 3 \\ 1 & 1 & 3-\lambda \end{vmatrix} =$$

$$-\lambda^3 + 6\lambda^2 - 21\lambda + 26 = -(\lambda - 2)(\lambda^2 - 4\lambda + 13).$$

The eigenvalues are:

$$\lambda_1 = 2,\ \lambda_2 = 2 + 3i,\ \lambda_3 = 2 - 3i.$$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -4 & -1 \\ 3 & 2-\lambda & 3 \\ 1 & 1 & 3-\lambda \end{pmatrix}$$

$\lambda_1 = 2$: Solve

$$\left(\begin{array}{ccc|c} -1 & -4 & -1 & 0 \\ 3 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & -4 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_2 = 0$
$x_1 = -x_3$ arb.

Set $x_3 = -1$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Soln $e^{2t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -4 & -1 \\ 3 & 2-\lambda & 3 \\ 1 & 1 & 3-\lambda \end{pmatrix}$$

For $\lambda_2 = 2 + 3i$: Solve

$$\left(\begin{array}{ccc|c} -1-3i & -4 & -1 & 0 \\ 3 & -3i & 3 & 0 \\ 1 & 1 & 1-3i & 0 \end{array}\right) \rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1-3i & 0 \\ 1 & -i & 1 & 0 \\ -1-3i & -4 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1-3i & 0 \\ 0 & -1-i & 3i & 0 \\ 0 & -3+3i & 10 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1-3i & 0 \\ 0 & -1-i & 3i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_2 = \frac{3i}{1+i} x_3 = \frac{3i}{1+i} \cdot \frac{1-i}{1-i} x_3 = \left(\frac{3}{2} - \frac{3}{2} i\right) x_3$$

$$x_1 = \left(-\frac{5}{2} + \frac{3}{2} i\right) x_3$$

The solution set is:

$$x_1 = \left(-\frac{5}{2} + \frac{3}{2}\,i\right) x_3,\; x_2 = \left(\frac{3}{2} - \frac{3}{2}\,i\right) x_3,\,,$$

x_3 arbitrary.

$$\mathbf{v}_2 = \begin{pmatrix} -5 + 3i \\ 3 - 3i \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}.$$

and

$$\mathbf{v}_3 = \begin{pmatrix} -5 - 3i \\ 3 + 3i \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - i \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}.$$

Now

$$\mathbf{u}_1 = e^{(2+3i)t}\left[\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}\right]$$

and

$$\mathbf{u}_2 = e^{(2-3i)t}\left[\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - i\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}\right]$$

convert to:

$$\mathbf{x}_1 = e^{2t}\left[\cos 3t\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - \sin 3t\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}\right]$$

and

$$\mathbf{x}_2 = e^{2t}\left[\cos 3t\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \sin 3t\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}\right]$$

Fundamental set of solution vectors:

$$\mathbf{x}_1 = e^{2t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$\mathbf{x}_2 = e^{2t}\left[\cos 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}\right],$$

$$\mathbf{x}_3 = e^{2t}\left[\cos 3t \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}\right].$$

General solution:

$$\mathbf{x} = C_1\mathbf{x}_1 + C_2\mathbf{x}_2 + C_3\mathbf{x}_3$$

On Exam 3, 2x2 Yes. !!

3x3 No.

2. Repeated Eigenvalues

Example 1: Find a fundamental set of solutions of

$$\mathbf{x}' = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} \mathbf{x}.$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix}$$

$$= 16 + 12\lambda - \lambda^3 = -(\lambda - 4)(\lambda + 2)^2.$$

Eigenvalues: $\lambda_1 = 4,\ \lambda_2 = \lambda_3 = -2$

$$\lambda_1 = 4: \quad (A - 4I) = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}$$

Solve:

$$(A - 4I)\mathbf{x} = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 + x_2 - x_3 = 0$

$2x_2 - x_3 = 0$

$x_2 = \frac{6}{2} x_3$

$x_1 = \frac{1}{2} x_3$

arb

Set $x_3 = 2$ $\quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

Soln $\vec{X}_1 = e^{4t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$\lambda_2 = \lambda_3 = -2$:

$$A - (-2)I = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right)$$

which row reduces to

$$\left(\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Solution set:

$$x_1 = x_2 - x_3,\ \ x_2,\ \ x_3 \quad \text{arbitrary}$$

Set $x_2 = 1,\ x_3 = 0:$ $\quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$

Set $x_2 = 0,\ x_3 = 1:$ $\quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$

Fundamental set:

$$\left\{ e^{4t}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},\ e^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ e^{-2t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$y''' - 12y' - 16y = 0$

Example 2: Find a fundamental set of solutions of $\mathbf{x}' = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix}\mathbf{x}$.

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 1 \\ -4 & -\lambda \end{vmatrix}$$

$$= \lambda^2 + 4\lambda + 4.$$

Characteristic equation:

$$\lambda^2 + 4\lambda + 4 = 0$$

Eigenvalues:

$$\lambda_1 = \lambda_2 = -2.$$

Eigenvectors: $A-\lambda I = \begin{pmatrix} -4-\lambda & 1 \\ -4 & -\lambda \end{pmatrix}$

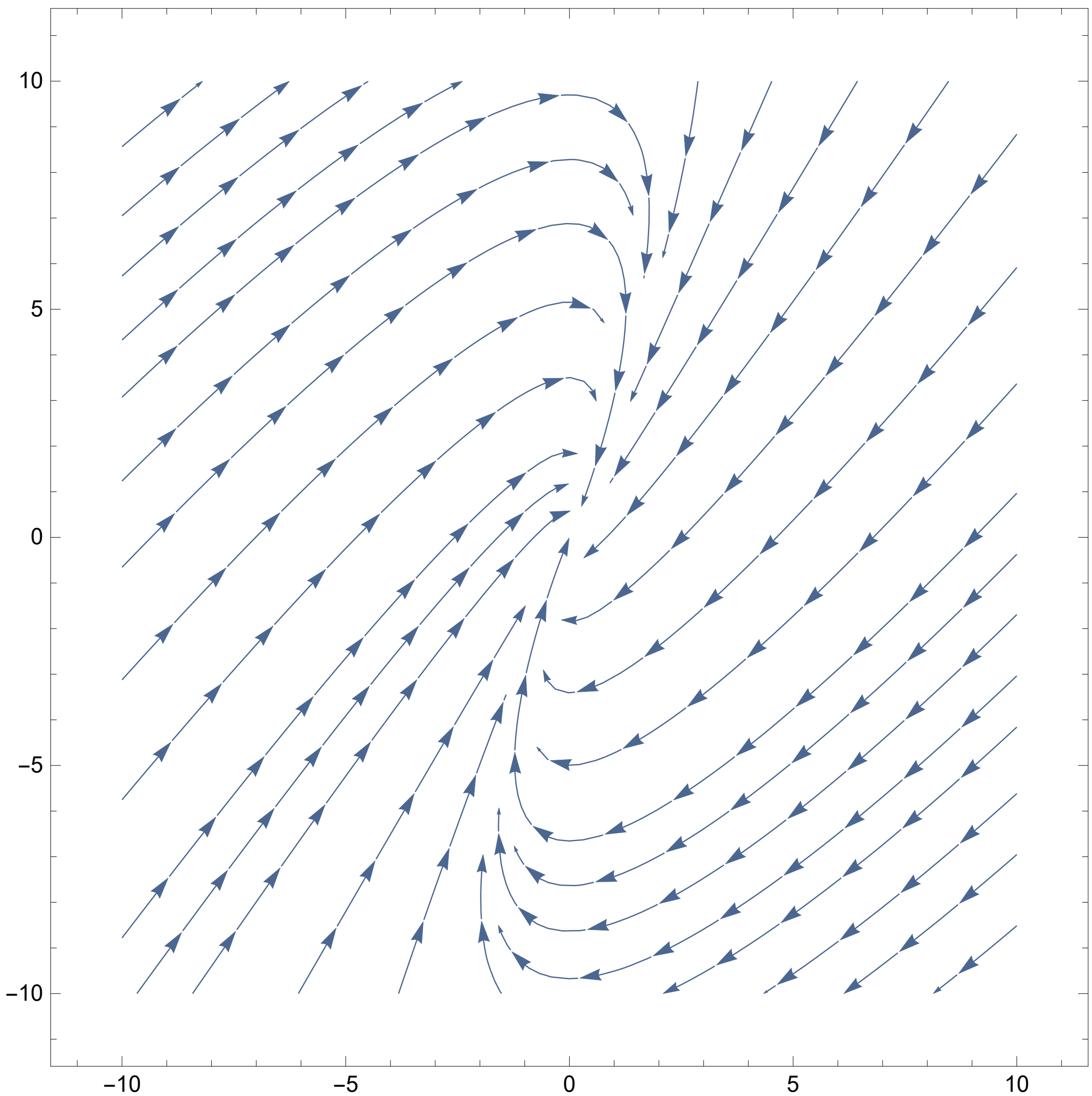
$\lambda_1 = \lambda_2 = -2$: Solve

$$(A-(-2)I)\mathbf{x} = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -2 & 1 & 0 \\ -4 & 2 & 0 \end{array}\right) \to$$

Problem: Only one eigenvector and only one solution! We need another solution.

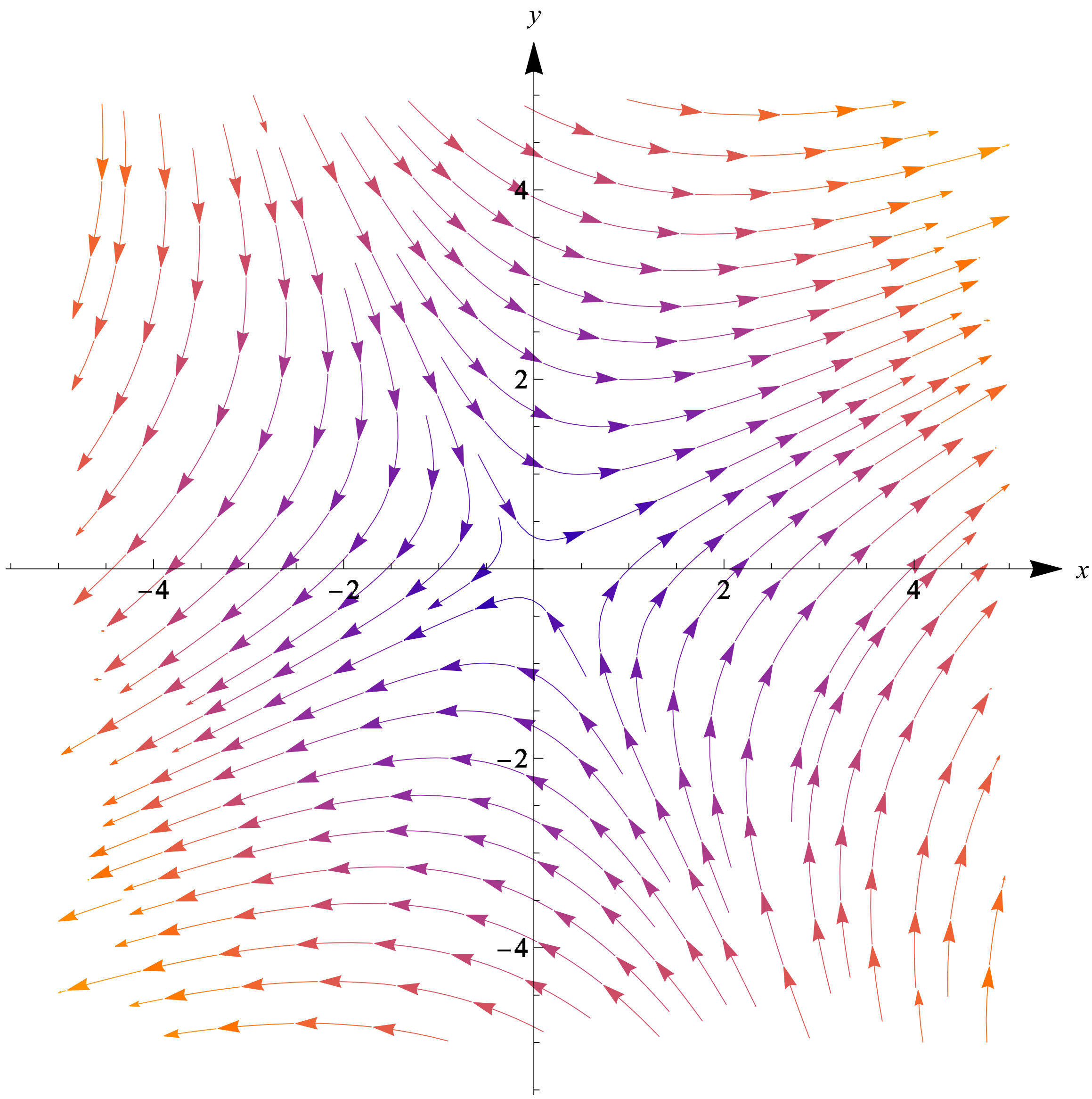
Graphs



Graphs – See Example 1, pg 61:

two independent eigenvectors and graph

pg. 65.



Example 3: Find a fundamental set of solutions of

$$\mathbf{x}' = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix} \mathbf{x}.$$

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 6 & 2 \\ 0 & -1-\lambda & -8 \\ 1 & 0 & -2-\lambda \end{vmatrix}$$

$$= -36 + 15\lambda + 2\lambda^2 - \lambda^3 = -(\lambda+4)(\lambda-3)^2.$$

Eigenvalues: $\lambda_1 = -4$, $\lambda_2 = \lambda_3 = 3$.

$$\lambda_1 = -4\text{:} \quad A - (-4)I = \begin{pmatrix} 9 & 6 & 2 \\ 0 & 3 & -8 \\ 1 & 0 & 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 9 & 6 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 0 & 6 & -18 & 0 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 = -2x_3,\ x_2 = \tfrac{8}{3}x_3, \quad x_3 \text{ arbitrary}$$

$$\text{Set } x_3 = -3: \quad \mathbf{v}_1 = \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}$$

$$\mathbf{x}_1 = e^{-4t}\begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}$$

$\lambda_2 = \lambda_3 = 3$:

$$A - 3I = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -4 & -8 \\ 1 & 0 & -5 \end{pmatrix}$$

Solve

$$(A - 3I)\mathbf{x} = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -4 & -8 \\ 1 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{rrr|r} 2 & 6 & 2 & 0 \\ 0 & -4 & -8 & 0 \\ 1 & 0 & -5 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & 0 & -5 & 0 \\ 0 & -4 & -8 & 0 \\ 2 & 6 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 2 & 6 & 2 & 0 \\ 0 & -4 & -8 & 0 \\ 1 & 0 & -5 & 0 \end{array}\right)$$

which row reduces to

$$\left(\begin{array}{rrr|r} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$x_1 = 5x_3,\ x_2 = -2x_3,\ x_3$ arbitrary

Set $x_3 = 1$: $\mathbf{v}_2 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$

NOTE: Only one eigenvector here!

Solutions:

$$\mathbf{x}_1 = e^{-4t}\begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}, \quad \mathbf{x}_2 = e^{3t}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}.$$

Problem: Only two solutions!

We need a third solution $\mathbf{x}_3$ which is independent of $\mathbf{x}_1, \mathbf{x}_2$.

Lesson from linear equations

Example:

$$y''' + y'' - 8y' - 12y = 0$$

Characteristic equation:

$$r^3 + r^2 - 8r - 12 = (r-3)(r+2)^2 = 0.$$

Fundamental set: $\left\{e^{3t},\ e^{-2t},\ te^{-2t}\right\}$

Write the equation in system form:

Equivalent system: $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & 8 & -1 \end{pmatrix} \mathbf{x}$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 12 & 8 & -1-\lambda \end{vmatrix}$$

$$= -\lambda^3 - \lambda^2 + 8\lambda + 12\lambda$$

characteristic equation:

$$\lambda^3 + \lambda^2 - 8\lambda - 12 = (\lambda - 3)(\lambda + 2)^2$$

Eigenvalues: $\lambda_1 = 3, \;\; \lambda_2 = \lambda_3 = -2$

Recall $y \to \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$

equation | System

$y_1 = e^{3t} \longrightarrow \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$

$y_2 = e^{-2t} \longrightarrow \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

$$y_3 = te^{-2t} \longrightarrow \begin{pmatrix} te^{-2t} \\ e^{-2t} - 2te^{-2t} \\ -4e^{-2t} + 4te^{-2t} \end{pmatrix}$$

$$= e^{-2t}\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + te^{-2t}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

eigenvector for -2

$\vec{x}_3$ has the form

$$e^{-2t}\vec{w} + te^{-2t}\vec{v}$$

Fundamental set:

$$\mathbf{x}_1 = e^{3t}\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \quad \mathbf{x}_2 = e^{-2t}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix},$$

$$\mathbf{x}_3 = e^{-2t}\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + te^{-2t}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

Look at the solution vector $\mathbf{x}_3$

$\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ is an eigenvector for -2

Question: What is the vector $\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$

$\vec{x}_3$ has the form

$e^{-2t}\vec{w} + te^{-2t}\vec{v}$ ↖ eigenvector

$$[A - (-2)I] \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 12 & 8 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$$

$$=$$

$A - (-2I)$ "maps" $\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$ onto

the eigenvector $\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$.

$$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$$

is called a **generalized eigenvector**.

The third solution has the form

$$\mathbf{x}_3 = e^{-2t}\mathbf{w} + te^{-2t}\mathbf{v}$$

Back to Example 3.

$$\mathbf{x}' = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix}\mathbf{x}.$$

Solutions:

$$\mathbf{x}' = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix}\mathbf{x}.$$

The third solution has the form

$$\mathbf{x}_3 = e^{3t}\mathbf{w} + te^{3t}\mathbf{v}$$

To find $\mathbf{w}$, solve

$$(A-3I)\mathbf{w} = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -4 & -8 \\ 1 & 0 & -5 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{rrr|r} 2 & 6 & 2 & 5 \\ 0 & -4 & -8 & -2 \\ 1 & 0 & -5 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 0 & -5 & 1 \\ 0 & -4 & -8 & -2 \\ 2 & 6 & 2 & 5 \end{array}\right) \longrightarrow$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Solution set:

$$w_1 = 1 + 5w_3,\ w_2 = \frac{1}{2} - 2w_3,\ w_3 \text{ arbitrary}$$

Set $w_3 = 0$: $\quad \mathbf{w} = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$

The third solution is:

$$\mathbf{x}_3 = e^{3t}\begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} + te^{3t}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Fundamental set:

$$\mathbf{x}_1 = e^{-4t}\begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}, \quad \mathbf{x}_2 = e^{3t}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix},$$

$$\mathbf{x}_3 = e^{3t}\begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} + te^{3t}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Back to Example 2.

$$\mathbf{x}' = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}$$

Solution: $\mathbf{x}_1 = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The second solution has the form

$$\mathbf{x}_2 = e^{2t}\mathbf{w} + te^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solve:

$$(A-(-2)I)\mathbf{w} = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -2 & 1 & 1 \\ -4 & 2 & 2 \end{array}\right)$$

Fundamental set:

$$\mathbf{x}_1 = e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

$$\mathbf{x}_2 = e^{-2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + te^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Eigenvalues of multiplicity 2:

Given $\mathbf{x}' = A\mathbf{x}$.

Suppose that A has an eigenvalue λ of multiplicity 2. Then exactly one of the following holds:

1. λ has two linearly independent eigenvectors, $\mathbf{v}_1$ and $\mathbf{v}_2$. Corresponding linearly independent solution vectors of the differential system are

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 \quad \text{and} \quad \mathbf{x}_2(t) = e^{\lambda t}\mathbf{v}_2.$$

(See Example 1, p. 12)

2. λ has only one eigenvector $\mathbf{v}$. (See Examples 2 and 3.) Then a linearly independent pair of solution vectors corresponding to λ is:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v} \quad \text{and} \quad \mathbf{x}_2(t) = e^{\lambda t}\mathbf{w} + te^{\lambda t}\mathbf{v}$$

where $\mathbf{w}$ is a vector that satisfies

$$(A - \lambda I)\mathbf{w} = \mathbf{v}.$$

The vector $\mathbf{w}$ is called a **generalized eigenvector** corresponding to the eigenvalue λ.

Examples

Find a fundamental set of solutions

and the general solution.

1. $\mathbf{x}' = \begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}\mathbf{x}.$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 5 \\ -1 & 4-\lambda \end{vmatrix}$$

$$= \lambda^2 - 6\lambda + 13$$

Eigenvalues: $3 + 2i, \quad 3 - 2i$

$$(A - \lambda I) = \begin{pmatrix} 2-\lambda & 5 \\ -1 & 4-\lambda \end{pmatrix}$$

$$\lambda_1 = 3 + 2i, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

Fundamental set:

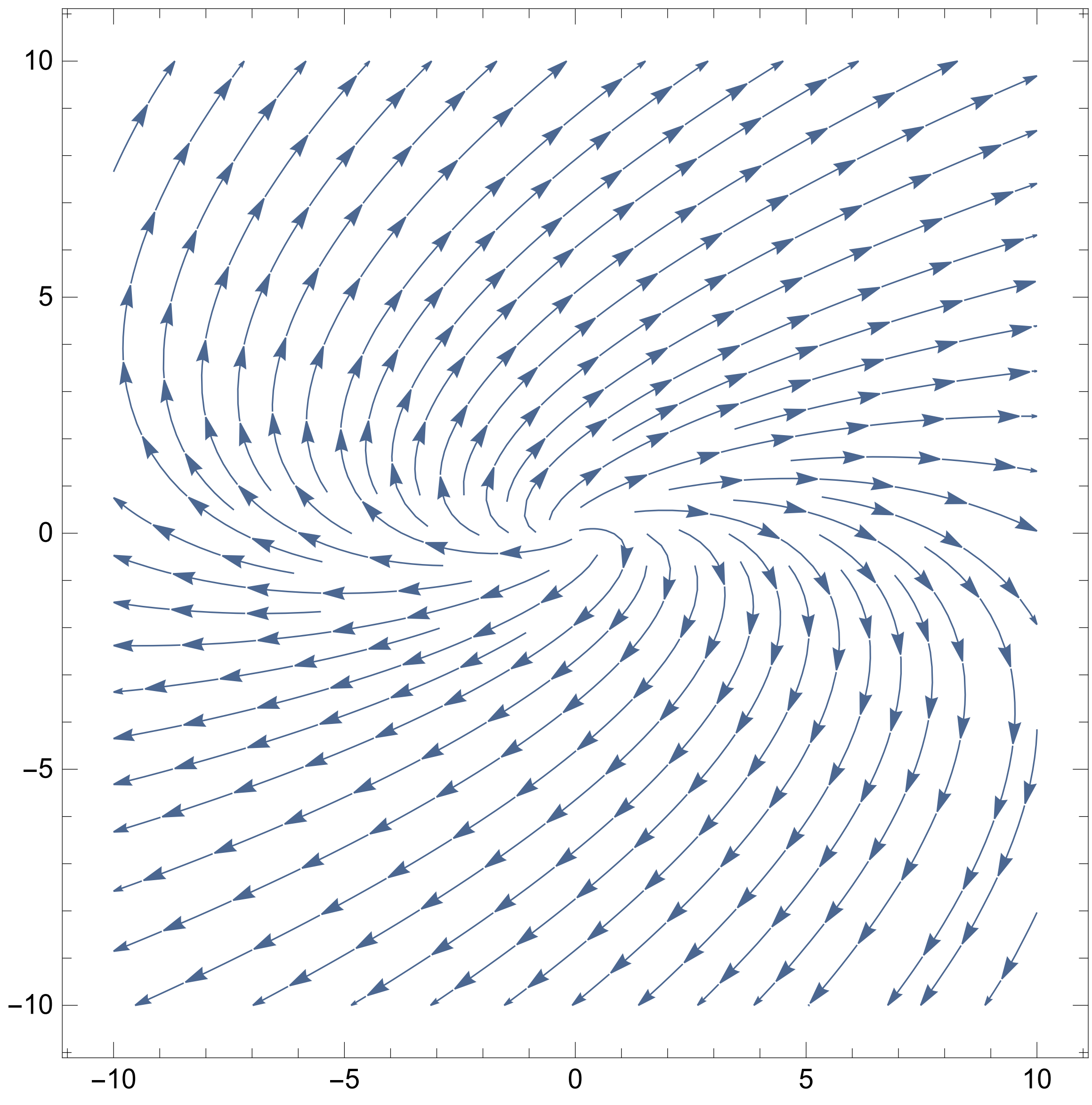
$$\mathbf{x}_1 = e^{3t}\left[\cos 2t\begin{pmatrix}1\\1\end{pmatrix} - \sin 2t\begin{pmatrix}-2\\0\end{pmatrix}\right]$$

$$\mathbf{x}_2 = e^{3t}\left[\cos 2t\begin{pmatrix}-2\\0\end{pmatrix} + \sin 2t\begin{pmatrix}1\\1\end{pmatrix}\right]$$

General solution:

$$\mathbf{x}(t) = C_1\mathbf{x}_1 + C_2\mathbf{x}_2$$

Graphs



2. $\mathbf{x}' = \begin{pmatrix} -4 & 1 & -2 \\ 2 & -3 & 2 \\ 2 & -1 & 0 \end{pmatrix}\mathbf{x}.$

HINT: -3 is an eigenvalue and -2

is an eigenvalue of multiplicity 2

Characteristic eqn: $(\lambda+3)(\lambda+2)^2=0$

$$(A - \lambda I) = \begin{pmatrix} -4-\lambda & 1 & -2 \\ 2 & -3-\lambda & 2 \\ 2 & -1 & -\lambda \end{pmatrix}$$

$$\lambda_1 = -3: \quad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$(A-\lambda I) = \begin{pmatrix} -4-\lambda & 1 & -2 \\ 2 & -3-\lambda & 2 \\ 2 & -1 & -\lambda \end{pmatrix}$$

$$\lambda_2 = \lambda_3 = -2: \qquad \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{rrr|r} -2 & 1 & -2 & 0 \\ 2 & -1 & 2 & 0 \\ 2 & -1 & 2 & 0 \end{array}\right) \rightarrow$$

Fundamental set:

$$e^{-3t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad e^{-2t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad e^{-2t}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

General solution:

$$\mathbf{x} =$$

$$C_1 e^{-3t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + C_3 e^{-2t}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\textbf{3.} \qquad \mathbf{x}' = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix} \mathbf{x}.$$

HINT: 4 is an eigenvalue and -2

is an eigenvalue of multiplicity 2

Characteristic eqn: $(\lambda - 4)(\lambda + 2)^2 = 0$

$$(A-\lambda I) = \begin{pmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{pmatrix}$$

$$\lambda_1 = 4: \qquad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$(A - \lambda I) = \begin{pmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{pmatrix}$$

$$\lambda_2 = \lambda_3 = -2: \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$(A - \lambda I) = \begin{pmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{pmatrix}$$

$$[A - (-2)I]\mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Fund. Set: $e^{4t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $e^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$e^{-2t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + te^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

General solution:

$$\mathbf{x} = C_1 e^{4t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} +$$

$$C_3\left[e^{-2t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + te^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right]$$

4. $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix} \mathbf{x}$

Characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix}$$

$$= \lambda^2 - 6\lambda + 9$$

Characteristic equation:

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0$$

Eigenvalues: $\lambda_1 = \lambda_2 = 3$

Eigenvectors:

$$(A-3I)\mathbf{x} = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{rr|r} -2 & -2 & 0 \\ 2 & 2 & 0 \end{array} \right) \rightarrow$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -2 & -2 & 1 \\ 2 & 2 & -1 \end{array} \right) \longrightarrow$$

Fundamental set:

$$\mathbf{x}_1 = e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

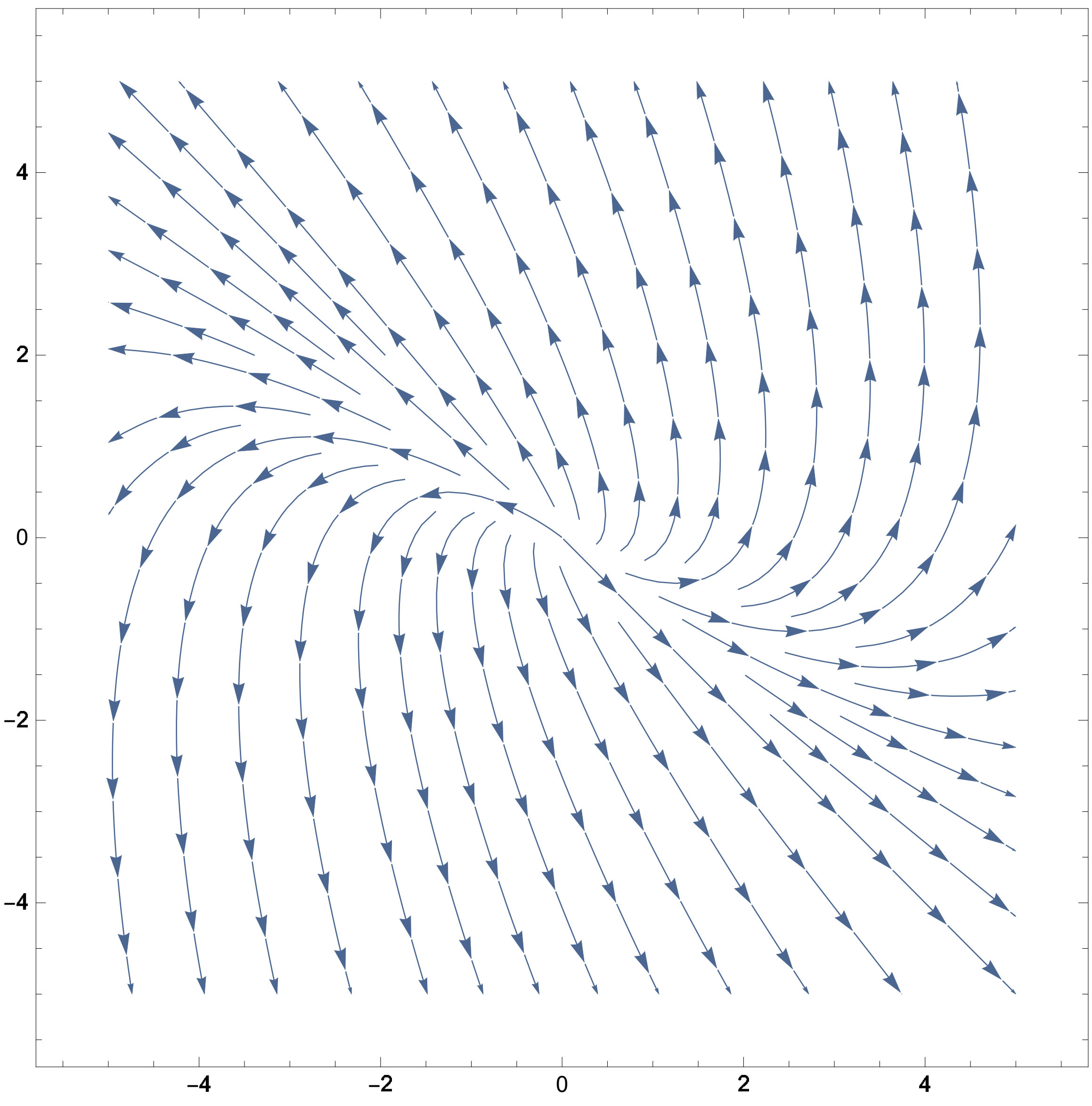
$$\mathbf{x}_2 = e^{3t}\begin{pmatrix} -1/2 \\ 0 \end{pmatrix} + te^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

General solution:

$$\mathbf{x} = C_1 e^{3x}\begin{pmatrix} 1 \\ -1 \end{pmatrix} +$$

$$C_2\left[e^{3t}\begin{pmatrix} -1/2 \\ 0 \end{pmatrix} + te^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

Graphs



5. $\mathbf{x}' = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix}\mathbf{x}.$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -2 \\ 4 & 1-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda + 5.$$

Characteristic equation:

$$\lambda^2 + 2\lambda + 5 = 0$$

Eigenvalues:

$$\lambda_1 = -1 + 2i, \;\; \lambda_2 = -1 - 2i.$$

Eigenvectors:

$$\lambda_1 = -1+2i, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

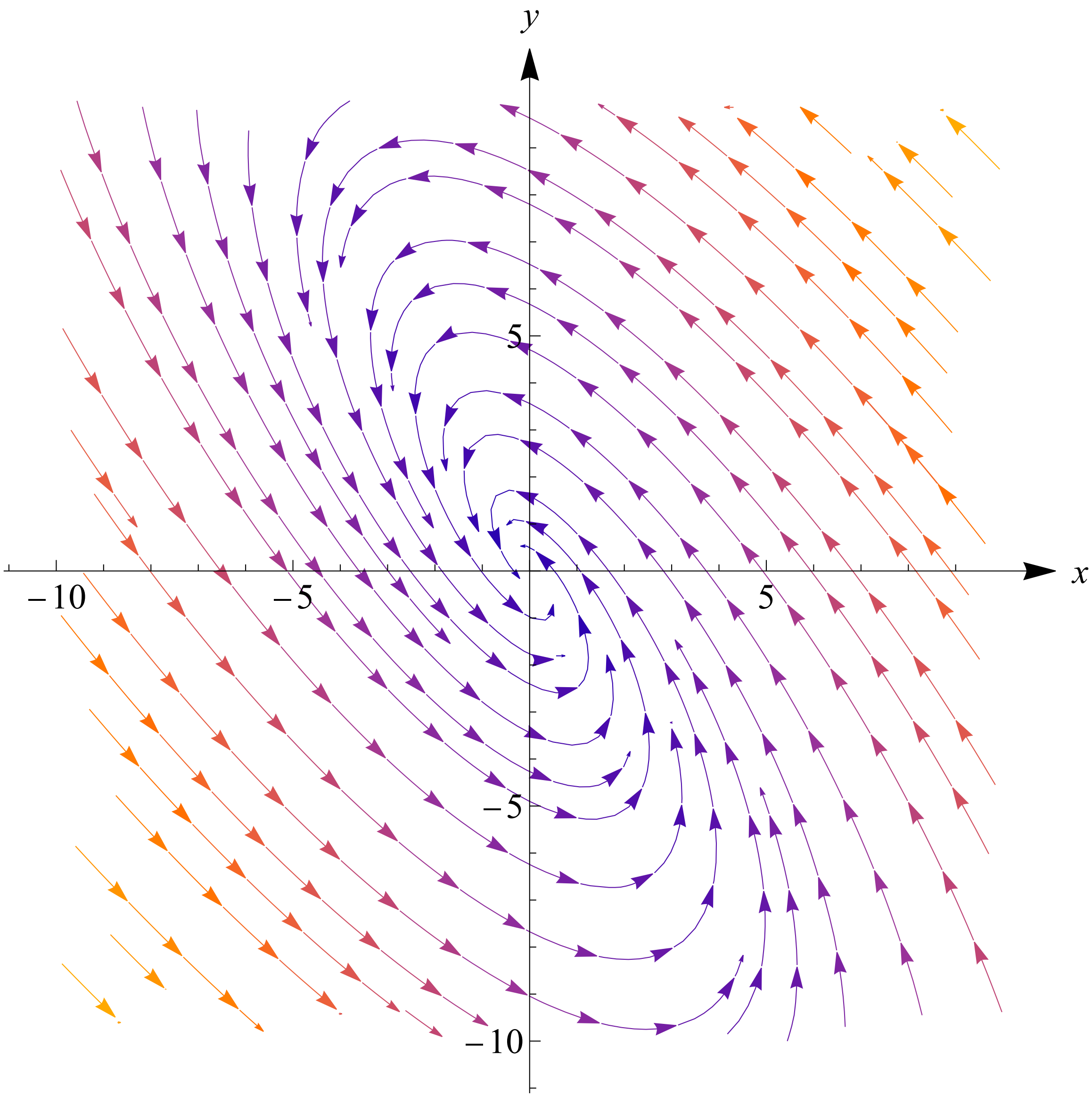
$$\lambda_2 = -1-2i, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

General solution:

$$\mathbf{x} = C_1\, e^{-t} \left[\cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] +$$

$$C_2\, e^{-t} \left[\cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$$

Graphs



Eigenvalues of Multiplicity 3.

Given the differential system

$$\mathbf{x}' = A\mathbf{x}.$$

Suppose that λ is an eigenvalue of A of multiplicity 3. Then exactly one of the following holds:

1. λ has three linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Then three linearly independent solution vectors of the system corresponding to λ are:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1, \quad \mathbf{x}_2(t) = e^{\lambda t}\mathbf{v}_2,$$

$$\mathbf{x}_3(t) = e^{\lambda t}\mathbf{v}_3.$$

2. λ has two linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$. Then three linearly independent solutions of the system corresponding to λ are:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1, \quad \mathbf{x}_2(t) = e^{\lambda t}\mathbf{v}_2$$

and

$$\mathbf{x}_3(t) = e^{\lambda t}\mathbf{w} + te^{\lambda t}\mathbf{v}$$

where $\mathbf{v}$ is an eigenvector corresponding to λ and $(A - \lambda I)\mathbf{w} = \mathbf{v}$. That is: $(A - \lambda I)^2\mathbf{w} = 0$.

3. λ has only one (independent) eigenvector $\mathbf{v}$. Then three linearly independent solutions of the system have the form:

$$\mathbf{x}_1 = e^{\lambda t}\mathbf{v}, \quad \mathbf{x}_2 = e^{\lambda t}\mathbf{w} + te^{\lambda t}\mathbf{v},$$

$$\mathbf{v}_3(t) = e^{\lambda t}\mathbf{z} + te^{\lambda t}\mathbf{w} + t^2 e^{\lambda t}\mathbf{v}$$

where

$$(A - \lambda I)\mathbf{z} = \mathbf{w} \ \& \ (A - \lambda I)\mathbf{w} = \mathbf{v}, \ i.e.$$

$$(A - \lambda I)^3\mathbf{z} = 0 \quad \& \quad (A - \lambda I)^2\mathbf{w} = 0$$

Example:

$$y''' - 6y'' + 12y' - 8y = 0$$

Char. eqn.: $(r-2)^3 = 0$

Char. roots: $r_1 = r_2 = r_3 = 2$

Fundamental set:

$$\left\{e^{2t},\ te^{2t},\ t^2e^{2t}\right\}$$

Corresponding system:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -12 & 6 \end{pmatrix} \mathbf{x}$$

Fundamental set:

$$\mathbf{x}_1 = e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{x}_2 = e^{2t} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + te^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},$$

$$\mathbf{x}_3 = e^{2t} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + te^{2t} \begin{pmatrix} 0 \\ 2 \\ 8 \end{pmatrix} + t^2 e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x}$$

given that $\lambda^3 - 3\lambda^2 + 4 = 0$ is the characteristic equation.

Eigenvectors

Examples

1. Find a fundamental set of solutions of

$$\mathbf{x}' = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} \mathbf{x}$$

given that $\lambda_1 = 10$, $\lambda_2 = \lambda_3 = 1$ are the eigenvalues of the coefficient matrix.

Eigenvectors

2. Find a fundamental set of solutions of

$$\mathbf{x}' = \begin{pmatrix} 1 & 5 \\ -4 & 5 \end{pmatrix} \mathbf{x}$$

3(a) Find a fundamental set of solutions of

$$y''' - 3y'' + 4y = 0 \qquad (*)$$

3(b) The linear differential system

equivalent to $(*)$ is

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 3 \end{pmatrix} \mathbf{x}$$

Find a fundamental set of solutions

of the system

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ -4 & 0 & 3-\lambda \end{vmatrix}$$

$$= \lambda^3 - 3\lambda^2 + 4$$

Eigenvalues:

$$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -2$$

Eigenvectors: