Chapter 12: VECTORS

1. Geometry: Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be points in 3-space:

A. **Distance Formula:** $d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

B. **Midpoint Formula:** The midpoint of the line segment joining P_1 and P_2 is the point

$$P\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

C. **Equation for the sphere of radius r and center $P(a, b, c)$:**

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

2. Vectors: A vector $\mathbf{a}$ in n-dimensional space is an ordered n-tuple of real numbers. In particular, a vector $\mathbf{a}$ in 3-space is an ordered triple of numbers: $\mathbf{a} = (a_1, a_2, a_3)$; a vector $\mathbf{a}$ in the plane (2-space) is an ordered pair of numbers: $\mathbf{a} = (a_1, a_2)$. The vector $\mathbf{0} = (0, 0, 0)$ is the *zero vector*

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors in 3-space and let α be a real number (scalar).

A. **Equality:** $\mathbf{a} = \mathbf{b}$ iff $a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$

B. **Vector Addition:** $\mathbf{a} + \mathbf{b} = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3)$

C. **Multiplication by a Scalar:** $\alpha\,\mathbf{a} = (\alpha\, a_1,\ \alpha\, a_2,\ \alpha\, a_3)$

NOTE: $\mathbf{a}$ and $\mathbf{b}$ are *parallel* iff $\mathbf{a} = \lambda\mathbf{b}$ for some number λ.

D. **Magnitude (Norm):** $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$; $\mathbf{a}$ is a *unit vector* if $\|\mathbf{a}\| = 1$. If $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{u_a} = \dfrac{\mathbf{a}}{\|\mathbf{a}\|}$ is a unit vector in the direction of $\mathbf{a}$.

(*i*) $\|\mathbf{a}\| \geq 0$; $\|\mathbf{a}\| = 0$ iff $\mathbf{a} = \mathbf{0}$.

(*ii*) $\|\alpha\mathbf{a}\| = |\,\alpha|\, \|\mathbf{a}\|$.

(*iii*) $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$.

NOTE: all of the above hold in n-dimensional space for any n.

E. **Unit Coordinate Vectors:** $\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0),\ \mathbf{k} = (0, 0, 1)$

F. **i**, **j**, *k*-**Representation:** $\mathbf{a} = (a_1, a_2, a_3) = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$.

3. Dot Product: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

The dot product of two n-dimensional vectors is defined similarly.

A. **Properties:**

(i) $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

(ii) $\mathbf{a} \cdot \mathbf{0} = 0$

(iii) $\mathbf{a} \cdot = \mathbf{b} \cdot \mathbf{a}$

(iv) $(\alpha \mathbf{a}) \cdot (\beta \mathbf{b}) = \alpha\beta\,(\mathbf{a} \cdot \mathbf{b})$

(v) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

B. **Geometric Interpretation:** $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \le \theta \le \pi$. $\mathbf{a}$ is *perpendicular* to $\mathbf{b}$ $(\mathbf{a} \perp \mathbf{b})$ iff $\mathbf{a} \cdot \mathbf{b} = 0$.

C. **Component & Projection of $\mathbf{a}$ on $\mathbf{b}$:**

$$\text{comp}_{\mathbf{b}}\, \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} = \mathbf{a} \cdot \mathbf{u_b} \qquad \text{proj}_{\mathbf{b}}\, \mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}\right) \frac{\mathbf{b}}{\|\mathbf{b}\|} = (\mathbf{a} \cdot \mathbf{u_b})\, \mathbf{u_b}$$

D. **Direction Angles; Direction Cosines:** $\mathbf{a} \cdot \mathbf{i} = \|\mathbf{a}\| \cos\alpha$, $\mathbf{a} \cdot \mathbf{j} = \|\mathbf{a}\| \cos\beta$, $\mathbf{a} \cdot \mathbf{k} = \|\mathbf{a}\| \cos\gamma$. The angles α, β, γ are called the *direction angles* of $\mathbf{a}$; $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the *direction cosines* of $\mathbf{a}$.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

4. Cross Product: This product is restricted to vectors in 3-space. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in 3-space such that $\mathbf{a} \neq \lambda \mathbf{b}$. The *cross product* of $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \times \mathbf{b}$, is the vector defined as follows:

(1) $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane determined by $\mathbf{a}$ and $\mathbf{b}$.

(2) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ (in this order) form a right-handed triple.

(3) $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

If $\mathbf{a} = \lambda \mathbf{b}$ for some number λ; that is, if $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

A. **Properties:**

(i) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

(ii) $(\alpha \mathbf{a}) \times (\beta \mathbf{b}) = \alpha\, \beta\, (\mathbf{a} \times \mathbf{b})$.

(iii) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

B. **Components of $\mathbf{a}\times\mathbf{b}$:** Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\,\mathbf{i} - (a_1b_3 - a_3b_1)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}$$

C. **Triple Scalar Product:** Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$. Then

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The volume of the parallelepiped having $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as sides is given by:

$$|\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\,|$$

5. Lines: There is one and only one line ℓ passing through a given point $P_0 : (x_0, y_0, z_0)$ parallel to a given vector $\mathbf{d} = (d_1, d_2, d_3)$. The vector $\mathbf{d}$ is called a *direction vector* for ℓ; the numbers d_1, d_2, d_3 are *direction numbers* for ℓ.

A. **Equations for ℓ:**

Vector Equation of ℓ:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{d} = (x_0 + td_1)\,\mathbf{i} + (y_0 + td_2)\,\mathbf{j} + (z_0 + td_3)\,\mathbf{k}$$

Scalar Parametric Equations of ℓ:

$$x(t) = x_0 + t\,d_1, \quad y(t) = y_0 + t\,d_2, \quad z(t) = z_0 + t\,d_3$$

Symmetric Equations of ℓ:

$$\frac{x - x_0}{d_1} = \frac{y - y_0}{d_2} = \frac{z - z_0}{d_3}$$

B. **Two Lines in Space:** Let ℓ and $\mathcal{L}$ be two lines in space, and let $\mathbf{d}$ and $\mathbf{D}$ be corresponding direction vectors. ℓ and $\mathcal{L}$ are either parallel or coincident if $\mathbf{d} = \lambda\mathbf{D}$; ℓ and $\mathcal{L}$ intersect in a point or are *skew* if $\mathbf{d} \neq \lambda\mathbf{D}$.

C. **Angle ϕ between ℓ and $\mathcal{L}$:** $\cos\phi = \dfrac{|\,\mathbf{d}\cdot\mathbf{D}\,|}{\|\mathbf{d}\|\,\|\mathbf{D}\|}$

D. Distance From a Point $P_1 : (x_1, y_1, z_1)$ to the Line ℓ:

$$d(P_1,\,\ell) = \frac{\|\overrightarrow{P_0P_1}\times\mathbf{d}\|}{\|\mathbf{d}\|}.$$

6. Planes: There is one and only one plane $\mathcal{P}$ passing through a given point $P_0 : (x_0, y_0, z_0)$ perpendicular to a given vector $\mathbf{N} = A\,\mathbf{i} + b\,\mathbf{j} + C\,\mathbf{k}$. The vector $\mathbf{N}$ is called a *normal vector* to the plane $\mathcal{P}$.

A. **Equations for** $\mathcal{P}$:

"Standard Form:" $\quad A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Removing the parentheses yields the equation

$$Ax + By + Cz + D = 0, \quad D = -Ax_0 - By_0 - Cz_0$$

which can also be written in the form

$$Ax + By + Cz = E.$$

B. **Two Planes in Space:** Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two planes in space, and let $\mathbf{N}_1$ and $\mathbf{N}_2$ be corresponding normal vectors. $\mathcal{P}_1$ and $\mathcal{P}_2$ are either parallel or coincident if $\mathbf{N}_1 = \lambda\mathbf{N}_2$; $\mathcal{P}_1$ and $\mathcal{P}_2$ intersect in a line if $\mathbf{N}_1 \neq \lambda\mathbf{N}_2$.

C. **Angle** θ **between** $\mathcal{P}_1$ **and** $\mathcal{P}_2$**:** $\quad \cos\theta = \dfrac{|\,\mathbf{N}_1 \cdot \mathbf{N}_2\,|}{\|\mathbf{N}_1\|\,\|\mathbf{N}_2\|}$

C. Distance From a Point $P_1 : (x_1, y_1, z_1)$ **to the Plane** $\mathcal{P}$:

$$d(P_1,\, \mathcal{P}) = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

NOTE: This formula requires the equation of the plane to be written in the form

$$Ax + By + Cz + D = 0.$$