Chapter 13: VECTOR CALCULUS

VECTOR FUNCTIONS: Let $f_1(t),\ f_2(t),\ f_3(t)$ be functions defined on some t-interval I. For each $t \in I$, form the vector

$$\mathbf{f}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}.$$

$\mathbf{f}$ is called a *vector-valued function* or, more simply, a *vector function*. Note: there is nothing special about 3 dimensions here; we can have vector functions in any number of coordinates. In particular, you will see many examples in two dimensions.

1. CALCULUS: Let $\mathbf{f}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$.

A. Limits: Let $c \in I$. Then

$$\lim_{t\to c} \mathbf{f}(t) = \mathbf{L} = L_1\,\mathbf{i} + L_2\,\mathbf{j} + L_3\,\mathbf{k} \quad \text{if and only if} \quad \lim_{t\to c} \|\mathbf{f}(t) - \mathbf{L}\| = 0$$
$$\text{if and only if} \quad \lim_{t\to c} f_1(t) = L_1, \quad \lim_{t\to c} f_2(t) = L_2, \quad \lim_{t\to c} f_3(t) = L_3$$

The usual limit theorems hold: limit of a sum, difference, etc.

B. Derivatives: $\mathbf{f}$ is *differentiable* at t if and only if

$$\lim_{h\to 0} \frac{1}{h}\,[f(t+h) - f(t)] \quad \text{exists.}$$

If the limit exists, it is called the *derivative of* $\mathbf{f}$ *at* t and is denoted $\mathbf{f}'(t)$. Because of the properties of limits indicated in (A),

$$\mathbf{f}'(t) = f_1'(t)\,\mathbf{i} + f_2'(t)\,\mathbf{j} + f_3'(t)\,\mathbf{k}.$$

C. Integrals:

$$\int \mathbf{f}(t)\,dt = \int f_1(t)\,dt\,\mathbf{i} + \int f_2(t)\,dt\,\mathbf{j} + \int f_1(t)\,dt\,\mathbf{k}$$

NOTE: In summary, limits derivatives and integrals are computed "component-wise."

2. GEOMETRY: Let $x = x(t),\ y = y(t),$ and $z = z(t)$ be differentiable functions on some t-interval I and let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. The tip of the vector $\mathbf{r}$ traces out a curve C in space. The equations:

$$x = x(t),\ y = y(t),\ z = z(t)$$

are *parametric equations* for C. NOTE: C is an *oriented curve* ; the "positive direction" on I induces a positive direction on C.

A. Tangent Vector: The vector

$$\mathbf{r}'(t) = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k},$$

if not $\mathbf{0}$, is a direction vector for the line tangent to C at the point $(x(t), y(t), z(t))$ on C; $\mathbf{r}'(t)$ points in the direction of increasing t.

B. Unit Tangent Vector: If $\mathbf{r}'(t) \neq \mathbf{0}$, then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

is the *unit tangent vector.*

C. Principal Normal Vector: If $\mathbf{T}'(t) \neq \mathbf{0}$, then

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

is the *principal normal vector;* $\mathbf{T} \perp \mathbf{N}$.

D. Osculating Plane: The plane determined by $\mathbf{T}$ and $\mathbf{N}$ is called the *osculating plane.* In particular, choose $c \in I$. Then $P\ (x(c), y(c), z(c))$ is a point on the curve C and $\mathbf{T}(c) \times \mathbf{N}(c)$ is a normal vector for the osculating plane at P. This is the information needed to write an equation for the osculating plane.

3. MECHANICS: Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ denote the position at time t of an object. As t ranges over the interval I, the object moves along the curve C.

A. Velocity Vector: The vector

$$\mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}$$

is called the *velocity vector* and

$$\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$$

is the *speed* of the object at time t.

B. Acceleration Vector: The vector

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j} + z''(t)\,\mathbf{k}$$

is called the *acceleration vector.*

4. ARC LENGTH/DISTANCE: Let $a, b \in I$, $a < b$. The length of the curve C for $a \leq t \leq b$ is given by:

$$L(C) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt = \int_a^b \|r'(t)\|\,dt$$

Distance: The distance $s(t)$ traveled by a particle moving along C over the time interval $[a, t]$ is given by:

$$s(t) = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du$$

The derivative of distance with respect to time, ds/dt, is

$$\frac{ds}{dt} = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \|\mathbf{r}'(t)\|,$$

see Section 5.2

and this is the speed of the object as noted above.

5. CURVATURE: The *curvature* of a curve is a measure of the rate at which the curve is "curving." The curvature of C is the magnitude of the change of the unit tangent vector with respect to arc length:

$$\kappa = \left\|\frac{\mathbf{T}}{ds}\right\| = \frac{\|\mathbf{T}/dt\|}{ds/dt} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

Special Cases:

(1) $C: \; y = f(x)$ the graph of a function:

$$\kappa = \frac{|y''(t)|}{(1 + [y']^2)^{3/2}}$$

(2) $C: \; x = x(t), y = y(t)$ a plane curve defined parametrically"

$$\kappa = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{([x']^2 + [y']^2)^{3/2}}$$

6. TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION: The acceleration vector

$$\mathbf{a}(t) = \mathbf{a}(t) = x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j} + z''(t)\,\mathbf{k}$$

is a linear combination of the unit tangent and principal normal vectors; i.e., the acceleration vector lies in the osculating plane:

$$\mathbf{a}(t) = a_T\,\mathbf{T} + a_N\,\mathbf{N}$$

The coefficients a_T and a_N are called the *tangential* and *normal* components of acceleration; a_T and a_N are given by:

$$a_T = \frac{d^2s}{dt^2}, \qquad a_N = \kappa\left(\frac{ds}{dt}\right)^2$$