

SUMMARY: CHAPTERS 16 and 17

CHAPTER 16 MULTIPLE INTEGRALS

I. DOUBLE INTEGRALS

- a. **Definition:** Let $f = f(x, y)$ be continuous on the rectangle $\mathcal{R} : a \leq x \leq b, c \leq y \leq d$. Let \mathcal{P} be a partition of \mathcal{R} and let m_{ij} and M_{ij} be the minimum and maximum values of f on the i, j sub-rectangle \mathcal{R}_{ij} . Then

(i) Lower sum:
$$L_f(\mathcal{P}) = \sum_{i=1}^n \sum_{j=1}^m m_{ij} \Delta x_i \Delta y_j.$$

(ii) Upper sum:
$$U_f(\mathcal{P}) = \sum_{i=1}^n \sum_{j=1}^m M_{ij} \Delta x_i \Delta y_j.$$

(iii) Riemann sum:
$$S_f(\mathcal{P}) = \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta x_i \Delta y_j$$
 where (x_i^*, y_j^*) is a point in \mathcal{R}_{ij} .

The *double integral of f over \mathcal{R}* is the unique number I that satisfies

$$L_f(\mathcal{P}) \leq I \leq U_f(\mathcal{P}) \quad \text{for all partitions } \mathcal{P}.$$

Notation:
$$I = \int \int_{\mathcal{R}} f(x, y) \, dx dy$$

Let Ω be an arbitrary closed bounded region in the plane. Then

$$\int \int_{\Omega} f(x, y) \, dx dy = \int \int_{\mathcal{R}} F(x, y) \, dx dy$$

where \mathcal{R} is a rectangle that contains Ω , and $F(x, y) = f(x, y)$ on Ω and $F(x, y) = 0$ on $\mathcal{R} - \Omega$.

- b. **Repeated Integrals:** If the region Ω is given by: $a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)$ (Type I region), then

$$\int \int_{\Omega} f(x, y) \, dx dy = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) \, dy dx$$

If the region Ω is given by: $c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)$ (Type II region), then

$$\int \int_{\Omega} f(x, y) \, dx dy = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) \, dx dy$$

- c. **Polar Coordinates:**
$$\int \int_{\Omega} f(x, y) \, dx dy = \int \int_{\Omega} f(r \cos \theta, r \sin \theta) r \, dr d\theta.$$

d. Applications:

- (i) **Volume:** If $f(x, y) \geq 0$ on Ω , then $V = \int \int_{\Omega} f(x, y) dx dy$ is the volume of the solid “cylinder” that has the surface $z = f(x, y)$ as its top, Ω as its base, and vertical sides.
- (ii) **Area:** $\int \int_{\Omega} 1 dx dy = \text{area of } \Omega$.
- (iii) **Mass of a Plate:** If the density of a “plate” at a point (x, y) in the closed, bounded region Ω is given by a continuous function $\lambda(x, y)$, then the mass of the plate is

$$M = \int \int_{\Omega} \lambda(x, y) dx dy$$

- (iv) **Center of Mass of a Plate:** Let the continuous function $\lambda(x, y)$ be the density function of a plate. Then the coordinates (x_M, y_M) of the center of mass of the plate are given by:

$$x_M = \frac{\int \int_{\Omega} x \lambda(x, y) dx dy}{M}, \quad y_M = \frac{\int \int_{\Omega} y \lambda(x, y) dx dy}{M}$$

where M is the mass of the plate.

II. TRIPLE INTEGRALS

- a. Definition:** Let $f = f(x, y, z)$ be continuous on the “box”

$$\mathcal{T} : a_1 \leq x \leq a_2, \quad b_1 \leq y \leq b_2, \quad c_1 \leq z \leq c_2.$$

Let \mathcal{P} be a partition of \mathcal{T} , and let m_{ijk} and M_{ijk} be the minimum and maximum values of f on the ijk sub-box \mathcal{R}_{ijk} . Then

- (i) Lower sum: $L_f(\mathcal{P}) = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l m_{ijk} \Delta x_i \Delta y_j \Delta z_k$.
- (ii) Upper sum: $U_f(\mathcal{P}) = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l M_{ijk} \Delta x_i \Delta y_j \Delta z_k$.
- (iii) Riemann sum: $S_f(\mathcal{P}) = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l f(x_i^*, y_j^*, z_k^*) \Delta x_i \Delta y_j \Delta z_k$ where (x_i^*, y_j^*, z_k^*) is a point in \mathcal{R}_{ijk} .

The *triple integral of f over \mathcal{T}* is the unique number I that satisfies

$$L_f(\mathcal{P}) \leq I \leq U_f(\mathcal{P}) \quad \text{for all partitions } \mathcal{P}.$$

Notation: $I = \int \int \int_{\mathcal{T}} f(x, y, z) dx dy dz$

Let Ω be an arbitrary closed bounded region in space. Then

$$\int \int \int_{\Omega} f(x, y, z) dx dy dz = \int \int \int_{\mathcal{R}} F(x, y, z) dx dy dz$$

where \mathcal{T} is a “box” that contains Ω , and $F(x, y, z) = f(x, y, z)$ on Ω and $F(x, y, z) = 0$ on $\mathcal{T} - \Omega$.

b. Repeated Integrals: If the region Ω is given by:

$$a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x), \psi_1(x, y) \leq z \leq \psi_2(x, y) \quad (\text{Type I region}),$$

then

$$\int \int \int_{\Omega} f(x, y, z) dx dy dz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x, y)}^{\psi_2(x, y)} f(x, y, z) dz dy dx$$

Note: There are *five* more types of special regions.

c. Cylindrical Coordinates:

$$\int \int \int_{\Omega} f(x, y, z) dx dy dz = \int \int \int_{\Omega} F(r, \theta, z) r dr d\theta dz.$$

d. Spherical Coordinates:

$$\int \int \int_{\Omega} f(x, y, z) dx dy dz = \int \int \int_{\Omega} F(\rho, \phi, \theta) \rho^2 \sin \phi d\rho d\phi d\theta.$$

CHAPTER 17. LINE INTEGRALS

Given a vector field

$$\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

and a smooth (or piecewise smooth) curve C :

$$C : x = x(u), y = y(u), \quad a \leq u \leq b \quad (\text{parametric form})$$

$$C : \mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j} \quad a \leq u \leq b \quad (\text{vector form})$$

Or, in three dimensions, a vector field

$$\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}$$

and a smooth (or piecewise smooth) curve C :

$$C : x = x(u), y = y(u), z = z(u), \quad a \leq u \leq b \quad (\text{parametric form})$$

$$C : \mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j} + z(u) \mathbf{k} \quad a \leq u \leq b \quad (\text{vector form})$$

DEFINITION: The *line integral* of F over C is the number given by

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}[\mathbf{r}(u)] \cdot \mathbf{r}'(u) du$$

Alternative notations:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C P(x, y) dx + Q(x, y) dy = \int_C \mathbf{F}_T ds \quad (2 \text{ dimensions})$$

or

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz = \int_C \mathbf{F}_T ds \quad (3 \text{ dimensions})$$

where \mathbf{F}_T is the component of \mathbf{F} on the unit tangent vector T .

THEOREM: Line integrals are invariant under orientation-preserving changes of parameter.

THEOREM: Reversing the orientation of C changes the sign of the integral:

$$\int_{-C} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = - \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}.$$

c.f. $\int_b^a f(x) dx = - \int_a^b f(x) dx.$

FUNDAMENTAL THEOREM OF LINE INTEGRALS:

Given a curve $C : \mathbf{r}(u)$, $a \leq u \leq b$ and a vector field \mathbf{F} . If $\mathbf{F} = \nabla f$ for some function $f(x, y)$, then

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = f(B) - f(A)$$

where $A = \mathbf{r}(a)$ and $B = \mathbf{r}(b)$.

DEFINITION: The curve C is *closed* if $\mathbf{r}(a) = \mathbf{r}(b)$.

COROLLARY 1. If \mathbf{F} is the gradient of some function f and the curve C is closed, then

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$$

COROLLARY 2. (Independence of Path) If \mathbf{F} is the gradient of some function f and if C_1 and C_2 are any two curves which begin at \mathbf{A} and end at \mathbf{B} , then

$$\int_{C_1} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

DEFINITION: The curve C is a *simple closed curve* if it is closed and $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all $a < t_1 < t_2 < b$. The positive direction on C is counterclockwise. A simple closed curve is also called a *Jordan curve*. The region enclosed by a simple closed curve is called a *Jordan region*.

GREEN'S THEOREM: Given a simple closed curve C oriented in the counterclockwise direction, and a vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$.

$$\oint_C P(x, y) dx + Q(x, y) dy = \int \int_{\Omega} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx dy$$

where Ω is the Jordan region enclosed by C .

COROLLARY TO GREEN'S THEOREM If C is a simple closed curve, and if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$; that is, if $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a gradient, then,

$$\oint_C P(x, y) dx + Q(x, y) dy = 0.$$

AREA OF Ω USING GREEN'S THEOREM: C a simple closed curve enclosing the region Ω :

$$\text{Area of } \Omega = \int \int_{\Omega} 1 dx dy = \oint_C -y dx = \oint_C x dy = \frac{1}{2} \oint_C -y dx + x dy.$$