CHAPTER 4

The Laplace Transform

4.1 Introduction

The Laplace transform provides an effective method of solving initial-value problems for linear differential equations with constant coefficients. However, the usefulness of Laplace transforms is by no means restricted to this class of problems. Some understanding of the basic theory is an essential part of the mathematical background of engineers, scientists and mathematicians.

The Laplace transform is defined in terms of an integral over the interval $[0, \infty)$. Integrals over an infinite interval are called *improper integrals,* a topic studied in Calculus II.

DEFINITION Let f be a continuous function on $[0, \infty)$. The Laplace transform of f, denoted by $\mathcal{L}[(f(x)]$, or by $F(s)$, is the function given by

$$\mathcal{L}[f(x)] = F(s) = \int_0^\infty e^{-sx} f(x)\, dx. \tag{1}$$

The domain of F is the set of all real numbers s for which the improper integral converges.

In more advanced treatments of the Laplace transform the parameter s assumes complex values, but the restriction to real values is sufficient for our purposes here. Note that $\mathcal{L}$ transforms a function $f = f(x)$ into a function $F = F(s)$ of the parameter s. The continuity assumption on f will hold throughout the first three sections. It is made for convenience in presenting the basic properties of $\mathcal{L}$ and for applying the Laplace transform method to solving initial-value problems. In the last two sections of this chapter we extend the definition of $\mathcal{L}$ to a larger class of functions, the piecewise continuous functions on $[0, \infty)$. There we will apply $\mathcal{L}$ to the problem of solving nonhomogeneous equations in which the nonhomogeneous term is piecewise continuous. This will involve some extension of our concepts of differential equation and solution.

As indicated above, the primary application of Laplace transforms of interest to us is solving linear differential equations with constant coefficients. Referring to our work in Chapter 3, the functions which arise naturally in the treatment of these equations are:

$$p(x)e^{rx}, \quad p(x)\cos\beta x, \quad p(x)\sin\beta x, \quad p(x)e^{rx}\cos\beta x, \quad p(x)e^{rx}\sin\beta x$$

where p is a polynomial.

We begin by calculating the Laplace transforms of some simple cases of these functions.

Example 1. Let $f(x) = 1 \cdot e^{0x} \equiv 1$ on $[0, \infty)$. By the Definition,

$$\begin{aligned}\mathcal{L}[1] &= \int_0^\infty e^{-sx} \cdot 1\,dx = \lim_{b\to\infty} \int_0^b e^{-sx}\,dx \\ &= \lim_{b\to\infty}\left[\left.\frac{e^{-sx}}{-s}\right|_0^b\right] = \lim_{b\to\infty}\left[\frac{e^{-sb}}{-s}\right] + \frac{1}{s} = \lim_{b\to\infty}\left[\frac{-1}{se^{sb}}\right] + \frac{1}{s}.\end{aligned}$$

Now, $\lim_{b\to\infty} -1/se^{sb}$ exists if and only if $s > 0$, and in this case

$$\lim_{b\to\infty} \frac{-1}{se^{sb}} = 0.$$

Thus,

$$\mathcal{L}[1] = \frac{1}{s}, \quad s > 0. \qquad \blacksquare$$

Example 2. Let $f(x) = e^{rx}$ on $[0, \infty)$. Then,

$$\begin{aligned}\mathcal{L}[e^{rx}] &= \int_0^\infty e^{-sx} \cdot e^{rx}\,dx = \lim_{b\to\infty} \int_0^b e^{-(s-r)x}\,dx \\ &= \lim_{b\to\infty}\left[\left.\frac{e^{-(s-r)x}}{-(s-r)}\right|_0^b\right] = \lim_{b\to\infty}\left[\frac{e^{-(s-r)b}}{-(s-r)}\right] + \frac{1}{s-r}.\end{aligned}$$

The limit exists (and has the value 0) if and only if $s - r > 0$. Therefore

$$\mathcal{L}[e^{rx}] = \frac{1}{s-r}, \quad s > r.$$

Note that if $r = 0$, then we have the result in Example 1. $\blacksquare$

Example 3. Let $f(x) = \cos\beta x$ on $[0, \infty)$. Then,

$$\begin{aligned}\mathcal{L}[\cos\beta x] &= \int_0^\infty e^{-sx} \cdot \cos\beta x\,dx = \lim_{b\to\infty} \int_0^b e^{-sx} \cos\beta x\,dx \\ &= \lim_{b\to\infty} \left.\frac{e^{-sx}[-s\cos\beta x - \beta\sin\beta x]}{s^2+\beta^2}\right|_0^b.\end{aligned}$$

(Note the integral was calculated using integration by parts; also, it is a standard entry in a table of integrals.)

Now,

$$\mathcal{L}[\cos\beta x] = -\left[\lim_{b\to\infty} \frac{1}{e^{-sb}} \cdot \frac{s\cos\beta b + \beta\sin\beta b}{s^2+\beta^2} + \frac{s}{s^2+\beta^2}\right].$$

Since $[s\cos\beta b + \beta\sin\beta b]/(s^2+\beta^2)$ is bounded, the limit exists (and has the value 0) if and only if $s > 0$. Therefore,

$$\mathcal{L}[\cos\beta x] = \frac{s}{s^2+\beta^2}, \quad s > 0. \qquad \blacksquare$$

The following table gives a basic list of the Laplace transforms of functions that we will encounter in this chapter. While the entries in the table can be verified using the Definition, some of the integrations involved are complicated. The properties of the Laplace transform presented in the next section provide a more efficient way to obtain many of the entries in the table. Handbooks of mathematical functions, for example the CRC Standard Mathematical Tables, give extensive tables of Laplace transforms.

Table of Laplace Transforms

$f(x)$	$F(s) = \mathcal{L}[f(x)]$
1	$\dfrac{1}{s}, \quad s > 0$
$e^{\alpha x}$	$\dfrac{1}{s-\alpha}, \quad s > \alpha$
$\cos \beta x$	$\dfrac{s}{s^2+\beta^2}, \quad s > 0$
$\sin \beta x$	$\dfrac{\beta}{s^2+\beta^2}, \quad s > 0$
$e^{\alpha x} \cos \beta x$	$\dfrac{s-\alpha}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$e^{\alpha x} \sin \beta x$	$\dfrac{\beta}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$x^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$x^n\, e^{rx}, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-r)^{n+1}}, \quad s > r$
$x \cos \beta x$	$\dfrac{s^2-\beta^2}{(s^2+\beta^2)^2}, \quad s > 0$
$x \sin \beta x$	$\dfrac{2\beta s}{(s^2+\beta^2)^2}, \quad s > 0$

Exercises 4.1

Use the definition of the Laplace transform to find the Laplace transform of the given function.

1. $f(x) = x$.
2. $f(x) = x^2$.

3. $f(x) = \sin x$.

4. $f(x) = xe^{rx}$.

5. $f(x) = \sinh x$.

6. $f(x) = \cosh x$.

7. Use the fact that $\displaystyle\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$ to find $\mathcal{L}[e^{rx} \cos \beta x]$ by the definition.

8. Use the fact that $\displaystyle\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$ to find $\mathcal{L}[e^{rx} \sin \beta x]$ by the definition.

9. Show that $\mathcal{L}[\sin x] = \dfrac{\displaystyle\int_0^{2\pi} e^{-sx} \sin x \, dx}{1 - e^{-2\pi s}}$.

10. Let f be a continuous function on $[0, \infty)$ and suppose that f is periodic with period p. That is f is continuous and $f(x+p) = f(x)$, $p > 0$, for all x. Show that

$$\mathcal{L}[f(x)] = \frac{\displaystyle\int_0^p e^{-sx} f(x) \, dx}{1 - e^{-ps}}.$$

4.2 Basic Properties of the Laplace Transform

In the preceding section we defined the Laplace transform and calculated the Laplace transforms of some of the functions that occur in solving linear differential equations with constant coefficients. In this section we consider the basic question of the existence of the Laplace transform of a function f, and we develop the properties of the Laplace transform that will be used in solving initial value problems.

To motivate the material in this section, consider the differential equation

$$y'' + ay' + by = f(x) \tag{2}$$

where a and b are constants and f is a continuous function on $[0, \infty)$. If we assume that $y = y(x)$ is a solution of (1) and formally apply $\mathcal{L}$, we obtain

$$\mathcal{L}\left[y''(x) + ay'(x) + by(x)\right] = \mathcal{L}[f(x)]. \tag{3}$$

The right-hand side of this equation suggests the basic question of the existence of $\mathcal{L}[f(x)]$. That is, for what functions f does $\mathcal{L}[f]$ exist?

DEFINITION A function f, continuous on $[0, \infty)$, is said to be of *exponential order* λ, λ a real number, if there exists a positive number M and a nonnegative number A such that

$$|f(x)| \leq Me^{\lambda x}$$

on $[A, \infty)$.

Example 1. (a) If f is a bounded function on $[0, \infty)$ [for example, $f(x) = \cos \beta x$ or $f(x) = \sin \beta x$], then f is of exponential order 0.

f bounded implies that there exists a positive number M such that $|f(x)| \leq M$ for all $x \in [0, \infty)$. Therefore,

$$|f(x)| \leq M = Me^{0x} \quad \text{on} \quad [0, \infty).$$

[Note: if $f(x) = \cos \beta x$ or $f(x) = \sin \beta x$, then we could take $M = 1$.]

(b) Let $f(x) = x$ on $[0, \infty)$. For any positive number λ,

$$\lim_{x \to \infty} \frac{x}{e^{\lambda x}} = 0$$

by L'Hôpital's rule. Therefore, there exists a nonnegative number A such that

$$\frac{x}{e^{\lambda x}} \leq 1 \quad \text{on} \quad [A, \infty).$$

This implies that

$$x \leq e^{\lambda x} = 1 \cdot e^{\lambda x} \quad \text{on} \quad [A, \infty)$$

and $f(x) = x$ is of exponential order λ. The same argument can be used to show that $f(x) = x^{\alpha}$, α any real number, is of exponential order λ for any positive number λ. In general, if $p = p(x)$ is a polynomial, then p is of exponential order λ for any positive number λ.

(c) If $f(x) = e^{rx}$, then f is of exponential order λ for any $\lambda \geq r$.

(d) Consider the function $f(x) = e^{x^2}$. If f is of exponential order λ for some λ, then there exists a positive number M and a nonnegative number A such that

$$e^{x^2} \leq Me^{\lambda x} \quad \text{on} \quad [A, \infty) \qquad \text{which implies} \quad e^{-\lambda x}e^{x^2} \leq M \quad \text{on} \quad [A, \infty).$$

But,

$$\lim_{x\to\infty} e^{-\lambda x}e^{x^2} = \lim_{x\to\infty} e^{x(x-\lambda)} = \infty,$$

a contradiction. Thus $f(x) = e^{x^2}$ is not of exponential order λ for any positive number λ. ■

Our first property of $\mathcal{L}$ is a sufficient condition for $\mathcal{L}[f(x)]$ to exist. We shall omit the proof.

THEOREM 1. Let f be a continuous function on $[0, \infty)$. If f is of exponential order λ, then the Laplace transform $\mathcal{L}[f(x)] = F(s)$ exists for $s > \lambda$.

We now turn to the left-hand side of equation (2) where we have $\mathcal{L}$ applied to the linear combination $y''(x) + ay'(x) + by(x)$.

THEOREM 2. The operator $\mathcal{L}$ is a linear operator. That is, if g and h are continuous functions on $[0, \infty)$, and if each of $\mathcal{L}[g(x)]$ and $\mathcal{L}[h(x)]$ exists for $s > \lambda$, then $\mathcal{L}[g(x) + h(x)]$ and $\mathcal{L}[c\,g(x)]$, c constant, each exist for $s > \lambda$, and

$$\begin{aligned} \mathcal{L}[g(x) + h(x)] &= \mathcal{L}[g(x)] + \mathcal{L}[h(x)] \\ \mathcal{L}[c\,g(x)] &= c\mathcal{L}[g(x)]. \end{aligned}$$

Proof:

$$\begin{aligned} \mathcal{L}[g(x) + h(x)] = \int_0^\infty e^{-sx}[g(x) + h(x)]\,dx &= \lim_{b\to\infty} \int_0^b e^{-sx}[g(x) + h(x)]\,dx \\ &= \lim_{b\to\infty} \left[\int_0^b e^{-sx}g(x)\,dx + \int_0^b e^{-sx}h(x)\,dx\right] \\ &= \lim_{b\to\infty} \int_0^b e^{-sx}g(x)\,dx + \lim_{b\to\infty} \int_0^b e^{-sx}h(x)\,dx \\ &= \int_0^\infty e^{-sx}g(x)\,dx + \int_0^\infty e^{-sx}h(x)\,dx \\ &= \mathcal{L}[g(x)] + \mathcal{L}[h(x)] \end{aligned}$$

The proof that

$$\mathcal{L}[c\,g(x)] = c\mathcal{L}[g(x)] \quad \text{for any constant } c$$

is left as an exercise. ■

COROLLARY Let $g_1(x),\ g_2(x),\ \ldots,\ g_n(x)$ be continuous functions on $[0,\infty)$. If $\mathcal{L}[g_1(x)],\ \mathcal{L}[g_2(x)],\ \ldots,\mathcal{L}[g_n(x)]$ all exist for $s>\lambda$, and if $c_1,\ c_2,\ \ldots,\ c_n$ are real numbers, then

$$\mathcal{L}\left[c_1g_1(x)+c_2g_2(x)+\cdots+c_ng_n(x)\right]$$

exists for $s>\lambda$ and

$$\mathcal{L}\left[c_1g_1(x)+c_2g_2(x)+\cdots+c_ng_n(x)\right]=c_1\mathcal{L}[g_1(x)]+c_2\mathcal{L}[g_2(x)]+\cdots+c_n\mathcal{L}[g_n(x)].$$

According to the Corollary, if $\mathcal{L}[y'']$, $\mathcal{L}[y']$, and $\mathcal{L}[y]$ all exist for $s>\lambda$ for some λ, then

$$\mathcal{L}[y''+ay'+by]=\mathcal{L}[y'']+a\mathcal{L}[y']+b\mathcal{L}[y].$$

The next property gives a relationship between the Laplace transform of the derivative of a function and the Laplace transform of the function itself.

THEOREM 3. Let g be a continuously differentiable function on $[0,\infty)$. If g is of exponential order λ, then $\mathcal{L}[g'(x)]$ exists for $s>\lambda$ and

$$\mathcal{L}[g'(x)]=s\mathcal{L}[g(x)]-g(0).$$

Proof: By the definition of $\mathcal{L}$, we have

$$\mathcal{L}[g'(x)]=\int_0^\infty e^{-sx}g'(x)\,dx=\lim_{b\to\infty}\int_0^b e^{-sx}g'(x)\,dx.$$

Now, using integration by parts,

$$\begin{aligned}\int_0^b e^{-sx}g'(x)\,dx &= e^{-sx}g(x)\Big|_0^b+s\int_0^b e^{-sx}g(x)\,dx\\ &= e^{-sb}g(b)-g(0)+s\int_0^b e^{-sx}g(x)\,dx\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}[g'(x)] &= \lim_{b\to\infty}\left[e^{-sb}g(b)-g(0)+s\int_0^b e^{-sx}g(x)\,dx\right]\\ &= \lim_{b\to\infty}e^{-sb}g(b)-g(0)+s\lim_{b\to\infty}\int_0^b e^{-sx}g(x)\,dx\\ &= \lim_{b\to\infty}e^{-sb}g(b)-g(0)+s\mathcal{L}[g(x)]\end{aligned}$$

provided $\lim_{b\to\infty}e^{-sb}g(b)$ exists.

Since g is of exponential order λ, there exist constants M and A such that

$$|\,g(x)|\le Me^{\lambda x}$$

on $[A, \infty)$. Therefore,

$$\left| e^{-sb} g(b) \right| \leq \left| e^{-sb} M e^{\lambda b} \right| = M e^{-(s-\lambda)b}$$

for all $b > A$, and it follows that

$$\lim_{b \to \infty} e^{-sb} g(b) = 0 \quad \text{for} \quad s > \lambda.$$

We can now conclude that

$$\mathcal{L}[g'(x)] = s\mathcal{L}[g(x)] - g(0). \qquad \blacksquare$$

COROLLARY Let g be a function which is n-times continuously differentiable on $[0, \infty)$. If each of the functions $g, g', \ldots, g^{(n-1)}$ is of exponential order λ, then $\mathcal{L}[g^{(n)}]$ exists for $s > \lambda$ and

$$\mathcal{L}[g^{(n)}(x)] = s^n \mathcal{L}[g(x)] - s^{n-1} g(0) - s^{n-2} g'(0) - \cdots - g^{(n-1)}(0).$$

In particular, if g'' is continuous on $[0, \infty)$, and if g and g' are of exponential order λ, then $\mathcal{L}[g''(x)]$ exists for $s > \lambda$ and

$$\mathcal{L}[g''(x)] = s^2 \mathcal{L}[g(x)] - sg(0) - g'(0).$$

The proof of the Corollary follows by induction.

Application to Initial-Value Problems

Consider the first order initial-value problem

$$y' + ay = f(x); \quad y(0) = \alpha$$

where α is a real number and the function $f = f(x)$ has Laplace transform $F = F(s)$.

Let $y = y(x)$ be the solution of the problem. Applying the properties of $\mathcal{L}$ established above, we have

$$\begin{aligned}
\mathcal{L}\left[y'(x) + ay(x)\right] &= \mathcal{L}[f(x)] \\
\mathcal{L}[y'(x)] + a\mathcal{L}[y(x)] &= F(s) \quad \text{(by linearity)} \\
s\mathcal{L}[y(x)] - y(0) + a\,\mathcal{L}[y(x)] &= F(s) \quad \text{(by Theorem 3)} \\
(s+a)\mathcal{L}[y(x)] - y(0) &= F(s)
\end{aligned}$$

Applying the initial condition $y(0) = \alpha$, and solving for $\mathcal{L}[y(x)] = Y(s)$, we get

$$Y(s) = \frac{F(s) + \alpha}{s + a}.$$

Next consider the second order initial-value problem

$$y'' + ay' + by = f(x); \quad y(0) = \alpha,\ y'(0) = \beta.$$

where α and β are real numbers, and f is a function with Laplace transform $\mathcal{L}[f(x)] = F(s)$.

Let $y = y(x)$ be the solution of this problem. Then

$$\begin{aligned}
\mathcal{L}\left[y''(x) + ay'(x) + by(x)\right] &= \mathcal{L}[f(x)] \quad \text{(equation (2))}\\
\mathcal{L}[y''(x)] + a\mathcal{L}[y'(x)] + b\mathcal{L}[y(x)] &= F(s) \quad \text{(by linearity)}\\
s^2\mathcal{L}[y(x)] - sy(0) - y'(0) + a\left\{s\mathcal{L}[y(x)] - y(0)\right\} + b\mathcal{L}[y(x)] &= F(s) \quad \text{(by the Corollary to Theorem 3)}\\
(s^2 + as + b)\mathcal{L}[y(x)] - sy(0) - y'(0) - ay(0) &= F(s)
\end{aligned}$$

Applying the initial conditions $y(0) = \alpha,\ y'(0) = \beta$ and solving for $\mathcal{L}[y(x)] = Y(s)$, we get

$$Y(s) = \frac{F(s) + \alpha s + \beta + a\alpha}{s^2 + as + b}.$$

Implicit in these derivations is the assumption that the Laplace transform of y and its derivatives exist. Assuming that this is the case, the importance of these results is that it gives us the Laplace transform of the solution of an initial-value problem directly. The question now is: Knowing $Y(s)$, what is $y(x)$? This is the topic of the next section, "inverting" the Laplace transform.

Example 2. Find the Laplace transform $\mathcal{L}[y(x)] = Y(s)$ of the solution of the initial-value problem

$$y' - 2y = 2e^{-3x}; \quad y(0) = -2.$$

SOLUTION If $y = y(x)$ is the solution, then

$$\begin{aligned}
\mathcal{L}[y'(x) - 2y(x)] &= \mathcal{L}[2e^{-3x}] = 2\mathcal{L}[e^{-3x}] = \frac{2}{s+3}\\
\mathcal{L}[y'(x)] - 2\mathcal{L}[y(x)] &= \frac{2}{s+3}\\
s\mathcal{L}[y(x)] - y(0) - 2\mathcal{L}[y(x)] &= \frac{2}{s+3}\\
(s-2)\mathcal{L}[y(x)] + 2 &= \frac{2}{s+3}\\
(s-2)\mathcal{L}[y(x)] &= \frac{2}{s+3} - 2
\end{aligned}$$

Therefore,

$$\mathcal{L}[y(x)] = Y(s) = \frac{2}{(s-2)(s+3)} - \frac{2}{s-2}. \qquad \blacksquare$$

Example 3. Find the Laplace transform $\mathcal{L}[y(x)] = Y(s)$ of the solution of the initial-value problem

$$y'' + 4y = 3e^{2x}; \quad y(0) = 5,\ y'(0) = -2.$$

SOLUTION If $y = y(x)$ is the solution, then

$$\begin{aligned}
\mathcal{L}[y''(x) + 4y(x)] &= \mathcal{L}[3e^{2x}] = 3\mathcal{L}[e^{2x}] = \frac{3}{s-2} \\
\mathcal{L}[y''(x)] + 4\mathcal{L}[y(x)] &= \frac{3}{s-2} \\
s^2\mathcal{L}[y(x)] - sy(0) - y'(0) + 4\mathcal{L}[y(x)] &= \frac{3}{s-2} \\
(s^2+4)\mathcal{L}[y(x)] - 5s - (-2) &= \frac{3}{s-2} \\
(s^2+4)\mathcal{L}[y(x)] &= \frac{3}{s-2} + 5s - 2
\end{aligned}$$

Therefore,

$$\mathcal{L}[y(x)] = Y(s) = \frac{3}{(s-2)(s^2+4)} + \frac{5s-2}{s^2+4}. \qquad \blacksquare$$

In the next section we will see how to go from $Y(s)$ to $y(x)$.

Additional Applications and Properties

Although the main use of Theorem 3 and its Corollary is in solving initial-value problems, the results can also be used to determine entries in a table of Laplace transforms. For example, if f is a continuously differentiable function and $\mathcal{L}[f(x)]$ is known, then $\mathcal{L}[f'(x)]$ can be determined: $\mathcal{L}[f'(x)] = s\mathcal{L}[f(x)] - f(0)$.

Example 4. In Example 3, Section 4.1, we showed that

$$\mathcal{L}[\cos \beta x] = \frac{s}{s^2+\beta^2}.$$

We could use essentially the same calculations to obtain $\mathcal{L}[\sin \beta x]$, but recall that the integrations involved are a little "messy."

Here is a simpler way to obtain $\mathcal{L}[\sin \beta x]$. Since $[\cos \beta x]' = -\beta \sin \beta x$, we can write

$$\begin{aligned}
\mathcal{L}[-\beta \sin \beta x] &= \mathcal{L}[(\cos \beta x)'] = s\mathcal{L}[\cos \beta x] - \cos(0) = \frac{s^2}{s^2+\beta^2} - 1 \\
-\beta\mathcal{L}[\sin \beta x] &= \frac{-\beta^2}{s^2+\beta^2}.
\end{aligned}$$

Therefore,

$$\mathcal{L}[\sin \beta x] = \frac{\beta}{s^2+\beta^2}. \qquad \blacksquare$$

Here are two more properties of the Laplace transform that are useful in determining entries in a table of transforms.

THEOREM 4. Let f be a continuous function on $[0,\infty)$, and be of exponential order λ. Then $F(s)=\mathcal{L}[f(x)]$ has derivatives of all orders, and for $s>\lambda$,

$$\frac{d^nF(s)}{ds^n}=(-1)^n\mathcal{L}\left[x^nf(x)\right],\quad n=1,\ 2,\ \ldots\ .$$

Justification By the definition of $\mathcal{L}[f(x)]=F(s)$, we have

$$F(s)=\int_0^\infty e^{-sx}f(x)\,dx.$$

Differentiation of this equation with respect to s can be justified and yields

$$\begin{aligned}\frac{dF}{ds}&=\int_0^\infty\frac{d}{ds}[e^{-sx}f(x)]\,dx=\int_0^\infty e^{-sx}[-xf(x)]\,dx\\&=\mathcal{L}[-xf(x)]=-\mathcal{L}[xf(x)]\end{aligned}$$

Therefore, $\mathcal{L}[xf(x)]=-\dfrac{dF}{ds}$.

Differentiating a second time, we have

$$\begin{aligned}\frac{d^2F}{ds^2}=\frac{d}{ds}\left[\frac{dF}{ds}\right]&=\int_0^\infty\frac{d}{ds}\left(e^{-sx}[-xf(x)]\right)\,dx\\&=\int_0^\infty e^{-sx}[x^2f(x)]\,dx=\mathcal{L}[x^2f(x)]\end{aligned}$$

The result stated in the theorem follows by mathematical induction. ■

Example 5. From the table at the end of Section 4.1, $\mathcal{L}[e^{rx}]=\dfrac{1}{s-r}$, $s>r$. Therefore,

$$\mathcal{L}[xe^{rx}]=-F'(s)=-\frac{d}{ds}\left(\frac{1}{s-r}\right)=\frac{1}{(s-r)^2},\ s>r,$$

$$\mathcal{L}[x^2e^{rx}]=F''(s)=\frac{d^2}{ds^2}\left(\frac{1}{s-r}\right)=\frac{2}{(s-r)^3},\ s>r,$$

$$\mathcal{L}[x^3e^{rx}]=-F'''(s)=-\frac{d^3}{ds^3}\left(\frac{1}{s-r}\right)=\frac{6}{(s-r)^4},\ s>r,$$

and so on. ■

The final property we'll consider is called the *translation property* of $\mathcal{L}$.

THEOREM 5. If f is a continuous function on $[0,\infty)$, and if $\mathcal{L}[f(x)]=F(s)$ exists for $s>\lambda$, then for any real number r,

$$\mathcal{L}[e^{rx}f(x)]=F(s-r)\quad\text{for}\quad s>\lambda+r.$$

Proof: From the definition of the Laplace transform,

$$\begin{aligned} F(s-r) &= \int_0^\infty e^{-(s-r)x} f(x)\,dx = \int_0^\infty e^{-sx} e^{rx} f(x)\,dx \\ &= \mathcal{L}[e^{rx} f(x)]. \qquad \blacksquare \end{aligned}$$

Example 6.

(a) Since $\mathcal{L}[x] = 1/s^2,\ s > 0,$ we have

$$\mathcal{L}[xe^{rx}] = F(s-r) = \frac{1}{(s-r)^2},\ s > r \qquad \text{(c.f. Example 5)}$$

(b) Since $\mathcal{L}[\cos\beta x] = \dfrac{s}{s^2+\beta^2},\ s > 0,$ we have

$$\mathcal{L}[e^{rx}\cos\beta x] = F(s-r) = \frac{s-r}{(s-r)^2+\beta^2},\ s > r \qquad \text{(as indicated in the Table, Section 4.1).} \qquad \blacksquare$$

Summary of the Properties of $\mathcal{L}$

1. If f is continuous on $[0,\infty)$, and of exponential order λ, then $\mathcal{L}[f(x)] = F(s)$ exists for $s > \lambda$. (Theorem 1)

2. $\mathcal{L}$ is a linear operator (Theorem 2): $\mathcal{L}[f+g] = \mathcal{L}[f] + \mathcal{L}[g]$; $\mathcal{L}[cf] = c\mathcal{L}[f]$.

3. If f is continuously differentiable on $[0,\infty)$, and of exponential order λ, then $\mathcal{L}[f'(x)]$ exists for $s > \lambda$ and

   $$\mathcal{L}[f'(x)] = s\mathcal{L}[f(x)] - f(0).$$

   If f is twice continuously differentiable on $[0,\infty)$, and of exponential order λ, then $\mathcal{L}[f''(x)]$ exists for $s > \lambda$ and

   $$\mathcal{L}[f''(x)] = s^2\mathcal{L}[f(x)] - sf(0) - f'(0).$$

   And so on. (Theorem 3)

4. If f is continuous on $[0,\infty)$, and of exponential order λ, then $F(s) = \mathcal{L}[f(x)]$ has derivatives of all orders, and for $s > \lambda$,

   $$\frac{d^nF(s)}{ds^n} = (-1)^n \mathcal{L}\left[x^n f(x)\right], \quad n = 1,\ 2,\ \ldots\ . \text{ (Theorem 4)}$$

5. If f is continuous on $[0,\infty)$, and if $\mathcal{L}[f(x)] = F(s)$ exists for $s > \lambda$, then for any real number r,

   $$\mathcal{L}[e^{rx}f(x)] = F(s-r) \quad \text{for } s > \lambda + r. \text{ (Theorem 5)}$$

Exercises 4.2

Use the properties of the Laplace transform and the Table to find $\mathcal{L}[f]$.

1. $f(x) = 3 - 2x + x^2$.
2. $f(x) = 2e^{-x} - 3\sin 4x$.
3. $f(x) = 3 + 4e^{3x} - 2\cos 2x$.
4. $f(x) = 2xe^{-2x} + 5e^{2x}\cos 3x$.
5. $f(x) = 5x^2 - 2e^{-3x}\sin 2x$.
6. $f(x) = 2 - 3x + 4x^2e^{2x}$.
7. $f(x) = x\sin x + 2x\cos 2x$.
8. Show that $\mathcal{L}[\cosh \beta x] = \dfrac{s}{s^2 - \beta^2}$.
9. Show that $\mathcal{L}[\sinh \beta x] = \dfrac{\beta}{s^2 - \beta^2}$.
10. Find $\mathcal{L}[3\cosh 2x - 5\sinh 3x]$.
11. Find $\mathcal{L}[e^{2x}\sinh x + e^{-x}\cosh 3x]$.
12. Find $\mathcal{L}[x\cosh 2x]$.
13. Show that $\mathcal{L}[xe^{rx}] = \dfrac{1}{(s-r)^2}$ by:
    (a) Using Property 4.
    (b) Using Property 5.
14. Use Property 4 to show that $\mathcal{L}[x\sin \beta x] = \dfrac{2\beta s}{(s^2 + \beta^2)^2}$.

    Find the Laplace transform $Y(s)$ of the solution of the given initial-value problem.

15. $y' - 2y = 0; \quad y(0 = 1$.
16. $y' - 2y = x; \quad y(0) = 1$.
17. $y' + 4y = 2e^{2x} - 3\sin 3x; \quad y(0) = -3$.
18. $y'' + 2y' - 8y = 0; \quad y(0) = 4,\ y'(0) = -2$.
19. $y'' + 6y' + 9y = 0; \quad y(0) = 0,\ y'(0) = 2$.
20. $y'' - 2y' + 5y = 2x + e^{-x}; \quad y(0) = -2,\ y'(0) = 0$.

21. $y'' - 2y' - 15y = 3 + 4e^{-3x}; \quad y(0) = 1,\ y'(0) = -3.$

22. $y'' - 4y' + 4y = 5e^{2x}; \quad y(0) = -3, y'(0) = 2.$

Here is another property of the Laplace transform.

23. Let f be a continuous function on $[0, \infty)$ and assume that both f and $\int_0^x f(t)\,dt$ are of exponential order λ on $[0, \infty)$. Show that if $F(s) = \mathcal{L}[f(x)]$, then

$$\mathcal{L}\left[\int_0^x f(t)\,dt\right] = \frac{1}{s}F(s).$$

24. Find $\mathcal{L}\left[\int_0^x \sin 2t\,dt\right]$ by using:

(a) The property given in Exercise 23.

(b) By first calculating the integral and then taking the Laplace transform of the result.

4.3 Inverse Laplace Transforms and Initial-Value Problems

In Section 4.2 we saw that the Laplace transform of the solution $y = y(x)$ of the initial-value problem

$$y'' + ay' + by = f(x); \quad y(0) = \alpha,\ y'(0) = \beta$$

is given by

$$\mathcal{L}[y(x)] = Y(s) = \frac{F(s) + \alpha s + \beta + a\alpha}{s^2 + as + b}$$

where $F(s) = \mathcal{L}[f(x)]$ is the Laplace transform of f.

Now that we know $\mathcal{L}[y(x)]$, the obvious question is: What is $y(x)$? The general problem of finding a function with a given Laplace transform is called the *inversion problem.* The inversion problem and its application to solving initial-value problems is the topic of this section.

If f is continuous on $[0, \infty)$, and if the Laplace transform, $\mathcal{L}[f(x)] = F(s)$ exists for $s > \lambda$, then the function F is uniquely determined by f; that is, the operator $\mathcal{L}$ is itself a function. Our first result states that $\mathcal{L}$ is a one-to-one function. A proof of this result is beyond the scope of this introductory treatment.

THEOREM 1. If f and g are continuous functions on $[0, \infty)$, and if $\mathcal{L}[f(x)] = \mathcal{L}[g(x)]$, then $f \equiv g$; that is $f(x) = g(x)$ for all $x \in [0, \infty)$.

The following definition gives the terminology and notation used in treating the inversion problem.

DEFINITION If $F(s)$ is a given transform and if the function f, continuous on $[0, \infty)$, has the property that $\mathcal{L}[f(x)] = F(s)$, then f is called the *inverse Laplace transform of* $F(s)$, and is denoted by

$$f(x) = \mathcal{L}^{-1}[F(s)].$$

The operator $\mathcal{L}^{-1}$ is called the *inverse operator of* $\mathcal{L}$.

There is a general formula for the inverse operator $\mathcal{L}^{-1}$ corresponding to (1), Section 4.1, but use of the formula requires a knowledge of complex-valued functions, a topic which is treated in more advanced courses.

The relationship between $\mathcal{L}$ and $\mathcal{L}^{-1}$ is given by the following equations:

$$\mathcal{L}^{-1}\left\{\mathcal{L}[f(x)]\right\} = f(x)$$

$$\mathcal{L}\left\{\mathcal{L}^{-1}[F(s)]\right\} = F(s)$$

for all functions f, continuous on $[0, \infty)$, such that $\mathcal{L}[f(x)] = F(s)$.

For convenience here, we reproduce the table of Laplace transforms given at the end of Section 4.1.

Table of Laplace Transforms

$f(x)$	$F(s) = \mathcal{L}[f(x)]$
1	$\frac{1}{s}, \quad s > 0$
$e^{\alpha x}$	$\frac{1}{s-\alpha}, \quad s > \alpha$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}, \quad s > 0$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}, \quad s > 0$
$e^{\alpha x} \cos \beta x$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$e^{\alpha x} \sin \beta x$	$\frac{\beta}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$x^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$x^n e^{rx}, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-r)^{n+1}}, \quad s > r$
$x \cos \beta x$	$\frac{s^2-\beta^2}{(s^2+\beta^2)^2}, \quad s > 0$
$x \sin \beta x$	$\frac{2\beta s}{(s^2+\beta^2)^2}, \quad s > 0$

A simple way to interpret Theorem 1 is that the table can be read either from left to right or from right to left. That is, the table is simultaneously a table of Laplace transforms and of inverse Laplace transforms.

Example 1. (a) If $\mathcal{L}[f(x)] = F(s) = \dfrac{1}{s-4}$, then $f(x) = e^{4x}$.

(b) If $\mathcal{L}[f(x)] = F(s) = \dfrac{s}{s^2+9}$, then $f(x) = \cos 3x$.

(c) If $\mathcal{L}[f(x)] = F(s) = \dfrac{s+2}{s^2+4s+13} = \dfrac{s+2}{(s+2)^2+9} = \dfrac{s-(-2)}{[s-(-2)]^2+9}$, then $f(x) = e^{-2x} \cos 3x$.

The properties of the Laplace transform operator $\mathcal{L}$ can be used to derive corresponding properties of its inverse operator $\mathcal{L}^{-1}$. For our purposes, the most important property is that of linearity.

THEOREM 2. The operator $\mathcal{L}^{-1}$ is linear; that is

$$\mathcal{L}^{-1}[F(s) + G(s)] = \mathcal{L}^{-1}[F(s)] + \mathcal{L}^{-1}[G(s)], \quad \text{and}$$

$$\mathcal{L}^{-1}[cF(s)] = c\mathcal{L}^{-1}[F(s)], \quad c \text{ any constant.}$$

The proof is left as an exercise.

Example 2. Find $\mathcal{L}^{-1}[F(s)]$ if

$$F(s) = \frac{7}{s+2} - \frac{6}{s^2+4}.$$

SOLUTION Since $\mathcal{L}^{-1}$ is a linear operator,

$$\begin{aligned}\mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}\left[\frac{7}{s+3} - \frac{6}{s^2+4}\right] \\ &= 7\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] - 3\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right]\end{aligned}$$

Now, reading the table from right to left, we see that

$$\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = e^{-3x} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] = \sin 2x.$$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{7}{s+3} - \frac{6}{s^2+4}\right] = 7e^{-3x} - 3\sin 2x. \qquad \blacksquare$$

The translation property of $\mathcal{L}$ (Theorem 5, Section 4.2) is also useful in finding inverse transforms. The analog of Theorem 5 for inverse transforms is:

THEOREM 3. If f is continuous on $[0, \infty)$, and if $\mathcal{L}[f(x)] = F(s)$ exists for $s > \lambda$, then, for any real number r,

$$\mathcal{L}^{-1}[F(s-r)] = e^{rx}f(x).$$

The following examples illustrate the kinds of manipulations that typically occur in calculating inverse Laplace transforms. The basic strategy is to try to re-write a given expression $F(s)$ as sum of terms which appear in the table.

Example 3. Find $\mathcal{L}^{-1}[F(s)]$ if

$$F(s) = \frac{4}{(s-3)^2} + \frac{1}{s^2-2s+10}.$$

SOLUTION

$$\mathcal{L}^{-1}[F(s)] = 4\mathcal{L}^{-1}\left[\frac{1}{(s-3)^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2-2s+10}\right]$$

From the table,

$$\mathcal{L}^{-1}\left[\frac{1}{(s-3)^2}\right] = xe^{3x}.$$

To put $\dfrac{1}{s^2-2s+10}$ in a form in the table, we complete the square in the denominator and "adjust" the numerator:

$$\frac{1}{s^2-2s+10} = \frac{1}{s^2-2s+1+9} = \frac{1}{(s-1)^2+9} = \frac{1}{3}\,\frac{3}{(s-1)^2+9}.$$

From the table,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-2s+10}\right] = \frac{1}{3}\,\mathcal{L}^{-1}\left[\frac{3}{(s-1)^2+9}\right] = \tfrac{1}{3}\,e^{x}\sin 3x.$$

Putting the two results together, we have

$$\mathcal{L}^{-1}\left[\frac{4}{(s-3)^2} + \frac{1}{s^2-2s+10}\right] = 4xe^{3x} + \tfrac{1}{3}\,e^{x}\sin 3x. \qquad \blacksquare$$

Example 4. Find $\mathcal{L}^{-1}[F(s)]$ if

$$F(s) = \frac{2s+1}{s^2-2s-8}.$$

SOLUTION By factoring the denominator, we can write

$$F(s) = \frac{2s+1}{s^2-2s-8} = \frac{2s+1}{(s+2)(s-4)}.$$

Now, by partial fraction decomposition,

$$\frac{2s+1}{(s+2)(s-4)} = \frac{\frac{3}{2}}{s-4} + \frac{\frac{1}{2}}{s+2}.$$

Therefore

$$\mathcal{L}^{-1}\left[\frac{2s+1}{s^2-2s-8}\right] = \frac{3}{2}\,\mathcal{L}^{-1}\left[\frac{1}{s-4}\right] + \frac{1}{2}\,\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = \tfrac{3}{2}\,e^{4x} + \tfrac{1}{2}\,e^{-2x}. \qquad \blacksquare$$

Example 5. Find $\mathcal{L}^{-1}[F(s)]$ if

$$F(s) = \frac{2s+4}{(s-2)(s^2-4s+8)}.$$

SOLUTION The quadratic factor in the denominator cannot be factored into linear factors.

By partial fraction decomposition

$$F(s) = \frac{2s+4}{(s-2)(s^2-4s+8)} = \frac{2}{s-2} + \frac{-2s+6}{s^2-4s+8}.$$

Next, we complete the square in the denominator of the second term:

$$\frac{2}{s-2} + \frac{-2s+6}{s^2-4s+8} = \frac{2}{s-2} + \frac{-2s+6}{s^2-4s+4+4} = \frac{2}{s-2} + \frac{-2s+6}{(s-2)^2+4}.$$

Finally, we "adjust" the numerator of the second term so that we can use the Table:

$$\frac{2}{s-2}+\frac{-2s+6}{(s-2)^2+4}=\frac{2}{s-2}+\frac{-2(s-2)+2}{(s-2)^2+4}=\frac{2}{s-2}+\frac{-2(s-2)}{(s-2)^2+4}+\frac{2}{(s-2)^2+4}.$$

Now

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{2s+4}{(s-2)(s^2-4s+8)}\right] &= \mathcal{L}^{-1}\left[2\,\frac{1}{s-2}-2\,\frac{s-2}{(s-2)^2+4}+\frac{2}{(s-2)^2+4}\right]\\ &= 2\,\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]-2\,\mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+4}\right]+\mathcal{L}^{-1}\left[\frac{2}{(s-2)^2+4}\right]\\ &= 2\,e^{2x}-2\,e^{2x}\cos 2x+e^{2x}\sin 2x. \quad \blacksquare\end{aligned}$$

Solution of Initial-Value Problems

Here we complete the application of Laplace transforms to the solution of initial-value problems. For our first example, we finish Example 2, Section 4.2.

Example 6. Find the solution of the initial-value problem

$$y'-2y=2e^{-3x};\quad y(0)=-2.$$

SOLUTION From the Example, if $y=y(x)$ is the solution, then

$$\begin{aligned}\mathcal{L}[y'(x)-2y(x)] &= \mathcal{L}[2e^{-3x}]=2\mathcal{L}[e^{-3x}]=\frac{2}{s+3}\\ \mathcal{L}[y'(x)]-2\mathcal{L}[y(x)] &= \frac{2}{s+3}\\ s\mathcal{L}[y(x)]-y(0)-2\mathcal{L}[y(x)] &= \frac{2}{s+3}\\ (s-2)\mathcal{L}[y(x)]+2 &= \frac{2}{s+3}\\ (s-2)\mathcal{L}[y(x)] &= \frac{2}{s+3}-2\end{aligned}$$

Therefore,

$$\mathcal{L}[y(x)]=Y(s)=\frac{2}{(s-2)(s+3)}-\frac{2}{s-2}.$$

Now, by partial fraction decomposition

$$\frac{2}{(s-2)(s+3)}=\frac{2/5}{s-2}-\frac{2/5}{s+3}.$$

Therefore,

$$Y(s)=\frac{2/5}{s-2}-\frac{2/5}{s+3}-\frac{2}{s-2}=\frac{-8/5}{s-2}-\frac{2/5}{s+3}$$

and

$$y(x)=\mathcal{L}^{-1}\left[\frac{-8/5}{s-2}-\frac{2/5}{s+3}\right]=-\tfrac{8}{5}\,e^{2x}-\tfrac{2}{5}\,e^{-3x}. \quad \blacksquare$$

Next, we finish Example 3 of Section 4.2.

Example 7. Find the solution of the initial-value problem

$$y'' + 4y = 3e^{2x}; \quad y(0) = 5,\ y'(0) = -2.$$

SOLUTION From the Example, if $y = y(x)$ is the solution, then

$$\begin{aligned}
\mathcal{L}[y''(x) + 4y(x)] &= \mathcal{L}[3e^{2x}] = 3\mathcal{L}[e^{2x}] = \frac{3}{s-2} \\
\mathcal{L}[y''(x)] + 4\mathcal{L}[y(x)] &= \frac{3}{s-2} \\
s^2\mathcal{L}[y(x)] - sy(0) - y'(0) + 4\mathcal{L}[y(x)] &= \frac{3}{s-2} \\
(s^2+4)\mathcal{L}[y(x)] - 5s - (-2) &= \frac{3}{s-2} \\
(s^2+4)\mathcal{L}[y(x)] &= \frac{3}{s-2} + 5s - 2
\end{aligned}$$

Therefore,

$$\mathcal{L}[y(x)] = Y(s) = \frac{3}{(s-2)(s^2+4)} + \frac{5s-2}{s^2+4}.$$

Now, by partial fraction decomposition,

$$\frac{3}{(s-2)(s^2+4)} = \frac{\frac{3}{8}}{s-2} - \frac{\frac{3}{8}s + \frac{3}{4}}{s^2+4}.$$

Therefore,

$$Y(s) = \frac{\frac{3}{8}}{s-2} - \frac{\frac{3}{8}s + \frac{3}{4}}{s^2+4} + \frac{5s-2}{s^2+4} = \frac{3}{8}\frac{1}{s-2} + \frac{37}{8}\frac{s}{s^2+4} - \frac{11}{8}\frac{2}{s^2+4}$$

and

$$\begin{aligned}
y(x) = \mathcal{L}^{-1}[Y(s)] &= \frac{3}{8}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{37}{8}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{11}{8}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] \\
&= \tfrac{3}{8}e^{2x} + \tfrac{37}{8}\cos 2x - \tfrac{11}{8}\sin 2x. \qquad \blacksquare
\end{aligned}$$

Example 8. Find the solution of the initial-value problem

$$y'' - 5y' + 6y = x - 1; \quad y(0) = 0,\ y'(0) = 1.$$

SOLUTION If $y = y(x)$ is the solution, then

$$\begin{aligned}
\mathcal{L}[y''(x) - 5y'(x) + 6y(x)] &= \mathcal{L}[x-1] = \mathcal{L}[x] - \mathcal{L}[1] = \frac{1}{s^2} - \frac{1}{s} \\
\mathcal{L}[y''(x)] - 5\mathcal{L}[y'(x)] + 6\mathcal{L}[y(x)] &= \frac{1-s}{s^2} \\
s^2\mathcal{L}[y(x)] - sy(0) - y'(0) - 5\left\{s\mathcal{L}[y(x)] - y(0)\right\} + 6\mathcal{L}[y(x)] &= \frac{1-s}{s^2} \\
(s^2 - 5s + 6)\mathcal{L}[y(x)] - 1 &= \frac{1-s}{s^2} \\
(s^2 - 5s + 6)\mathcal{L}[y(x)] &= \frac{1-s}{s^2} + 1.
\end{aligned}$$

Therefore,

$$\mathcal{L}[y(x)] = Y(s) = \frac{1-s}{s^2(s^2 - 5s + 6)} + \frac{1}{s^2 - 5s + 6} = \frac{1-s}{s^2(s-2)(s-3)} + \frac{1}{(s-2)(s-3)}.$$

Now, by partial fraction decomposition,

$$\frac{1-s}{s^2(s-2)(s-3)} = -\frac{1}{36}\left(\frac{1}{s}\right) + \frac{1}{6}\left(\frac{1}{s^2}\right) + \frac{1}{4}\left(\frac{1}{s-2}\right) - \frac{2}{9}\left(\frac{1}{s-3}\right)$$

and

$$\frac{1}{(s-2)(s-3)} = \frac{-1}{s-2} + \frac{1}{s-3}.$$

Therefore,

$$Y(s) = -\frac{1}{36}\left(\frac{1}{s}\right) + \frac{1}{6}\left(\frac{1}{s^2}\right) - \frac{3}{4}\left(\frac{1}{s-2}\right) + \frac{7}{9}\left(\frac{1}{s-3}\right)$$

and

$$y(x) = -\tfrac{1}{36} + \tfrac{1}{6}\,x - \tfrac{3}{4}\,e^{2x} + \tfrac{7}{9}\,e^{3x}. \qquad \blacksquare$$

In many applications of differential equations it is not required to determine the solutions explicitly. Instead what is needed is information about the solutions. Often such information can be obtained by analyzing their Laplace transforms. The next example illustrates this type of application.

Example 9. Consider the differential equation

$$y'' - y' - 6y = 2e^{-x} \quad \text{on } [0, \infty)$$

together with the condition $y(0) = -1$. From Chapter 3, we know that the general solution of the differential equation has the form

$$y(x) = C_1e^{-2x} + C_2e^{3x} + Ae^{-x} \qquad (*)$$

where C_1, C_2 are arbitrary constants and A is a constant which can be determined

The question we want to examine is: Can we choose a value for $\alpha = y'(0)$ so that solution of the resulting initial value-problem

$$y'' - y' - 6y = 2e^{-x}; \quad y(0) = -1,\ y'(0) = \alpha$$

has limit 0 as $x \to \infty$? Since $e^{-2x} \to 0$ and $e^{-x} \to 0$ as $x \to \infty$, we want to choose α so that the coefficient of the e^{3x} term is 0.

If $y = y(x)$ is the solution of the initial-value problem, then

$$\begin{aligned}
\mathcal{L}[y''(x) - y'(x) - 6y(x)] &= \mathcal{L}[2e^{-x}] = \frac{2}{s+1} \\
\mathcal{L}[y''(x)] - \mathcal{L}[y'(x)] - 6\mathcal{L}[y(x)] &= \frac{2}{s+1} \\
s^2\mathcal{L}[y(x)] - sy(0) - y'(0) - \{s\mathcal{L}[y(x)] - y(0)\} - 6\mathcal{L}[y(x)] &= \frac{2}{s+1} \\
(s^2 - s - 6)\mathcal{L}[y(x)] + s - \alpha - 1 &= \frac{2}{s+1} \\
(s^2 - s - 6)\mathcal{L}[y(x)] &= \frac{2}{s+1} + 1 + \alpha - s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{L}[y(x)] = Y(s) &= \frac{2}{(s+1)(s^2 - s - 6)} + \frac{1 + \alpha - s}{s^2 - s - 6} \\
&= \frac{2}{(s+1)(s+2)(s-3)} + \frac{1 + \alpha - s}{(s+2)(s-3)}.
\end{aligned}$$

Now, by partial fraction decomposition,

$$\frac{2}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3} = \frac{-\frac{1}{2}}{s+1} + \frac{\frac{2}{5}}{s+2} + \frac{\frac{1}{10}}{s-3}$$

and

$$\frac{1 + \alpha - s}{(s+2)(s-3)} = \frac{D}{s-2} + \frac{E}{s-3} = \frac{-\frac{\alpha+3}{5}}{s+2} + \frac{\frac{\alpha-2}{5}}{s-3}.$$

Combining these results, we have

$$Y(s) = -\frac{\alpha + 1}{5}\left(\frac{1}{s+2}\right) + \frac{2\alpha - 3}{10}\left(\frac{1}{s-3}\right) - \frac{1}{2}\left(\frac{1}{s+1}\right).$$

Clearly, if $2\alpha - 3 = 0$, that is, if $\alpha = 3/2$, then the e^{3x} term in $(*)$ is eliminated. The resulting solution is:

$$y(x) = -\tfrac{1}{2}\, e^{-2x} - \tfrac{1}{2} e^{-x}.$$

This solution has initial values: $y(0) = -1,\ y'(0) = \frac{3}{2}$, and $\lim_{x\to\infty} y(x) = 0$. ■

Exercises 4.3

Find $\mathcal{L}^{-1}[F(s)]$

1. $F(s) = \dfrac{6}{s+7}$.
2. $F(s) = \dfrac{1}{2s+2}$.
3. $F(s) = \dfrac{1}{s^2+25}$.
4. $F(s) = \dfrac{4}{s} - \dfrac{3}{s-4}$.
5. $F(s) = \dfrac{s+4}{s^2+8s+17}$.
6. $F(s) = \dfrac{4}{s^2-6s+13}$.
7. $F(s) = \dfrac{s+4}{s^2+4s+8}$.
8. $F(s) = \dfrac{2}{s} - \dfrac{5}{s^2} + \dfrac{1}{s^3}$.
9. $F(s) = \dfrac{2}{(s+2)^2} - \dfrac{s}{s^2-2s+2}$.
10. $F(s) = \dfrac{s+3}{s^2-2s+9}$.

Use partial fraction decomposition to find the inverse Laplace transform.

11. $F(s) = \dfrac{1}{(s+1)(s^2+1)}$.
12. $F(s) = \dfrac{s+3}{s^2-s-2}$.
13. $F(s) = \dfrac{1}{s(s^2+4)}$.
14. $F(s) = \dfrac{1}{s^2-1}$.
15. $F(s) = \dfrac{s^2-3s-1}{s^3+s^2-2s}$.
16. $F(s) = \dfrac{s}{s^4-1}$.
17. $F(s) = \dfrac{4}{s^2(s-1)(s-2)}$.

Find the solution of the initial-value problem.

18. $y' - 4y = 0; \quad y(0) = -2.$

19. $y' + 2y = e^x; \quad y(0) = 1.$

20. $y' - 3y = e^{2x}; \quad y(0) = 2.$

21. $y' + y = \sin x; \quad y(0) = 1.$

22. $y'' + 4y = 0; \quad y(0) = 2,\ y'(0) = 2.$

23. $y'' - 2y' + 2y = 0; \quad y(0) = 0,\ y'(0) = 1.$

24. $y'' - y = \sin x; \quad y(0) = 1,\ y'(0) = 1.$

25. $y'' - y = e^x; \quad y(0) = 1,\ y'(0) = 0.$

26. $y'' - y' - 2y = \sin 2x; \quad y(0) = 1,\ y'(0) = 1.$

27. $y'' + 2y' + y = x + e^x; \quad y(0) = -\frac{7}{4},\ y'(0) = \frac{9}{4}.$

28. $y'' + 2y' + y = 4e^{-x}; \quad y(0) = 2,\ y'(0) = -1.$

29. $y'' + 3y' + 2y = 6e^x; \quad y(0) = 2,\ y'(0) = -1.$

30. $y'' - 2y' + 5y = 3e^{-2x}; \quad y(0) = 1,\ y'(0) = 1.$

31. Given the initial-value problem

$$y'' - y' - 6y = 2e^{-x}; \quad y(0) = \alpha,\ y'(0) = -1.$$ (See Example 9)

What value should be assigned to α so that the resulting solution will have limit 0 as $x \to \infty$?

32. What initial conditions should be assigned with the differential equation

$$y'' + y = e^{-x}$$

so that $\lim_{x\to\infty} y(x) = 0$ where $y = y(x)$ is the solution.

33. Consider the differential equation: $y'' + y' - 2y = 3\sin 2x$ together with the initial value $y(0) = 2$. For what value(s) of $\beta = y'(0)$ will the resulting solution(s) be bounded?

34. Consider the differential equation: $y'' + 3y' + 2y = x$ together with the initial value $y'(0) = -2$. For what value(s) of $\alpha = y(0)$ will the resulting solution(s) be bounded?

The Laplace transform method applies to initial-value problems in which the initial values are specified at $x = 0$. Actually, the method can be applied when the initial conditions are specified at some point $a \neq 0$. All that is required is a simple change of independent variable; a translation. For example, the initial-value problem

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x; \quad y(1) = 0,\ y'(1) = 1$$

is transformed into

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = t + 1, \quad y(0) = 0,\ y'(0) = 1$$

by setting $t = x - 1$. With this transformation, we have: $t = 0$ when $x = 1$; $x = t + 1$; and

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt} \cdot 1 = \frac{dy}{dt} \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right]\frac{dt}{dx} = \frac{d}{dt}\left[\frac{dy}{dt}\right] \cdot 1 = \frac{d^2y}{dt^2}. \end{aligned}$$

35. Find the solution of the initial-value problem: $y'' - 3y' + 2y = x$; $y(1) = 0,\ y'(1) = 1$ by first solving the transformed problem.

36. Use the Laplace transform method to solve the initial-value problem

$$y' - 2y = 1 + x; \quad y(2) = -1.$$

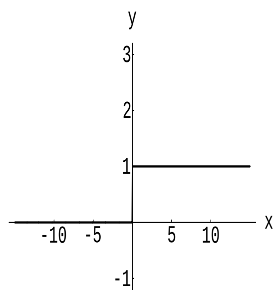
4.4 Applications to Discontinuous Functions

In the preceding sections we assumed that the functions under consideration were continuous on $[0, \infty)$. In this section we consider Laplace transforms of certain types of discontinuous functions that occur in various applications.

A particular example of the type of discontinuous function that we will be considering in this section is the *unit step function,* or *Heaviside function* u. This function is defined on $(-\infty, \infty)$ by

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases} \tag{1}$$

The graph of u is



It is clear that u is continuous on $(-\infty, \infty)$ except at $x = 0$. At $x = 0$, $\lim_{x \to 0} u(x)$ does not exist. However, note that the left-hand and right-hand limits of u at $x = 0$ do exist:

$$\lim_{x \to 0^-} u(x) = 0 \qquad \text{and} \qquad \lim_{x \to 0^+} u(x) = 1.$$

Recall from calculus that a function f is continuous at a point c if and only if $\lim_{x \to c} f(x) = f(c)$, and $\lim_{x \to c} f(x)$ exists if and only if the left-hand and right-hand limits of f at c both exist and are equal.

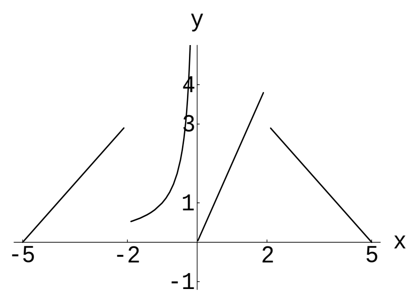
DEFINITION 1. (Jump Discontinuity) Let the function $f = f(x)$ be defined on an interval I and continuous except at a point $c \in I$, c not an endpoint of I. If the left-hand and right-hand limits of f at c both exist but are not equal, then f is said to have a *jump* (or *finite*) *discontinuity* at c.

Example 1. (a) The unit step function u is continuous on $(-\infty, \infty)$ except at $x = 0$ where it has a jump discontinuity.

(b) Let f be defined on $[-5, 5]$ by

$$f(x) = \begin{cases} x + 5 & -5 \leq x < -2 \\ -\dfrac{1}{x} & -2 \leq x < 0 \\ 2x & 0 \leq x \leq 2 \\ -(x - 5) & 2 < x \leq 5 \end{cases}$$

The graph of f is



Clearly f is discontinuous at $x = -2$, $x = 0$, and $x = 2$. Since

$$\lim_{x \to -2^-} f(x) = 3 \quad \text{and} \quad \lim_{x \to -2^+} f(x) = \tfrac{1}{2},$$

and

$$\lim_{x \to 2^-} f(x) = 4 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 3,$$

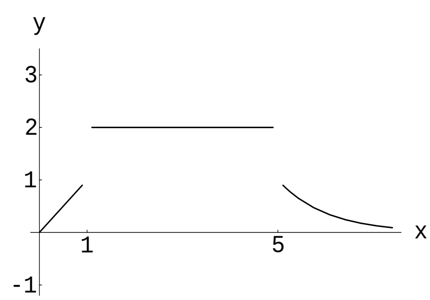
f has jump discontinuities at $x = -2$ and $x = 2$.

The discontinuity at $x = 0$ is not a jump discontinuity because $\lim_{x \to 0^-} f(x)$ does not exist ($f(x) \to \infty$ as $x \to 0^-$).

(c) Let g be defined on $[0, \infty)$ by

$$g(x) = \begin{cases} x & 0 \le x < 1 \\ 3 & x = 1 \\ 2 & 1 \le x < 5 \\ e^{-(x-5)} & x \ge 5 \end{cases}$$

The graph of g is



This function has jump discontinuities at $x = 1$ and $x = 5$. Note that the value of g at $x = 1$ is independent of the left-hand and right-hand limits of g at $x = 1$. ■

DEFINITION 2. (Piecewise Continuous) A function f defined on an interval I is *piecewise continuous on* I if it is continuous on I except for at most a finite number of points c_1, c_2, $\ldots$, c_n of I at which it has a jump discontinuity.

Remark A continuous function is also piecewise continuous. ■

Referring to Example 1, the unit step function u is piecewise continuous on $(-\infty, \infty)$; the function g in (c) is piecewise continuous on $[0, \infty)$. the function f in (b) is not piecewise continuous on $[-5, 5]$ because the discontinuity at $x = 0$ is not a jump discontinuity.

In this section we want to apply Laplace transform methods to piecewise continuous functions. Before we can calculate the Laplace transform of a piecewise function we have to know how to integrate such a function.

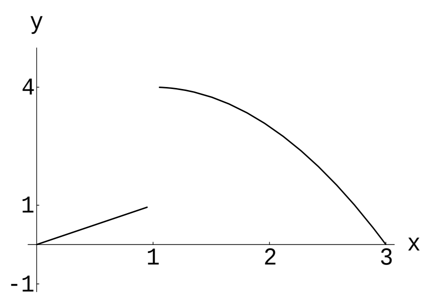
Suppose that f is piecewise continuous on $[a, b]$ with jump discontinuities at $c_1 < c_2 < \cdots < c_n$. It follows from the definition of the definite integral that

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots + \int_{c_n}^b f(x)\,dx.$$

Example 2. Let f be defined on $[0, 3]$ by

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 4 - (x-1)^2 & 1 < x \le 3 \end{cases}$$

The graph of f is



We have

$$\begin{aligned} \int_0^3 f(x)\,dx &= \int_0^1 x\,dx + \int_1^3 (4 - (x-1)^2)\,dx \\ &= \left[\frac{x^2}{2}\right]_0^1 + \left[4x - \frac{(x-1)^3}{3}\right]_1^3 = \frac{35}{6}. \end{aligned}$$ ■

The integration of piecewise continuous functions can be extended to include (improper) integrals on an infinite interval.

Example 3. Consider the function g of Example 1:

$$\int_0^\infty g(x)\,dx = \int_0^1 x\,dx + \int_1^5 2\,dx + \int_5^\infty e^{-(x-5)}\,dx$$

$$\begin{aligned} &= \left[\frac{x^2}{2}\right]_0^1 + [2x]_1^5 + \lim_{b\to\infty}\int_5^b e^{-(x-5)}\,dx \\ &= \frac{1}{2} + 8 + \lim_{b\to\infty}\left[-e^{-(x-5)}\right]_5^b \\ &= \frac{17}{2} + [0+1] = \frac{19}{2}. \qquad \blacksquare \end{aligned}$$

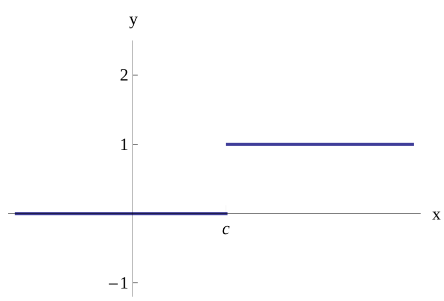
The following theorem is an extension of Theorem 1, Section 4.2.

THEOREM 1. If the function f is piecewise continuous on $[0,\infty)$, and of exponential order λ, then the Laplace transform $\mathcal{L}[f(x)]$ exists for $s > \lambda$.

Let c be a real number. The translation of the unit step function u by c is the function $u_c = u(x-c)$ defined on $(-\infty,\infty)$ by

$$u(x-c) = \begin{cases} 0 & x < c \\ 1 & x \geq c. \end{cases} \tag{2}$$

The graph of u_c for $c > 0$ is



Example 4. The Laplace transform of u_c, $c > 0$ is

$$\mathcal{L}[u(x-c)] = \frac{e^{-cs}}{s}, \ s > 0. \tag{3}$$

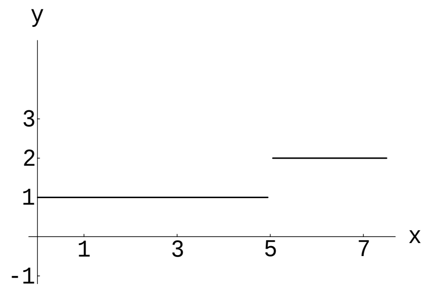
Proof: By definition

$$\begin{aligned} \mathcal{L}[u(x-c)] &= \int_0^\infty e^{-sx}u(x-c)\,dx \\ &= \int_0^c e^{-sx}\cdot 0\,dx + \int_c^\infty e^{-sx}\cdot 1\,dx \\ &= \lim_{b\to\infty}\int_c^b e^{-sx}\,dx = \lim_{b\to\infty}\left[\frac{e^{-sx}}{-s}\right]_c^b \\ &= \lim_{b\to\infty}\frac{e^{-sb}}{-s} + \frac{e^{-sc}}{s} = \frac{e^{-cs}}{s}, \ s > 0. \end{aligned}$$

Note that if $c = 0$, then $u_0 = u(x-0) = u(x) \equiv 1$ on $[0, \infty)$, and $\mathcal{L}[u(x)] = \mathcal{L}[1] = 1/s,\ s > 0$ as we saw in Section 4.1. ■

Example 5. (a) Let f be the function defined on $[0, \infty)$ by

$$f(x) = \begin{cases} 1 & 0 \leq x < 5 \\ 2 & 5 \leq x < \infty. \end{cases}$$



Then

$$\begin{aligned}
\mathcal{L}[f(x)] &= \int_0^\infty e^{-sx} f(x)\,dx = \int_0^5 e^{-sx} \cdot 1\,dx + \int_5^\infty e^{-sx} \cdot 2\,dx \\
&= \left[\frac{-e^{-sx}}{s}\right]_0^5 + \lim_{b\to\infty} \int_0^b 2e^{-sx}\,dx \\
&= \frac{-e^{-5s}}{s} + \frac{1}{s} + \lim_{b\to\infty}\left[\frac{2e^{-sx}}{-s}\right]_5^b \\
&= \frac{-e^{-5s}}{s} + \frac{1}{s} + \lim_{b\to\infty} \frac{2e^{-sb}}{-s} + \frac{2e^{-5s}}{s} \\
&= \frac{-e^{-5s}}{s} + \frac{1}{s} + \frac{2e^{-5s}}{s} = \frac{1}{s} + \frac{e^{-5s}}{s}, \quad s > 0.
\end{aligned}$$

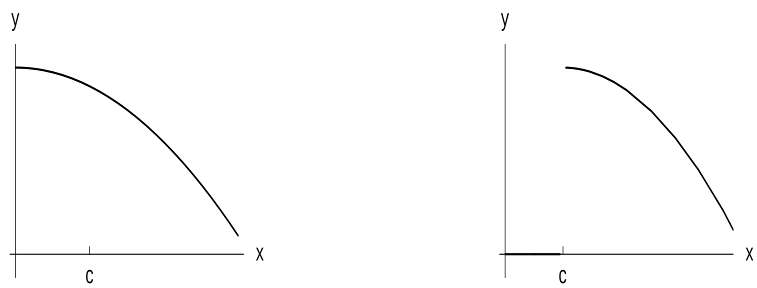
(b) We calculate $\mathcal{L}[g(x)]$ where g is the function given in Example 1(c).

$$\begin{aligned}
\mathcal{L}[g(x)] &= \int_0^\infty e^{-sx} g(x)\,dx = \int_0^1 e^{-sx} \cdot x\,dx + \int_1^5 e^{-sx} \cdot 2\,dx + \int_5^\infty e^{-sx} e^{-(x-5)}\,dx \\
&= \left[\frac{-xe^{-sx}}{s} - \frac{e^{-sx}}{s^2}\right]_0^1 + \left[\frac{-2e^{-sx}}{s}\right]_1^5 + \lim_{b\to\infty} \int_0^b e^{-[x(s+1)-5]}\,dx \\
&= \frac{-e^{-s}}{s} + \frac{-e^{-s}}{s^2} + \frac{1}{s^2} + \frac{-2e^{-5s}}{s} + \frac{2e^{-s}}{s} + \lim_{b\to\infty}\left[\frac{-e^{-[x(s+1)-5]}}{s+1}\right]_5^\infty \\
&= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} - \frac{2e^{-5s}}{s} + \frac{2e^{-s}}{s} + \frac{e^{-5s}}{s+1} \\
&= \frac{1}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} - \frac{2e^{-5s}}{s} + \frac{e^{-5s}}{s+1}.
\end{aligned}$$ ■

The unit step function and its translations can be used to obtain translations of arbitrary functions. For example, if f is defined on $[0, \infty)$ and $c > 0$, then the function

$$f_c(x) = f(x-c)u(x-c) = \begin{cases} 0 & x < c \\ f(x-c)u(x-c) & x \geq c \end{cases}$$

is the function f shifted c units to the right as illustrated in the figure below.



THEOREM 2. Let f be defined on $[0, \infty)$ and suppose $\mathcal{L}[f(x)] = F(s)$ exists for $s > \lambda$. If $c > 0$, then $\mathcal{L}[f_c(x)]$ exists for $s > \lambda$ and is given by

$$\mathcal{L}[f_c(x)] = \mathcal{L}[f(x-c)u(x-c)] = e^{-cs}F(s).$$

Proof: By the definition,

$$\begin{aligned} \mathcal{L}[f_c(x)] &= \mathcal{L}[f(x-c)u(x-c)] = \int_0^\infty e^{-sx} f(x-c)u(x-c)\, dx \\ &= \lim_{b\to\infty} \int_c^b e^{-sx} f(x-c)\, dx. \end{aligned}$$

Now let $t = x - c$. Then

$$x = t + c, \qquad dx = dt, \qquad \text{and} \qquad t = 0 \quad \text{when} \quad x = c.$$

With this change of variable,

$$\begin{aligned} \lim_{b\to\infty} \int_c^b e^{-sx} f(x-c)\, dx &= \lim_{b\to\infty} \int_0^{b-c} e^{-s(t+c)} f(t)\, dt \\ &= e^{-cs} \lim_{b\to\infty} \left(\int_0^{b-c} e^{-st} f(t)\, dt \right) \\ &= e^{-cs} \int_0^\infty e^{-st} f(t)\, dt = e^{-cs}F(s) \end{aligned}$$

since $b - c \to \infty$ as $b \to \infty$. ■

This theorem can be expressed in an equivalent manner using the inverse Laplace transform.

COROLLARY If $\mathcal{L}^{-1}[F(s)] = f(x)$ and $c > 0$, then

$$\mathcal{L}^{-1}[e^{-cs}F(s)] = f(x-c)u(x-c) = f_c(x).$$

We now illustrate how to use translations of the unit step function and Theorem 2 to calculate Laplace transforms of piecewise continuous functions.

Example 6. We calculate $\mathcal{L}[f(x)]$ where f is the function given in Example 4:

$$f(x) = \begin{cases} 1 & 0 \le x < 5 \\ 2 & 5 \le x < \infty. \end{cases}$$

Since f has a jump discontinuity at $x = 5$, we'll write f in terms of $u(x-5)$. Define $f_1(x)$ by

$$f_1(x) = \begin{cases} 1 & 0 \le x < 5 \\ 0 & 5 \le x < \infty \end{cases} = 1 - u(x-5)$$

and $f_2(x)$ by

$$f_2(x) = \begin{cases} 0 & 0 \le x < 5 \\ 2 & 5 \le x < \infty \end{cases} = 2u(x-5).$$

Then

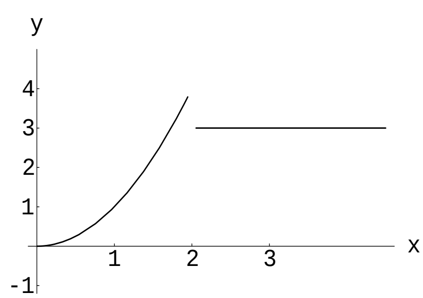
$$f(x) = f_1(x) + f_2(x) = 1 - u(x-5) + 2u(x-5) = 1 + u(x-5)$$

and

$$\mathcal{L}[f(x)] = \mathcal{L}[1 + u(x-5)] = \mathcal{L}[1] + \mathcal{L}[u(x-5)] = \frac{1}{s} + \frac{e^{-5s}}{s}, \quad s > 0. \quad \blacksquare$$

Example 7. Let $h(x) = \begin{cases} x^2 & 0 \le x < 2 \\ 3 & 2 \le x < \infty. \end{cases}$. Calculate $\mathcal{L}[h(x)]$.

SOLUTION The graph of h is



Let $h_1(x) = x^2 - x^2\, u(x-2)$ and $h_2(x) = 3u(x-2)$. Then

$$h(x) = h_1(x) + h_2(x) = x^2 - x^2\, u(x-2) + 3u(x-2)$$

and

$$\mathcal{L}[h(x)] = \mathcal{L}[x^2] - \mathcal{L}[x^2\,u(x-2)] + 3\mathcal{L}[u(x-2)] = \frac{2}{s^3} - \mathcal{L}[x^2\,u(x-2)] + 3\frac{e^{-2s}}{s}.$$

Before we can apply Theorem 2 to calculate $\mathcal{L}[x^2\,u(x-2)]$ we must have the coefficient x^2 of $u(x-2)$ expressed as a function of $(x-2)$. Since

$$x^2 = [(x-2)+2]^2 = (x-2)^2 + 4(x-2) + 4$$

we have

$$\begin{aligned}
\mathcal{L}[x^2\,u(x-2)] &= \mathcal{L}\left[(x-2)^2u(x-2) + 4(x-2)u(x-2) + 4u(x-2)\right] \\
&= \mathcal{L}[(x-2)^2u(x-2)] + 4\mathcal{L}[(x-2)u(x-2)] + 4\mathcal{L}[u(x-2)] \\
&= e^{-2s}\frac{2}{s^3} + 4e^{-2s}\frac{1}{s^2} + 4e^{-2s}\frac{1}{s}.
\end{aligned}$$

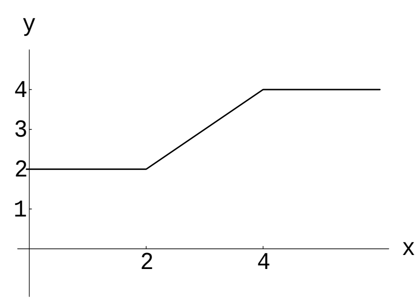
It now follows that

$$\mathcal{L}[h(x)] = \frac{2}{s^3} - \left[e^{-2s}\frac{2}{s^3} + 4e^{-2s}\frac{1}{s^2} + 4e^{-2s}\frac{1}{s}\right] + 3\frac{e^{-2s}}{s} = \frac{2}{s^3} - \frac{2e^{-2s}}{s^3} - \frac{4e^{-2s}}{s^2} - \frac{e^{-2s}}{s}. \qquad \blacksquare$$

Example 8. Let

$$g(x) = \begin{cases} 2 & 0 \le x < 2 \\ x & 2 \le x < 4 \\ 4 & x \ge 4 \end{cases}.$$

The graph of g is



Set

$$\begin{aligned}
g_1(x) &= 2 - 2\,u(x-2) \\
g_2(x) &= x\,u(x-2) - x\,u(x-4) = [(x-2)+2)]u(x-2) - [(x-4)+4]u(x-4) \\
&= (x-2)u(x-2) + 2u(x-2) - (x-4)u(x-4) - 4u(x-4) \\
g_3(x) &= 4\,u(x-4).
\end{aligned}$$

Then

$$g(x) = g_1(x) + g_2(x) + g_3(x) = 2 + (x-2)u(x-2) - (x-4)u(x-4).$$

Therefore,

$$\mathcal{L}[g(x)] = \frac{2}{s} + e^{-2s}\frac{1}{s^2} - e^{-4s}\frac{1}{s^2} \qquad \blacksquare$$

The Corollary to Theorem 2 is used to calculate inverse Laplace transforms.

Example 9. Suppose

$$F(s) = \frac{1}{s^2} + \frac{e^{-4s}}{s-2}.$$

Then

$$\mathcal{L}^{-1}\left[\frac{1}{s^2} + \frac{e^{-4s}}{s-2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{e^{-4s}}{s-2}\right]$$

From the Table of Laplace transforms $\mathcal{L}^{-1}\left[1/s^2\right] = x$. To calculate $\mathcal{L}^{-1}\left[e^{-4s}/(s-2)\right]$, let $F(s) = 1/(s-2)$. Then $\mathcal{L}^{-1}[F(s)] = e^{2x}$. Therefore, by the Corollary to Theorem 2,

$$\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s-2}\right] = e^{2(x-4)}u(x-4).$$

It now follows that

$$\mathcal{L}^{-1}[F(s)] = x + e^{2(x-4)}u(x-4) = \begin{cases} x & 0 \le x < 4 \\ x + e^{2(x-4)} & 4 \le x < \infty \end{cases}. \qquad \blacksquare$$

Example 10. Suppose

$$F(s) = \frac{e^{-s}}{(s+1)^2} + \frac{e^{-\pi s}}{s^2+4}.$$

Then

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{e^{-s}}{(s+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+4}\right].$$

To calculate $\mathcal{L}^{-1}[e^{-s}/(s+1)^2]$, let $F_1(s) = 1/(s+1)^2$. Then, from the Table of Laplace transforms, $\mathcal{L}^{-1}[F_1(s)] = xe^{-x}$. Therefore,

$$\mathcal{L}^{-1}\left[\frac{e^{-s}}{(s+1)^2}\right] = (x-1)e^{-(x-1)}\,u(x-1).$$

To calculate $\mathcal{L}^{-1}[e^{-\pi s}/(s^2+4)]$, let

$$F_2(s) = \frac{1}{s^2+4} = \frac{1}{2}\,\frac{2}{s^2+4}.$$

From the Table of Laplace transforms, $\mathcal{L}^{-1}[F_2(s)] = \frac{1}{2}\sin 2x$. Therefore,

$$\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+4}\right] = \tfrac{1}{2}\sin 2(x-\pi)\;u(x-\pi) = \tfrac{1}{2}\sin 2x\;u(x-\pi). \qquad (\sin 2(x-\pi) = \sin 2x)$$

Finally,

$$\begin{aligned} \mathcal{L}^{-1}[F(s)] &= (x-1)e^{-(x-1)}\,u(x-1) + \tfrac{1}{2}\sin 2x\;u(x-\pi) \\ &= \begin{cases} 0 & 0 \le x < 1 \\ (x-1)e^{-(x-1)} & 1 \le x < \pi \\ (x-1)e^{-(x-1)} + \frac{1}{2}\sin 2x & x \ge \pi \end{cases} \end{aligned}$$

Exercises 4.4

Use the definition of the Laplace transform to find $\mathcal{L}[f]$.

1. $f(x) = \begin{cases} x & 0 \le x < 1 \\ x - 1 & 1 \le x < 2 \\ 0 & x \ge 2 \end{cases}$.

2. $f(x) = \begin{cases} -1 & 0 \le x < 1 \\ x - 1 & x \ge 1 \end{cases}$.

3. $f(x) = \begin{cases} 0 & 0 \le x < 5 \\ 2 & x \ge 5 \end{cases}$.

4. $f(x) = \begin{cases} 2x & 0 \le x < 3 \\ 1 & x \ge 3 \end{cases}$.

5. $f(x) = \begin{cases} 1 & 0 \le x < 2 \\ x - 2 & 2 \le x < 4 \\ e^{-(x-4)} & x \ge 4 \end{cases}$.

6. $f(x) = \begin{cases} 1 & 0 \le x < 1 \\ -1 & 1 \le x < 2 \end{cases}$
   and extended periodically on $[2, \infty)$. See Exercises 4.1, Problem 10.

7. $f(x) = \begin{cases} 2x & 0 \le x < 2 \\ 2 & 2 \le x < 4 \end{cases}$
   and extended periodically on $[4, \infty)$. (Sketch the graph of f.)

8. $f(x) = x - n$ on $[n, n+1)$, $n = 0,\ 1,\ 2,\ \ldots$. (Sketch the graph of f.)

Express the function f in terms of the unit step function and its translations. Then find $\mathcal{L}[f]$.

9. $f(x) = \begin{cases} x^2 & 0 \le x < 3 \\ 3x & x \ge 3 \end{cases}$.

10. $f(x) = \begin{cases} \sin x & 0 \le x < \pi \\ \sin 2x & \pi \le x < 2\pi \\ \sin 3x & x \ge 2\pi \end{cases}$. Sketch the graph of f.

11. $f(x) = \begin{cases} 0 & 0 \le x < \pi \\ 1 + \cos x & \pi \le x < 2\pi \\ 2\cos x & x \ge 2\pi \end{cases}$.

12. $f(x) = \begin{cases} x & 0 \le x < 2 \\ 0 & 2 \le x < 4 \\ (x-4)^2 & x \ge 4 \end{cases}$.

13. $f(x) = \begin{cases} x & 0 \le x < 1 \\ 2 - x & 1 \le x < 3 \\ x - 4 & 3 \le x < 4 \\ 0 & x \ge 4 \end{cases}$. Sketch the graph of f.

Calculate the inverse Laplace transform of $F(s)$.

14. $F(s) = \dfrac{e^{-s}}{s} + \dfrac{2e^{-3s}}{s} - \dfrac{6e^{-4s}}{s}$.

15. $F(s) = \dfrac{1 + e^{-\pi s}}{s^2 + 1}$.

16. $F(s) = \dfrac{1}{s+1} - \dfrac{e^{-2s}}{s+1}$.

17. $F(s) = \dfrac{s + (s-1)e^{-\pi s}}{s^2 + 1}$.

18. $F(s) = \dfrac{2}{s^2} + \dfrac{e^{-2s}}{s^2} + \dfrac{2e^{-2s}}{s^3} - \dfrac{4e^{-3s}}{s}$.

19. $F(s) = \dfrac{e^{-2s}}{s(s+1)}$.

20. $F(s) = \dfrac{1 - e^{-2s}}{s^2 + \pi^2}$.

4.5 Initial-Value Problems with Piecewise Continuous Nonhomogeneous Terms

In this section we illustrate the application of Laplace transforms to the solution of nonhomogeneous initial-value problems in which the forcing function f is piecewise continuous.

Example 1. Find the solution of the initial-value problem

$$y' + 2y = f(x); \quad y(0) = 4$$

where $f(x) = \begin{cases} x & 0 \le x < 3 \\ 5 & x \ge 3. \end{cases}$

SOLUTION The function f has a jump discontinuity at $x = 3$. Therefore, the first step is to express f in terms of $u_3(x) = u(x-3)$. You should verify that

$$f(x) = x - x\,u(x-3) + 5u(x-3) = x - [(x-3)+3]u(x-3) + 5u(x-3) = x - (x-3)u(x-3) + 2u(x-3).$$

Now, taking the Laplace transform of the equation, we get

$$\mathcal{L}[y' + 2y] = s\mathcal{L}[y] - y(0) + 2\mathcal{L}[y] = \frac{1}{s^2} - \frac{e^{-3s}}{s^2} + 2\,\frac{e^{-3s}}{s}$$

Applying the initial condition and solving for $\mathcal{L}[y]$, we find that

$$\begin{aligned} \mathcal{L}[y] = Y(s) &= \frac{1}{s^2(s+2)} + \frac{e^{-3s}}{s^2(s+2)} + 2\,\frac{e^{-3s}}{s(s+2)} + \frac{4}{s+2} \\ &= \frac{4s^2+1}{s^2(s+2)} + \frac{(2s+1)e^{-3s}}{s^2(s+2)}. \end{aligned}$$

By partial fraction decomposition

$$\begin{aligned} \frac{4s^2+1}{s^2(s+2)} &= -\frac{1}{4}\,\frac{1}{s} + \frac{1}{2}\,\frac{1}{s^2} + \frac{17}{4}\,\frac{1}{s+2} \\ \frac{(2s+1)e^{-3s}}{s^2(s+2)} &= \frac{3}{4}\,\frac{e^{-3s}}{s} + \frac{1}{2}\,\frac{e^{-3s}}{s^2} - \frac{3}{4}\,\frac{e^{-3s}}{s+2} \end{aligned}$$

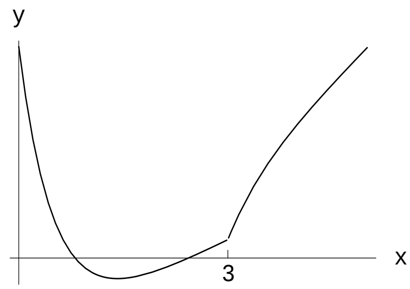
Therefore,

$$Y(s) = -\frac{1}{4}\,\frac{1}{s} + \frac{1}{2}\,\frac{1}{s^2} + \frac{17}{4}\,\frac{1}{s+2} + \frac{3}{4}\,\frac{e^{-3s}}{s} + \frac{1}{2}\,\frac{e^{-3s}}{s^2} - \frac{3}{4}\,\frac{e^{-3s}}{s+2}$$

and

$$\begin{aligned} y(x) = \mathcal{L}^{-1}[Y(s)] &= -\tfrac{1}{4} + \tfrac{1}{2}\,x + \tfrac{17}{4}e^{-2x} + \tfrac{3}{4}\,u(x-3) + \tfrac{1}{2}\,(x-3)\,u(x-3) - \tfrac{3}{4}\,e^{-2(x-3)}\,u(x-3) \\ &= \begin{cases} -\tfrac{1}{4} + \tfrac{1}{2}x + \tfrac{17}{4}e^{-2x} & 0 \le x < 3 \\ x - 1 + \tfrac{17}{4}e^{-2x} - \tfrac{3}{4}e^{-2(x-3)} & x \ge 3 \end{cases} \end{aligned}$$

The graph of y is



Example 2. Find the solution of the initial-value problem

$$y'' + 4y = f(x); \quad y(0) = 1, \; y'(0) = 0$$

where $f(x) = \begin{cases} 1 & 0 \leq x < \pi/2 \\ -2 & x \geq \pi/2. \end{cases}$

SOLUTION The function f has a jump discontinuity at $x = \pi/2$. Therefore, we express f in terms of $u(x - \pi/2)$:

$$f(x) = 1 - 3u(x - \pi/2).$$

Applying the Laplace transform to the equation and using the initial conditions, we get

$$\begin{aligned} \mathcal{L}[y'' + 4y] = s^2\mathcal{L}[y] - sy(0) - y'(0) + 4\mathcal{L}[y] &= \frac{1}{s} - \frac{3e^{-\pi s/2}}{s} \\ (s^2 + 4)\mathcal{L}[y] - s &= \frac{1}{s} - \frac{3e^{-\pi s/2}}{s}. \end{aligned}$$

Solving for $\mathcal{L}[y] = Y(s)$, we have

$$Y(s) = \frac{1}{s(s^2 + 4)} - 3e^{-\pi s/2}\frac{1}{s(s^2 + 4)} + \frac{s}{s^2 + 4}.$$

By partial fraction decomposition,

$$\frac{1}{s(s^2 + 4)} = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2 + 4}.$$

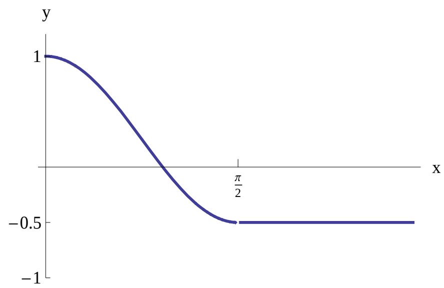
Therefore,

$$Y(s) = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2 + 4} - \frac{3}{4}\frac{e^{-\pi s/2}}{s} + \frac{3}{4}e^{-\pi s/2}\frac{s}{s^2 + 4} + \frac{s}{s^2 + 4}$$

and

$$\begin{aligned} y(x) = \mathcal{L}^{-1}[Y(s)] &= \tfrac{1}{4} - \tfrac{1}{4}\cos 2x - \tfrac{3}{4}u(x - \pi/2) + \tfrac{3}{4}\cos 2(x - \pi/2)u(x - \pi/2) + \cos 2x \\ &= \begin{cases} \tfrac{1}{4} + \tfrac{3}{4}\cos 2x & 0 \leq x < \pi/2 \\ -\tfrac{1}{2} & x \geq \pi/2. \end{cases} \end{aligned}$$

The graph of y is



Exercises 4.5

Solve the initial-value problem.

1. $y' - 2y = f(x); y(0) = 2$ where

$$f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 2 & x \ge 1 \end{cases}.$$

2. $y' + 3y = f(x); y(0) = 1$ where

$$f(x) = \begin{cases} \sin x & 0 \le x < \pi \\ 0 & x \ge \pi \end{cases}.$$

3. $y'' + y = f(x); \quad y(0) = 0,\ y'(0) = 1$ where

$$f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & x \ge 1 \end{cases}.$$

4. $y'' + 4y = f(x); \quad y(0) = 1,\ y'(0) = 0$ where

$$f(x) = \begin{cases} 0 & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}.$$

5. $y'' + 2y' + y = f(x); \quad y(0) = 0,\ y'(0) = 0$ where

$$f(x) = \begin{cases} 1 & 0 \le x < 2 \\ x - 1 & x \ge 2 \end{cases}.$$

6. $y'' + 4y = f(x); \quad y(0) = 0,\ y'(0) = 0$ where

$$f(x) = \begin{cases} \sin x & 0 \le x < 2\pi \\ 0 & x \ge 2\pi \end{cases}.$$

7. $y'' - 4y' + 3y = f(x); \quad y(0) = 0,\ y'(0) = 0$ where

$$f(x) = \begin{cases} -1 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}.$$

8. $y'' - 3y' + 2y = f(x); \quad y(0) = 0,\ y'(0) = 0$ where

$$f(x) = \begin{cases} 0 & 0 \le x < 2 \\ 2x - 4 & x \ge 2 \end{cases}.$$

9. $y'' + 2y' + y = f(x); \quad y(0) = 3,\ y'(0) = -1$ where

$$f(x) = \begin{cases} e^x & 0 \le x < 1 \\ e^x - 1 & x \ge 1 \end{cases}.$$

10. $y'' - 4y' + 4y = f(x); \quad y(0) = 1,\ y'(0) = -1$ where

$$f(x) = \begin{cases} e^{2x} & 0 \le x < 2 \\ -e^{2x} & x \ge 2 \end{cases}.$$