Determinants

Associated with each $n \times n$ matrix A is a number called its *determinant*. We will give an inductive development of this concept, beginning with the determinant of a 2×2 matrix. Then we'll express a 3×3 determinant as a sum of 2×2 determinants, a 4×4 determinant as a sum of 3×3 determinants, and so on.

Consider the system of two linear equations in two unknowns

$$\begin{aligned} ax + by &= \alpha \\ cx + dy &= \beta \end{aligned}$$

We eliminate the y unknown by multiplying the first equation by d, the second equation by $-b$, and adding. This gives

$$(ad - bc)x = d\alpha - b\beta.$$

This equation has the solution $x = \dfrac{d\alpha - b\beta}{ad - bc}$, provided $ad - bc \neq 0$.

Similarly, we can solve the system for the y unknown by multiplying the first equation by $-c$, the second equation by a, and adding. This gives

$$(ad - bc)y = a\beta - c\alpha$$

which has the solution $y = \dfrac{a\beta - c\alpha}{ad - bc}$, again provided $ad - bc \neq 0$.

The matrix of coefficients of the system is $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The number $ad - bc$ is called the *determinant of* A. The determinant of A is denoted by $\det A$ and by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

The determinant has a geometric interpretation in this 2×2 case.

The graph of the first equation $ax + by = \alpha$ is a straight line with slope $-a/b$, provided $b \neq 0$. The graph of the second equation is a straight line with slope $-c/d$, provided $d \neq 0$. (If $b = 0$ or $d = 0$, then the corresponding line is vertical.) Assume that $b,\ d \neq 0$. If

$$\frac{-a}{b} \neq \frac{-c}{d},$$

then the lines have different slopes and the system of equations has a unique solution. However,

$$\frac{-a}{b} \neq \frac{-c}{d} \quad \text{is equivalent to} \quad ad - bc \neq 0.$$

Thus, $\det A \neq 0$ implies that the system has a unique solution.

On the other hand, if $ad - bc = 0$, then $\dfrac{-a}{b} = \dfrac{-c}{d}$ (assuming $b,\ d \neq 0$), and the two lines have the same slope. In this case, the lines are either parallel (the system has no solutions), or the lines coincide (the system has infinitely many solutions).

In general, an $n \times n$ matrix A is said to be *nonsingular* if $\det A \neq 0$; A is *singular* if $\det A = 0$.

Look again at the solutions

$$x = \frac{d\alpha - b\beta}{ad - bc}, \quad y = \frac{a\beta - c\alpha}{ad - bc}, \qquad ad - bc \neq 0.$$

The two numerators also have the form of a determinant of a 2×2 matrix. In particular, these solutions can be written as

$$x = \frac{\begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

This representation of the solutions of a system of two equations in two unknowns is the $n = 2$ version of a general result known as *Cramer's rule.*

Example 1. Given the system of equations

$$\begin{aligned} 5x - 2y &= 8 \\ 3x + 4y &= 10 \end{aligned}$$

Verify that the determinant of the matrix of coefficients is nonzero and solve the system using Cramer's rule.

SOLUTION The matrix of coefficients is $A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$ and $\det A = 26$. According to Cramer's rule,

$$x = \frac{\begin{vmatrix} 8 & -2 \\ 10 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ 2 & 4 \end{vmatrix}} = \frac{52}{26} = 2, \quad y = \frac{\begin{vmatrix} 5 & 8 \\ 3 & 10 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ 2 & 4 \end{vmatrix}} = \frac{26}{26} = 1$$

The solution set is $x = 2$, $y = 1$. ■

Now we'll go to 3×3 matrices.

The determinant of a 3×3 matrix

If

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

then

$$\det A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

The problem with this definition is that it is hard to remember. Fortunately the expression on the right can be written conveniently in terms of 2×2 determinants as follows:

$$\begin{aligned}
\det A &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\
&= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \\
&= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}
\end{aligned}$$

This representation of a 3×3 determinant is called the *expansion of the determinant across the first row.* Notice that the coefficients are the entries $a_1,\ a_2,\ a_3$ of the first row, that they occur alternately with $+$ and $-$ signs, and that each is multiplied by a 2×2 determinant. You can remember the determinant that goes with each entry a_i as follows: in the original matrix, mentally cross out the row and column containing a_i and take the determinant of the 2×2 matrix that remains.

Example 2. Let $A = \begin{pmatrix} 3 & -2 & -4 \\ 2 & 5 & -1 \\ 0 & 6 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 6 & 5 \\ 1 & 2 & 1 \\ 3 & -2 & 1 \end{pmatrix}$. Calculate $\det A$ and $\det B$.

SOLUTION

$$\begin{aligned}
\det A &= 3\begin{vmatrix} 5 & -1 \\ 6 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + (-4)\begin{vmatrix} 2 & 5 \\ 0 & 6 \end{vmatrix} \\
&= 3[(5)(1) - (-1)(6)] + 2[(2)(1) - (-1)(0)] - 4[(2)(6) - (5)(0)] \\
&= 3(11) + 2(2) - 4(12) = -11
\end{aligned}$$

$$\begin{aligned}
\det B &= 7\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} - 6\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} \\
&= 7[(2)(1) - (1)(-2)] - 6[(1)(1) - (1)(3)] + 5[(1)(-2) - (2)(3)] \\
&= 7(4) - 6(-2) + 5(-8) = 0. \qquad \blacksquare
\end{aligned}$$

There are other ways to group the terms in the definition. For example

$$\begin{aligned}
\det A &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\
&= -a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1c_3 - a_3c_1) \\
&= -a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + -c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}
\end{aligned}$$

This is called the *expansion of the determinant down the second column.*

In general, depending on how you group the terms in the definition, you can expand across any row or down any column. The signs of the coefficients in the expansion across a row or down a column are alternately $+$, $-$, starting with a $+$ in the (1,1)-position. The pattern of signs is:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Example 3. Let $A = \begin{pmatrix} 3 & -2 & -4 \\ 2 & 5 & -1 \\ 0 & 6 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 7 & 0 & 5 \\ 1 & 0 & 1 \\ 3 & -2 & 1 \end{pmatrix}$.

1. Calculate $\det A$ by expanding down the first column.

$$\begin{aligned} \det A &= 3\begin{vmatrix} 5 & -1 \\ 6 & 1 \end{vmatrix} - 2\begin{vmatrix} -2 & -4 \\ 6 & 1 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & -1 \end{vmatrix} \\ &= 3[(5)(1) - (-1)(6)] - 2[(-2)(1) - (-4)(6)] + 0 \\ &= 3(11) - 2(22) + 0 = -11 \end{aligned}$$

2. Calculate $\det A$ by expanding across the third row.

$$\begin{aligned} \det A &= 0\begin{vmatrix} -2 & -4 \\ 5 & -1 \end{vmatrix} - 6\begin{vmatrix} 3 & -4 \\ 2 & -1 \end{vmatrix} + (1)\begin{vmatrix} 3 & -2 \\ 2 & 5 \end{vmatrix} \\ &= 0 - 6[(3)(-1) - (-4)(2)] + [(3)(5) - (-2)(2)] \\ &= -6(5) + (19) = -11 \end{aligned}$$

3. Calculate $\det C$ by expanding down the second column.

$$\begin{aligned} \det C &= -0\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 7 & 5 \\ 3 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 7 & 5 \\ 1 & 1 \end{vmatrix} \\ &= 0 + 0 + 2(2) = 14 \end{aligned}$$

Notice the advantage of expanding across a row or down a column that contains one or more zeros. ■

Now consider the system of three equations in three unknowns

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_2 \end{aligned}$$

Writing this system in vector-matrix form, we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

It can be shown that if $\det A \neq 0$, then the system has a unique solution which is given by

$$x_1 = \frac{\det A_1}{\det A}, \qquad x_2 = \frac{\det A_2}{\det A}, \qquad x_3 = \frac{\det A_3}{\det A}$$

where

$$A_1 = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}, \quad \text{and} \quad A_3 = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}$$

This is Cramer's rule in the 3×3 case.

If $\det A = 0$, then the system either has infinitely many solutions or no solutions.

Example 4. Given the system of equations

$$\begin{aligned} 2x + y - z &= 3 \\ x + y + z &= 1 \\ x - 2y - 3z &= 4 \end{aligned}.$$

Verify that the determinant of the matrix of coefficients is nonzero and find the value of y using Cramer's rule.

SOLUTION The matrix of coefficients is $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{pmatrix}$ and $\det A = 5$.

According to Cramer's rule

$$y = \frac{\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix}}{5} = \frac{-5}{5} = -1. \qquad \blacksquare$$