DIFFERENTIAL EQUATIONS

Basic Terminology

A **differential equation** is an equation that contains an unknown function together with one or more of its derivatives.

Examples

1. $y' = 2x + \cos x$

2. $\dfrac{dy}{dt} = ky$

3. $x^2y'' - 2xy' + 2y = 4x^3$

4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

5. $\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} = 0$

TYPE:

If the unknown function depends on a single independent variable, then the equation is an **ordinary differential equation**; if the unknown function depends on more than one independent variable, then the equation is a **partial differential equation**.

ORDER:

The **order** of a differential equation is the order of the highest derivative of the unknown function appearing in the equation.

Examples:

1. $$\frac{dy}{dt} = ky$$

2. $$x^2y'' - 2xy' + 2y = 4x^3$$

3. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

4. $$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$$

5. $$y'' + 2x \sin y' + 3e^{xy} = \frac{d^3}{dx^3}(e^{2x})$$

SOLUTION:

A **solution of a differential equation** is a function defined on some interval I (in the case of an ordinary differential equation) or on some domain D in two or higher dimensional space (in the case of a partial differential equation) with the property that the equation reduces to an identity when the function is substituted into the equation.

Examples:

1. $y' = 2x + \cos x$

$y = x^2 + \sin x,$

$y = x^2 + \sin x + C$

2. $y' = ky$

$y = e^{kt}, \quad y = Ce^{kt}$

3. $x^2y'' - 2xy' + 2y = 4x^3$

$y = x^2 + 2x^3$;

$y = 2x + x^2$ not a solution.

4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$u = \ln\sqrt{x^2 + y^2}$,

$u = \cos x \sinh y$

$u = \cosh x \sinh y$ is not a solution

n-PARAMETER FAMILY OF SOLUTIONS:

Intuitively, to find a set of solutions of an n-th order differential equation we “integrate” n times, with each integration step producing an arbitrary constant of integration. Thus, ”in theory,” an n-th order differential equation has an **n-parameter family of solutions**.

SOLVING A DIFFERENTIAL EQUATION:

To **solve** an n-th order differential equation means to find an n-parameter family of solutions. (Same n.)

Examples: n-parameter family of solutions:

1. $y' = 3x^2 - 2x + 4$

Answer: $y = x^3 - x^2 + 4x + C$

2. $y'' = 2x + \sin 2x$

Answer:

$$y = \frac{1}{3}x^3 - \frac{1}{4}\sin 2x + C_1 x + C_2$$

3. $y''' - 3y'' + 3y' - y = 0$

Answer: $y = C_1 e^x + C_2 x e^x + C_3 x^2 e^x$

4. $x^2 y'' - 2xy' + 2y = 4x^3$

Answer: $y = C_1 x + C_2 x^2 + 2x^3$

GENERAL SOLUTION/SINGULAR SOLUTIONS:

An "n-parameter family of solutions" is also called the **general solution**.

Solutions of an n-th order differential equation which are not included in an n-parameter family of solutions are called **singular solutions**.

Example:

$$\frac{dy}{dx} = (4x + 2)\sqrt{y - 2}$$

General solution:

$$\sqrt{y - 2} = x^2 + x + C$$

Singular solution: $y \equiv 2$

PARTICULAR SOLUTION:

If specific values are assigned to the arbitrary constants in the general solution of a differential equation, then the resulting solution is called a **particular solution** of the equation.

Examples:

1. $x^2y'' - 2xy' + 2y = 4x^3$

General solution:

$$y = C_1x + C_2x^2 + 2x^3$$

Particular solutions:

$$y = 2x^3 \quad (C_1 = C_2 = 0)$$

$$y = 3x - 2x^2 + 2x^3 \quad (C_1 = 3,\ C_2 = -2$$

$$\frac{dy}{dx} = (4x+2)\sqrt{y-2}$$

General solution:

$$\sqrt{y-2} = x^2 + x + C$$

Particular solutions:

$$\sqrt{y-2} = x^2 + x \quad (C = 0)$$

$$\sqrt{y-2} = x^2 + x - 5 \quad (C = -5)$$

Singular solution: $y \equiv 2$

THE DIFFERENTIAL EQUATION OF AN n-PARAMETER FAMILY:

Given an n-parameter family of curves. The **differential equation of the family**, is an n-th order differential equation that has the given family as its general solution.

Strategy for finding the differential equation

Step 1. Differentiate the family n times. This produces a system of $n+1$ equations.

Step 2 Choose any n of the equations and solve for the parameters.

Step 3. Substitute the "values" for the constants in the remaining equation.

Examples:

1. $y^2 = Cx^3 + 4$

Answer: $2xyy' = 12 - 3y^2$

or $$y' = \frac{12 - 3y^2}{2xy}$$

2. $y = C_1x + C_2x^3$

Answer: $x^2y'' - 3xy' + 3y = 0$

3. $y = C_1 e^{2x} + C_2 e^{3x} + C_3$

Answer: $y''' - 5y'' + 6y' = 0$

4. $y = C_1 \cos 3x + C_2 \sin 3x$

Answer: $y'' + 9y = 0$

n-th ORDER INITIAL-VALUE PROBLEMS:

Examples:

1. Find a solution of

$$y' = 3x^2 + 2x + 1$$

which passes through the point $(-2, 4)$.

Answer: $y = x^3 + x^2 + x + 10$

2. Given the differential equation

$$y'' + 9y = 0.$$

a. Find a solution which satisfies

$y(0) = 3$

Answer:

$y = C_1 \sin 3x + 3\cos 3x$ for <u>any</u> C_1.

b. Find a solution which satisfies

$$y(0) = 4, \; y(\pi) = 4$$

Answer: No solution!!

c. Find a solution which satisfies

$$y(\pi/4) = 1,\ y'(\pi/4) = 2$$

Answer: $$\frac{1}{3\sqrt{2}} \sin 3x - \frac{5}{3\sqrt{2}} \cos 3x$$

An **n-th order initial-value problem** consists of an n-th order differential equation

$$F\left[x, y, y', y'', \ldots, y^{(n)}\right] = 0$$

together with n (initial) conditions of the form

$$y(c) = k_0,\ y'(c) = k_1,\ y''(c) = k_2,\ \ldots,$$

$$y^{(n-1)}(c) = k_{n-1}$$

where c and $k_0,\ k_1,\ \ldots,\ k_{n-1}$ are given numbers.

NOTES:

1. An n-th order differential equation can be written in the form

$$F\left[x, y, y', y'', \cdots, y^{(n)}\right] = 0$$

by bringing all the terms to the left-hand side of the equation.

2. The initial conditions determine a particular solution of the differential equation.

Strategy for Solving an Initial-Value Problem:

Step 1. Find the general solution of the differential equation.

Step 2. Use the initial conditions to solve for the arbitrary constants in the general solution.

Examples:

Given that $y = C_1 x + C_2 x^3$ is the general solution of

$$x^2 y'' - 3xy' + 3y = 0$$

a. find the solution which satisfies

$$y(1) = 2, \quad y'(1) = -4.$$

Answer: $y = 5x - 3x^3$

b. find the solution that satisfies

$$y(0) = 0, \quad y'(0) = 2.$$

Answer: $y = 2x + C_2 x^3$ for any C_2

c. find the solution that satisfies

$$y(0) = 2, \quad y'(0) = 3.$$

Answer: No solution.

EXISTENCE AND UNIQUENESS:

The fundamental questions in a course on differential equations are:

1. Does a given initial-value problem *have* a solution? That is, do solutions to the problem *exist*?

2. If a solution does exist, is it *unique*? That is, is there exactly one solution to the problem or is there more than one solution?