

Systems of Linear Differential Equations

I. Converting a linear equation to a linear system

Consider the third-order equation

$$y''' + p(t)y'' + q(t)y' + r(t)y = f(t)$$

or

$$y''' = -r(t)y - q(t)y' - p(t)y'' + f(t).$$

Introduce new dependent variables

x_1, x_2, x_3 , as follows:

$$x_1 = y$$

$$x_2 = x'_1 (= y')$$

$$x_3 = x'_2 (= y'')$$

Then

$$y''' = x'_3 = -r(t)x_1 - q(t)x_2 - p(t)x_3 + f(t)$$

The third-order equation can be written equivalently as the system of three first-order equations:

$$x'_1 = x_2$$

$$x'_2 = x_3$$

$$x'_3 = -r(t)x_1 - q(t)x_2 - p(t)x_3 + f(t).$$

Note that this system is just a very special case of the “general” system of three, first-order differential equations:

$$\begin{aligned}x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + a_{13}(t)x_3(t) + b_1(t) \\x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + a_{23}(t)x_3(t) + b_2(t) \\x_3' &= a_{31}(t)x_1 + a_{32}(t)x_2 + a_{33}(t)x_3(t) + b_3(t).\end{aligned}$$

Example 1:

$$y''' - 3y'' - 4y' + 12y = 3e^t.$$

or

$$y''' = -12y + 4y' + 3y'' + 3e^t.$$

Set

$$x_1 = y, \quad x'_1 = x_2 (= y'), \quad x'_2 = x_3 (= y'').$$

Then

$$y''' = x'_3 = -12x_1 + 4x_2 + 3x_3 + 3e^t$$

Equivalent system:

$$x'_1 = x_2$$

$$x'_2 = x_3$$

$$x'_3 = -12x_1 + 4x_2 + 3x_3 + 3e^t.$$

II. General Linear Systems

Let $a_{11}(t), a_{12}(t), \dots, a_{nn}(t), b_1(t), b_2(t), \dots$ be continuous functions on the interval I . The system of n first-order differential equations

$$x'_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t)$$

$$x'_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t)$$

⋮

⋮

$$x'_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t)$$

is called a *first-order linear differential system*.

The system is **homogeneous** if

$$b_1(t) \equiv b_2(t) \equiv \cdots \equiv b_n(t) \equiv 0 \quad \text{on } I.$$

It is **nonhomogeneous** if the functions $b_i(t)$ are not all identically zero on I .

Set

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

The system can be written in the
vector-matrix form

$$\mathbf{x}' = A(t) \mathbf{x} + \mathbf{b}(t). \quad (\text{S})$$

The matrix $A(t)$ is called the **matrix of coefficients** or the **coefficient matrix**.

Example 2:

$$y''' - 3y'' - 4y' + 12y = 3e^t.$$

Set

$$x_1 = y, \quad x'_1 = x_2 (= y'), \quad x'_2 = x_3 (= y'').$$

Equivalent system:

$$x'_1 = x_2$$

$$x'_2 = x_3$$

$$x'_3 = -12x_1 + 4x_2 + 3x_3 + 3e^t.$$

Vector-matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3e^t \end{pmatrix}$$

A **solution** of the linear differential system (S) is a differentiable vector function

$$\mathbf{v} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

that satisfies (S) on the interval I .

Example 3:

$$y''' - 3y'' - 4y' + 12y = 3e^t.$$

Fundamental set:

$$\{y_1 = e^{3t}, \quad y_2 = e^{2t}, \quad y_3 = e^{-2t}\}$$

Particular solution: $z = \frac{1}{2}e^t$

Corresponding solutions of the system:

$$\mathbf{x}_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} + \begin{pmatrix} e^t/2 \\ e^t/2 \\ e^t/2 \end{pmatrix}, \quad \mathbf{x}_2 = ? \quad \mathbf{x}_3 = ?$$

THEOREM (Existence and Uniqueness Theorem) Let a be any point on the interval I , and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be any n real numbers. Then the initial-value problem

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{b}(t), \quad \mathbf{x}(a) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

has a unique solution.

III. Homogeneous Systems: General Theory:

$$x'_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n(t)$$

$$x'_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n(t)$$

$$\vdots \qquad \qquad \qquad \vdots$$

$$x'_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n(t)$$

$$\mathbf{x}' = A(t)\mathbf{x}. \qquad \qquad \qquad (\text{H})$$

Note: The zero vector $\mathbf{z}(t) \equiv \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a solution of (H). This solution is called the **trivial solution**.

THEOREM v is a solution of (H) and α is any real number, then $u = \alpha v$ is also a solution of (H); any constant multiple of a solution of (H) is a solution of (H).

THEOREM If v_1 and v_2 are solutions of (H), then $u = v_1 + v_2$ is also a solution of (H); the sum of any two solutions of (H) is a solution of (H).

In general,

THEOREM If v_1, v_2, \dots, v_k are solutions of (H), and if c_1, c_2, \dots, c_k are real numbers, then

$$c_1v_1 + c_2v_2 + \dots + c_kv_k$$

is a solution of (H); any linear combination of solutions of (H) is also a solution of (H).

Linear Dependence/Independence

– in general

Let

$$\mathbf{v}_1(t) = \begin{pmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{pmatrix}, \quad \mathbf{v}_2(t) = \begin{pmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2} \end{pmatrix},$$

$$\dots, \quad \mathbf{v}_k(t) = \begin{pmatrix} v_{1k} \\ v_{2k} \\ \vdots \\ v_{nk} \end{pmatrix}$$

be vector functions defined on some interval I .

The vectors are **linearly dependent** on I if there exist k real numbers c_1, c_2, \dots, c_k , not all zero, such that

$$c_1\mathbf{v}_1(t) + c_2\mathbf{v}_2(t) + \dots + c_k\mathbf{v}_k(t) \equiv 0 \quad \text{on } I.$$

Otherwise the vectors are **linearly independent** on I .

THEOREM Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be n, n -component vector functions defined on an interval I . If the vectors are linearly dependent, then

$$\begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{vmatrix} \equiv 0 \quad \text{on } I.$$

This determinant is called the **Wronskian** of the vector functions $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

Special case: n solutions of (H)

THEOREM Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be n solutions of (H). Exactly one of the following holds:

1. $W(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)(t) \equiv 0$ on I and the solutions are linearly dependent.
2. $W(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)(t) \neq 0$ for all $t \in I$ and the solutions are linearly independent.

THEOREM Let v_1, v_2, \dots, v_n be n linearly independent solutions of (H). Let u be *any* solution of (H). Then there exists a unique set of constants c_1, c_2, \dots, c_n such that

$$u = c_1 v_1 + c_2 v_2 + \dots + c_n v_n.$$

That is, every solution of (H) can be written as a unique linear combination of v_1, v_2, \dots, v_n .

A set of n linearly independent solutions of (H) is called a **fundamental set of solutions**. A fundamental set is also called a **solution basis** for (H).

Let v_1, v_2, \dots, v_n be a fundamental set of solutions of (H). Then

$$\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n,$$

c_1, c_2, \dots, c_n arbitrary constants, is the **general solution** of (H).

Example

$$\mathbf{x}_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

$$\mathbf{x}_3 = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \end{pmatrix}$$

is a fundamental set of solutions of

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

IV. Homogeneous Systems with Constant Coefficients

$$x'_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$x'_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

\vdots \vdots

$$x'_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

where $a_{11}, a_{12}, \dots, a_{nn}$ are constants.

The system in vector-matrix form is

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

or

$$\mathbf{x}' = A\mathbf{x}.$$

Solutions of $\mathbf{x}' = A\mathbf{x}$:

Example:

$$\mathbf{x}_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

is a solution of

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Note that

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 27 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

3 is an eigenvalue of A and

$$\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

is a corresponding eigenvector.

Given the homogeneous system with constant coefficients

$$\mathbf{x}' = A\mathbf{x}.$$

If λ is an eigenvalue of A and \mathbf{v} is a corresponding eigenvector, then

$$\mathbf{x} = e^{\lambda t} \mathbf{v}$$

is a solution.

If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$, then

$$\mathbf{x}_1 = e^{\lambda_1 t} \mathbf{v}_1, \mathbf{x}_2 = e^{\lambda_2 t} \mathbf{v}_2, \dots, \mathbf{x}_k = e^{\lambda_k t} \mathbf{v}_k$$

are linearly independent solutions of

$$\mathbf{x}' = A\mathbf{x}.$$

Examples:

1. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{x}.$$

Step 1. Find the eigenvalues of A :

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 4 - \lambda & -1 \\ 2 & 1 - \lambda \end{vmatrix} \\ &= \lambda^2 - 5\lambda + 6. \end{aligned}$$

Characteristic equation:

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0.$$

Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 3$.

Step 2. Find the eigenvectors:

$$\lambda_1 = 2, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \lambda_2 = 3, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solutions:

Fundamental set of solution vectors:

$$\mathbf{x}_1 = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution of the system

is:

$$\mathbf{x} = C_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

2. Solve $\mathbf{x}' = \begin{pmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{pmatrix} \mathbf{x}$.

Step 1. Find the eigenvalues of A :

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3 - \lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1 - \lambda \end{vmatrix} \\ &= -\lambda^3 + 2\lambda^2 + \lambda - 2. \end{aligned}$$

Characteristic equation:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 2)(\lambda - 1)(\lambda + 1) = 0.$$

Eigenvalues:

$$\lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = -1.$$

Step 2. Find the eigenvectors:

$$\lambda_1 = 2 : \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$$

$$\lambda_2 = 1 : \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix},$$

$$\lambda_3 = -1 : \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

Fundamental set of solution vectors:

$$\mathbf{x}_1 = e^{2t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = e^t \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix},$$

$$\mathbf{x}_3 = e^{-t} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

The general solution of the system
is:

$$\mathbf{x} = C_1 e^{2t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + C_2 e^t \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

3. Solve

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

To find the solution vector satisfying the initial condition, solve

$$C_1 \mathbf{v}_1(0) + C_2 \mathbf{v}_2(0) + C_3 \mathbf{v}_3(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

which is:

$$C_1 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 3 & 1 \\ -1 & -1 & 2 \\ 2 & 7 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Solution:

$$C_1 = 3, \quad C_2 = -1, \quad C_3 = 1.$$

The solution of the initial-value problem is:

$$\mathbf{x} = 3e^{2t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - e^t \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + e^{-t} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

Two difficulties:

1. A has complex eigenvalues and complex eigenvectors.
2. A has an eigenvalue of multiplicity greater than 1.

Complex eigenvalues/eigenvectors

4. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix} \mathbf{x}.$$

$$\det(A - \lambda I) = \begin{vmatrix} -3 - \lambda & -2 \\ 4 & 1 - \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5.$$

The eigenvalues are:

$$\lambda_1 = -1 + 2i, \quad \lambda_2 = -1 - 2i.$$

For $\lambda_1 = -1 + 2i$: Solve

$$\left(\begin{array}{cc|c} -2 - 2i & -2 & 0 \\ 4 & 2 - 2i & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 + i & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

The solution set is:

$$x_2 = -(1 + i)x_1, \quad x_1 \text{ arbitrary}$$

Set $x_1 = 1$. Then, for $\lambda_1 = -1 + 2i$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 - i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

and, for $\lambda_2 = -1 - 2i$:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Solutions

$$\begin{aligned} \mathbf{u}_1 &= e^{(-1+2i)t} \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\ &= e^{-t} (\cos 2t + i \sin 2t) \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\ &= e^{-t} \left[\cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] + \\ &\quad i e^{-t} \left[\cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]. \end{aligned}$$

$$\begin{aligned}
\mathbf{u}_2 &= e^{(-1-2i)t} \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} - i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\
&= e^{-t} (\cos 2t + i \sin 2t) \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\
&= e^{-t} \left[\cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] - \\
&\quad i e^{-t} \left[\cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right].
\end{aligned}$$

Fundamental set:

$$\begin{aligned} \mathbf{x}_1 &= \frac{\mathbf{u}_1 + \mathbf{u}_2}{2} \\ &= e^{-t} \left[\cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \end{aligned}$$

$$\begin{aligned} \mathbf{x}_2 &= \frac{\mathbf{u}_1 + \mathbf{u}_2}{2i} \\ &= e^{-t} \left[\cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \end{aligned}$$

General solution:

$$\begin{aligned} \mathbf{x} &= C_1 e^{-t} \left[\cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] + \\ &C_2 e^{-t} \left[\cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \end{aligned}$$

5. Determine a fundamental set of solution vectors of

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 & -1 \\ 3 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \mathbf{x}.$$

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -4 & -1 \\ 3 & 2 - \lambda & 3 \\ 1 & 1 & 3 - \lambda \end{vmatrix} =$$

$$-\lambda^3 + 6\lambda^2 - 21\lambda + 26 = -(\lambda - 2)(\lambda^2 - 4\lambda + 13).$$

The eigenvalues are:

$$\lambda_1 = 2, \quad \lambda_2 = 2 + 3i, \quad \lambda_3 = 2 - 3i.$$

$$\text{For } \lambda_1 : \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For $\lambda_2 = 2 + 3i$: Solve

$$\left(\begin{array}{ccc|c} -1 - 3i & -4 & -1 & 0 \\ 3 & -3i & 3 & 0 \\ 1 & 1 & 1 - 3i & 0 \end{array} \right) \rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} - \frac{3}{2}i & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The solution set is:

$$x_1 = \left(-\frac{5}{2} + \frac{3}{2}i \right) x_3, \quad x_2 = \left(\frac{3}{2} + \frac{3}{2}i \right) x_3, \quad x_3,$$

x_3 arbitrary.

Set $x_3 = 2$. Then

$$\mathbf{v}_2 = \begin{pmatrix} -5 + 3i \\ 3 + 3i \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}.$$

and

$$\mathbf{v}_3 = \begin{pmatrix} -5 - 3i \\ 3 - 3i \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}.$$

Now

$$e^{(2+3i)t} \left[\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \right]$$

$$= e^{2t} (\cos 3t + i \sin 3t) \left[\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \right]$$

$$= e^{2t} \left[\cos 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \right]$$

$$+ i e^{2t} \left[\cos 3t \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} \right].$$

Fundamental set of solution vectors:

$$\mathbf{v}_1 = e^{2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$\mathbf{v}_2 = e^{2t} \left[\cos 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \right],$$

$$\mathbf{v}_3 = e^{2t} \left[\cos 3t \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} \right].$$

Repeated eigenvalues

6. Find a fundamental set of solutions of

$$\mathbf{x}' = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} \mathbf{x}.$$

$$\begin{aligned} \det(A - \lambda I_3) &= \begin{vmatrix} 1 - \lambda & -3 & 3 \\ 3 & -5 - \lambda & 3 \\ 6 & -6 & 4 - \lambda \end{vmatrix} \\ &= 16 + 12\lambda - \lambda^3 = -(\lambda - 4)(\lambda + 2)^2. \end{aligned}$$

The eigenvalues of A are:

$$\lambda_1 = 4, \lambda_2 = \lambda_3 = -2.$$

$$\lambda_1 = 4 : \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

$$\lambda_2 = \lambda_3 = -2:$$

$$[A - (-2)I_3]\mathbf{x} = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solve

$$\left(\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{array} \right)$$

which row reduces to

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Solution set:

$$x_1 = a - b, \quad x_2 = a, \quad x_3 = b$$

a, b any real numbers.

$$\text{Set } a = 1, b = 0 : \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$$

$$\text{Set } a = 0, b = -1 : \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Fundamental set:

$$\left\{ e^{4t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

7. Find a fundamental set of solutions of

$$\mathbf{x}' = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix} \mathbf{x}.$$

$$\begin{aligned} \det(A - \lambda I_3) &= \begin{vmatrix} 5 - \lambda & 6 & 2 \\ 0 & -1 - \lambda & -8 \\ 1 & 0 & -2 - \lambda \end{vmatrix} \\ &= -36 + 15\lambda + 2\lambda^2 - \lambda^3 = -(\lambda + 4)(\lambda - 3)^2. \end{aligned}$$

$$\lambda_1 = -4, \quad \lambda_2 = \lambda_3 = 3.$$

$$\lambda_1 = -4 : \quad \mathbf{v}_1 = \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}.$$

$\lambda_2 = \lambda_3 = 3$: Solve

$$(A - 3I_3)\mathbf{x} = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -4 & -8 \\ 1 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|c} 2 & 6 & 2 & 0 \\ 0 & -4 & -8 & 0 \\ 1 & 0 & -5 & 0 \end{array} \right)$$

which row reduces to

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Solution set:

$$x_1 = 5a, \quad x_2 = -2a, \quad x_3 = a,$$

a any real number.

$$\text{Set } a = 1: \quad \mathbf{v}_2 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Problem: Only one eigenvector!

Solutions:

$$\mathbf{x}_1 = e^{-4t} \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}, \quad \mathbf{x}_2 = e^{3t} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}.$$

$$8. \quad y''' + y'' - 8y' - 12y = 0$$

Char.eqn.

$$r^3 + r^2 - 8r - 12 = (r - 3)(r + 2)^2 = 0.$$

Fundamental set: $\{e^{3t}, e^{-2t}, te^{-2t}\}$

$$\text{Equivalent system: } \mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & 8 & -1 \end{pmatrix} \mathbf{x}$$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 12 & 8 & -1 - \lambda \end{vmatrix} \\ &= -\lambda^3 - \lambda^2 + 8\lambda + 12\lambda \end{aligned}$$

Characteristic eqn.

$$\lambda^3 + \lambda^2 - 8\lambda - 12 = (\lambda - 3)(\lambda + 2)^2 = 0$$

Fundamental set:

$$\mathbf{v}_1 = e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \quad \mathbf{v}_2 = e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix},$$

$$\mathbf{v}_3 = e^{-2t} \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + te^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

Question: What is the vector $\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$
?

$$\begin{aligned} [A - (-2)I] \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} &= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 12 & 8 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$$

is called a **generalized eigenvector**.

7. continued. Solve

$$(A - 3I)\mathbf{w} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 6 & 2 & 5 \\ 0 & -4 & -8 & -2 \\ 1 & 0 & -5 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Solution set:

$$x_1 = 1 + 5a, \quad x_2 = \frac{1}{2} - 2a, \quad x_3 = a,$$

a any real number.

$$\text{Set } a = 0: \quad \mathbf{w} = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$$

Fundamental set:

$$\mathbf{v}_1 = e^{-4t} \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = e^{3t} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix},$$

$$\mathbf{v}_3 = e^{3t} \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} + te^{3t} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Eigenvalues of multiplicity 2

Given $\mathbf{x}' = A\mathbf{x}$.

Suppose that A has an eigenvalue λ of multiplicity 2. Then exactly one of the following holds:

1. λ has two linearly independent eigenvectors, \mathbf{v}_1 and \mathbf{v}_2 . Corresponding linearly independent solution vectors of the differential system are

$$\mathbf{x}_1(t) = e^{\lambda t} \mathbf{v}_1 \quad \text{and} \quad \mathbf{x}_2(t) = e^{\lambda t} \mathbf{v}_2.$$

(See Example 6.)

2. λ has only one (independent) eigenvector \mathbf{v} . (See Examples 7 and 8.) Then a linearly independent pair of solution vectors is:

$$\mathbf{x}_1(t) = e^{\lambda t} \mathbf{v} \quad \text{and} \quad \mathbf{x}_2(t) = e^{\lambda t} \mathbf{w} + t e^{\lambda t} \mathbf{v}$$

where \mathbf{w} is a vector that satisfies

$$(A - \lambda I)\mathbf{w} = \mathbf{v}.$$

The vector \mathbf{w} is called a **generalized eigenvector** corresponding to the eigenvalue λ .

Nonhomogeneous Linear Differential Systems

Let $A(t)$ be the $n \times n$ matrix

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

and let \mathbf{x} and $\mathbf{b}(t)$ be the vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

Nonhomogeneous system:

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{b}(t) \quad (\text{N})$$

Reduced system:

$$\mathbf{x}' = A(t)\mathbf{x} \quad (\text{H})$$

Theorem 1. If $\mathbf{z}_1(t)$ and $\mathbf{z}_2(t)$ are solutions of (N), then

$$\mathbf{x}(t) = \mathbf{z}_1(t) - \mathbf{z}_2(t)$$

is a solution of (H). (C.f. Theorem 1, Section 3.4, and Theorem 7, Section 6.1.)

Theorem 2. Let $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$ be a fundamental set of solutions the reduced system (H) and let $\mathbf{z} = \mathbf{z}(t)$ be a particular solution of (N). If $\mathbf{u} = \mathbf{u}(t)$ is *any* solution of (N), then there exist constants c_1, c_2, \dots, c_n such that

$$\mathbf{u}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \dots + c_n\mathbf{x}_n(t) + \mathbf{z}(t)$$

(C.f. Theorem 2, Section 3.4, and Theorem 8, Section 6.1.)

General Solution of (N):

$$\mathbf{x} = \underbrace{C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \cdots + C_n\mathbf{x}_n(t)}_{\text{general solution of (H)}} +$$

$$\underbrace{\mathbf{z}(t)}_{\text{particular solution of (N)}}$$

Variation of Parameters

$$\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$$

a fundamental set of solutions of (H).

$V(t)$ the corresponding fundamental matrix.

$$\mathbf{x}(t) = V(t)\mathbf{C} \quad \text{where} \quad \mathbf{C} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}.$$

is the general solution of (H).

V satisfies the matrix differential system

$$X' = A(t)X.$$

That is, $V'(t) = A(t)V(t)$. (Exercises 6.3, Problem 19.)

Replace the constant vector \mathbf{C} by a vector function $\mathbf{u}(t)$ which is to be determined so that

$$\mathbf{z}(t) = V(t)\mathbf{u}(t)$$

is a solution of (N).

$$\mathbf{u}'(t) = V^{-1}(t)\mathbf{b}(t)$$

$$\mathbf{u}(t) = \int V^{-1}(t)\mathbf{b}(t) dt.$$

and

$$\mathbf{z}(t) = V(t) \int V^{-1}(t)\mathbf{b}(t) dt$$

is a solution of (N).

General solution of (N):

$$\mathbf{x}(t) = V(t)\mathbf{C} + V(t) \int V^{-1}(t)\mathbf{b}(t) dt.$$