Exam 1 Review: Questions and Answers

Part I. Finding solutions of a given differential equation.

1. Find the numbers r such that $y = e^{rx}$ is a solution of $y'' - y' - 30y = 0$.

   Answer: $r = 6, \ -5$

2. Find the numbers r such that $y = e^{rx}$ is a solution of $y'' + 8y' + 16y = 0$.

   Answer: $r = -4$

3. Find the numbers r such that $y = e^{rx}$ is a solution of $y'' + 2y' + 10y = 0$.

   Answer: $r = -1 \pm 3i$

4. Find the numbers r such that $y = x^r$ is a solution of $x^2y'' - 5xy' + 8y = 0$.

   Answer: $r = 2, \ 4$

5. Find the numbers r such that $y = x^r$ is a solution of $y'' - \dfrac{5}{x}\, y' + \dfrac{9}{x^2} y = 0$.

   Answer: $r = 3$

6. Find the numbers r such that $y = x^r$ is a solution of $y'' - \dfrac{1}{x}\, y' - \dfrac{15}{x^2} y = 0$.

   Answer: $r = -3, \ 5$

Part II. Find the differential equation for an n-parameter family of curves.

1. $y^2 = Cx^3 - 2$.

   Answer: $y' = \dfrac{3y^2 + 6}{2xy}$

2. $y = C_1\, x + C_2 x^3 + 4$.

   Answer: $x^2y'' - 3xy' + 3y = 12$

3. $y = C_1\, e^{-2x} + C_2\, xe^{-2x}$.

   Answer: $y'' + 4y' + 4y = 0$

4. $y^3 = C(x-2)^2 + 4$

   Answer: $y' = \dfrac{2y^3 - 8}{3(x-2)y^2}$

5. $y = C_1\, e^{-2x} + C_2\, e^{5x}$.

   Answer: $y'' - 3y' - 10y = 0$

6. $y = C_1 + C_2 e^{4x} + 2x$

   Answer: $y'' - 4y' = -8$

7. $y^4 = Cx^2 - 3x$

   Answer: $y' = \dfrac{2y^4 + 3x}{4xy^3}$

Part III. Identify each of the following first order differential equations as linear, separable, Bernoulli, homogeneous, or none of these.

1. $x(1+y^2) + y(1+x^2)y' = 0.$ **Answer:** separable
2. $xdy - \dfrac{2y}{x}\,dx = x^3 e^{-x} dx$ **Answer:** linear
3. $(xy + y)y' = x - xy.$ **Answer:** separable
4. $xy^2 \dfrac{dy}{dx} = x^3 e^{y/x} - x^2 y$ **Answer:** homogeneous
5. $y' = -\dfrac{3y}{x} + x^4 y^{1/3}.$ **Answer:** Bernoulli
6. $y^2 e^{xy} \dfrac{dy}{dx} = x^3 - xy \sin y + y^2.$ **Answer:** None of these
7. $(3x^2 + 1)y' - 2xy = 6x.$ **Answer:** linear
8. $x^2 y' = x^2 + 3xy + y^2$ **Answer:** homogeneous
9. $x(1-y) + y(1+x^2)\dfrac{dy}{dx} = 0.$ **Answer:** separable
10. $xy' = x^2 y + y^2 \, \ln x.$ **Answer:** Bernoulli

Part IV. First order linear equations; find general solution, solve an initial-value problem.

1. Find the general solution of

$$x^2\, y' = x^4 \, \cos 2x + 2xy$$

   Answer: $y = \frac{1}{2}x^2 \sin 2x + Cx^2$

2. Find the general solution of

$$x^2 y' - 3xy = 3x^5 e^{2x}$$

   Answer: $y = \frac{3}{2}\, x^3 e^{2x} + Cx^3$

3. Find the general solution of

$$(1+x^2)\, y' + 1 + 2x\, y = 0$$

   Answer: $y = \dfrac{C - x}{1 + x^2}$

4. Find the general solution of

$$xy' - y = 2x\ \ln\ x$$

Answer: $y = x(\ln x)^2 + Cx$

5. Find the solution of the initial-value problem

$$xy' + 3y = \frac{2e^{2x}}{x^2}, \quad y(1) = 2$$

Answer: $y = \dfrac{e^{2x}}{x^3} - \dfrac{e^2}{x^3} + \dfrac{2}{x^3}$

Part V. Separable equations; find general solution

1. Find the general solution of

$$y' = xe^{x+y}$$

Answer: $y = -\ln(e^x - xe^x + C)$

2. Find the general solution of

$$(xy^2 - 4x)y' = x^3y + 2y$$

Answer: $\frac{1}{2}\,y^2 - 4\ln|y| = \frac{1}{3}x^3 + 2\ln|x| + C$

3. Find the general solution of

$$(x^2y + 2y)\frac{dy}{dx} = 2xy^2 + 8x$$

Answer: $y^2 = C(x+2)^2 - 4$

4. Find the general solution of

$$\ln x\frac{dy}{dx} = \frac{y}{x}$$

Answer: $y = C\ln x$

Part VI. Bernoulli equations; find general solution.

1. Find the general solution of

$$y' + xy = xy^3$$

Answer: $y^2 = \dfrac{1}{1 + Ce^{x^2}}$

2. Find the general solution of

$$xy' = -4y + 4x^3\sqrt{y}$$

Answer: $y^{1/2} = \frac{2}{5}\,x^3 + Cx^{-2}$

3. Find the general solution of

$$xy' - y = y^2 \ln x$$

Answer: $y = \dfrac{x}{x - x\ln x + C}$

Part VII. Homogeneous equations; find general solution.

1. Find the general solution of

$$xy^2y' = x^3 + y^3$$

Answer: $y^3 = x^3\left(\ln x^3 + C\right)$

2. Find the general solution of

$$xy\,y' = x^2\sec(y/x) + y^2$$

Answer: $y = x\sin^{-1}(\ln x + C)$

x

3. Find the general solution of

$$x^2y' = xy + x^2 + y^2$$

Answer: $y = x\tan(\ln x + C)$

Part VIII. Applications

1. Given the family of curves

$$y = Ce^{2x} + 1$$

Find the family of orthogonal trajectories.

Answer: $y^2 + x - 2y = C$

2. Given the family of curves

$$y^3 = Cx^2 + 2$$

Find the family of orthogonal trajectories.

Answer: $3x^2 + 2y^2 + \dfrac{8}{y} = C$

3. A 200 gallon tank, initially full of water, develops a leak at the bottom. Given that 20% of the water leaks out in the first 4 minutes, find the amount of water left in the tank t minutes after the leak develops if:

   (i) The water drains off a rate proportional to the amount of water present.

   (ii) The water drains off a rate proportional to the product of the time elapsed and the amount of water present.

   (iii) The water drains off a rate proportional to the square root of the amount of water present.

   Answer: *(i)* $V(t) = 200(4/5)^{t/4}$. *(ii)* $V(t) = 200(4/5)^{t^2/16}$. *(iii)* $V(t) = \left[\left(2\sqrt{10} - 5\sqrt{2}\right) t + \sqrt{200}\right]^2$

4. A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 100 grams of the material was present initially and after 2 hours the sample lost 20% of its mass, find:

   (a) An expression for the mass of the material remaining at any time t.

   (b) The mass of the material after 4 hours.

   (c) The half-life of the material.

   Answer: (a) $A(t) = 100(4/5)^{t/2}$ (b) $A(4) = 32$ grams (c) 6.21 hrs

5. A biologist observes that a certain bacterial colony triples every 4 hours and after 12 hours occupies 1 square centimeter. Assume that the colony obeys the population growth law.

   (a) How much area did the colony occupy when first observed?

   (b) What is the doubling time for the colony?

   Answer: (a) 1/27 sq. cm. (b) $T = \dfrac{4 \ln 2}{\ln 3}$.

6. An advertising company designs a campaign to introduce a new product to a metropolitan area of population M. Let $P = P(t)$ denote the number of people who become aware of the product by time t. Suppose that P increases at a rate proportional to the number of people still unaware of the product. The company determines that no one was aware of the product at the beginning of the campaign and that 30% of the people were aware of the product after 10 days of advertising.

   (a) Give the mathematical model (differential equation and initiation condition).

   (b) Determine the solution of the initial-value problem in (a).

   (c) Determine the value of the proportionality constant.

   (d) How long does it take for 90% of the population to become aware of the product?

   Answer: (a) $\dfrac{dP}{dt} = k(M-P)$, $P(0) = 0$. (b) $P(t) = M - Me^{-kt}$ (c) $k = \dfrac{\ln(7/10)}{-10}$,
   (d) $t = \dfrac{10 \ln(1/10)}{\ln(7/10)}$.

7. A disease is spreading through a paddle of 100 platypuses. Let $P(t)$ be the number of sick platypuses t days after the outbreak. The disease is spreading at a rate proportional to the product of the time elapsed and number of platypuses that do not have the disease.

(a) Suppose that 10 platypuses had the disease initially. The mathematical model is:

(b) Suppose that 40 platypuses have the disease after 6 days. Then the number of healthy platypuses (rounded off to the nearest platypus) after 12 days is:

Answer: (a) $\dfrac{dP}{dt} = kt(100 - P),\ P(0) = 10,$ (b) 18

Part IX. Second order linear equations; general theory.

1. The equation $y'' + (2/x)y' - (6/x^2)y = 0$ has two solutions of the form $y = x^r$. Find the Wronskian of your solutions and give the general solution of the equation.

   Answer: $W = 5x^{-3},\ y = C_1x^{-3} + C_2x^2$

2. Given the differential equation

$$x^2y'' - 2x\,y' - 10\,y = 0$$

   (a) Find two values of r such that $y = x^r$ is a solution of the equation.

   (b) Determine a fundamental set of solutions and give the general solution of the equation.

   (c) Find the solution of the equation satisfying the initial conditions $y(1) = 6,\ y'(1) = 2$.

   Answer:

   (a) $r_1 = -2,\ r_2 = 5$.

   (b) Fundamental set: $\{y_1 = x^{-2}, y_2 = x^5\}$; $W(y_1, y_2) = \begin{vmatrix} x^{-2} & x^5 \\ -2x^{-3} & 5x^4 \end{vmatrix} = 7x^2 \neq 0$. General solution: $y = C_1x^5 + C_2x^{-2}$.

   (c) $y = 2\,x^5 + 4\,x^{-2}$.

3. Given the differential equation

$$y'' - \left(\frac{6}{x}\right)y' + \left(\frac{12}{x^2}\right)y = 0$$

   (a) Find two values of r such that $y = x^r$ is a solution of the equation.

   (b) Determine a fundamental set of solutions and give the general solution of the equation.

   (c) Find the solution of the equation satisfying the initial conditions $y(1) = 2,\ y'(1) = -1$.

   (d) Find the solution of the equation satisfying the initial conditions $y(2) = y'(2) = 0$.

   Answer:

   (a) $r_1 = 3,\ r_2 = 4$.

   (b) Fundamental set: $\{y_1 = x^3, y_2 = x^4\}$; $W(y_1, y_2) = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} = x^6 \neq 0$. General solution: $y = C_1x^3 + C_2x^4$.

   (c) $y = 9x^3 - 7x^4$; (d) $y \equiv 0$.

Part X. Homogeneous equations with constant coefficients.

1. Find the general solution of
$$y'' + 10y' + 25\,y = 0$$

   Answer: $y = C_1 e^{-5x} + C_2 x e^{-5x}$

2. Find a fundamental set of solutions of
$$y'' - 8\,y' + 15\,y = 0$$

   Answer: $\left\{e^{5x},\ e^{3x}\right\}$

3. Find the general solution of
$$y'' + 4y' + 20y = 0$$

   Answer: $y = C_1 e^{-2x} \cos\,4x + C_2 e^{-2x} \sin\,4x = e^{-2x}\,(C_1\,\cos\,4x + C_2\,\sin\,4x)$

4. Find the general solution of
$$y'' + 3\,y' - 28\,y = 0$$

   Answer: $y = C_1 e^{-7x} + C_2 e^{4x}$

5. Find the solution of the initial-value problem:
$$y'' - 2y' + 2y = 0; \quad y(0) = -1,\ y'(0) = -1$$

   Answer: $y = -e^{x} \cos\,x$

6. Find the solution of the initial-value problem:
$$y'' + 4y' + 4y = 0; \quad y(-1) = 2,\ y'(-1) = 1$$

   Answer: $y = 7e^{-2(x+1)} + 5xe^{-2(x+1)}$

7. Find a fundamental set of solutions of
$$y'' - 6y' + 9y = 0$$

   Answer: $\left\{e^{3x},\ xe^{3x}\right\}$

8. Find the general solution of
$$y'' - 2\alpha\,y' + \alpha^2 y = 0,$$

   α a constant.

   Answer: $y = C_1 e^{\alpha x} + C_2 x e^{\alpha x}$

9. Find the general solution of
$$y'' - 2\alpha\, y' + (\alpha^2 + \beta^2)y = 0,$$
$\alpha,\ \beta$ constants.

   Answer: $y = C_1 e^{\alpha x} \cos\, \beta x + C_2 e^{\alpha x} \sin\, \beta x$

10. $y = C_1 e^{-2x} + C_2 e^{5x}$ is the general solution of a second order linear differential equation with constant coefficients. What is the equation?

    Answer: $y'' - 3y' - 10y = 0$

11. The function $y = 2e^{3x} - 5e^{-4x}$ is a solution of a second order linear differential equation with constant coefficients. What is the equation?

    Answer: $y'' + y' - 12y = 0$

12. The function $y = 7e^{-3x} \cos\, 2x$ is a solution of a second order linear differential equation with constant coefficients. What is the equation?

    Answer: $y'' + 6y' + 13y = 0$

13. The function $y = 2e^{4x} - 6xe^{4x}$ is a solution of a second order linear differential equation with constant coefficients. What is the equation?

    Answer: $y'' - 8y' + 16y = 0$

14. Find a second order linear homogeneous differential equation with constant coefficients that has $y = e^{4x}$ as a solution.

    Answer: $y'' - (\alpha + 4)y' + 4\alpha\, y = 0$

15. Find a second order linear homogeneous differential equation with constant coefficients that has $y = 3e^{2x} - 4xe^{-3x}$ as a solution.

    Answer: Not possible (Characteristic polynomial is $(r-2)(r+3)^2$ which corresponds to a third order equation)