MATH 3321
Sample Questions for Exam 2

Second Order Nonhomogeneous Differential Equations: Section 3.4, 3.5

1. $z_1(x) = 2x^3 + x\ln x$, $z_2(x) = x\ln x - x^3$ are solutions of a second order, linear nonhomogeneous equation $L[y] = f(x)$. $y_1(x) = x^{-2}$ is a solution of the corresponding reduced equation $L[y] = 0$.

   (a) Give a fundamental set of solutions of the reduced equation $L[y] = 0$.

   (b) Give the general solution of the nonhomogeneous equation $L[y] = f(x)$.

   Answer: (a) $\{y_1(x) = x^{-2},\ y_2(x) = x^3\}$ (b) $y = C_1x^{-2} + C_2x^3 + x\ln x$

2. $z_1(x) = 2x^2 + \tan x$, $z_2(x) = x^2 - 2x + \tan x$, $z_3(x) = x^2 - 3x + \tan x$ are solutions of a second order, linear nonhomogeneous equation $L[y] = f(x)$.

   (a) Give a fundamental set of solutions of the corresponding reduced equation $L[y] = 0$.

   (b) Give the general solution of the nonhomogeneous equation $L[y] = f(x)$.

   Answer: (a) $\{y_1(x) = x^2,\ y_2(x) = x\}$ (b) $y = C_1x^2 + C_2x + \tan x$

3. Given the differential equation $y'' + p(x)y' + q(x)y = 4x$. $\{y_1 = x^2,\ y_2 = x^2\ln x\}$ is a fundamental set of solutions of the reduced equation. Find the general solution of the given equation

   Answer: $y = C_1x^2 + C_2x^2\ln x + 4x^3$.

4. Given the differential equation $y'' - \dfrac{5}{x}y' + \dfrac{8}{x^2}y = 2x^2$. The reduced equation has solutions of the form $y = x^r$. Find the general solution of the given equation

   Answer: $y = C_1x^2 + C_2x^4 + x^4\ln x$.

5. Find a particular solution of $y'' - \dfrac{4}{x}y' + \dfrac{6}{x^2}y = 6x + 2$. The reduced equation has solutions of the form $y = x^r$.

   Answer: $z = 6x^3\ln x - 2x^2\ln x$.

6. Find the general solution of $y'' + 8y' + 16y = \dfrac{2e^{-4x}}{x^2}$.

   Answer: $y = C_1e^{-4x} + C_2xe^{-4x} - 2e^{-4x}\ln x$

7. Find the general solution of $y'' - 6\,y' + 9\,y = \dfrac{e^{3x}}{x}$.

   Answer: $y = C_1\,e^{3x} + C_2\,x\,e^{3x} + x\,e^{3x}\ln x$.

8. Find the general solution of $y'' + 9\,y = 4\,\cos\,2x + 6x$.

   Answer: $y = C_1 \cos 3x + C_2 \sin 3x + \frac{4}{5}\,\cos 2x + \frac{2}{3}x$

9. Find the general solution of $y'' + 4\,y = 2\,\sin\,2x - 8$.

   Answer: $y = -\frac{1}{2}x \cos 2x - 2$

10. Find the general solution of $y'' - 6\,y' + 8\,y = 2\,e^{4x} + 5x - 3$.

    Answer: $y = C_1 e^{2x} + C_2 e^{4x} + xe^{4x} + \frac{5}{8}x + \frac{3}{32}$

11. A particular solution of the nonhomogeneous differential equation

$$y'' - 2\,y' - 15\,y = 2\,\cos 3x + 5e^{5x} + 2$$

    will have the form:

    Answer: $z = A\,\cos\,3x + B\,\sin\,3x + C\,x\,e^{5x} + D$.

12. A particular solution of the nonhomogeneous differential equation

$$y'' - 8\,y' + 16\,y = e^{2x} \sin 4x + 2e^{4x} + 5x$$

    will have the form:

    Answer: $A\,e^{2x}\,\cos\,4x + B\,e^{2x}\,\sin\,4x + C\,x^2\,e^{4x} + Dx + E$.

13. The general solution of

$$y'' + 4y' + 20y = 2xe^{-2x} + 4e^{-2x} \sin 4x + 2e^{4x} + 5x$$

    will have the form:

    Answer: $y = C_1 e^{-2x} \cos 4x + C_2 e^{-2x} \sin 4x + (Ax + B) + Cxe^{-2x} \cos 4x + Dxe^{-2x} \sin 4x + Ee^{4x} + Fx + G$

Higher Order Linear Equations: Section 3.7

1. The general solution of $y''' - 4y'' + y' + 6y = 0$ is: (Hint: $7e^{2x}$ is a solution of the reduced equation)

   Answer: $y = C_1 e^{2x} + C_2 e^{3x} + C_3 e^{-x}$

2. The general solution of $y''' + y'' - 8y' - 12y = 0$ is: (Hint: $r = 3$ is a root of the characteristic equation)

   Answer: $y = C_1 e^{3x} + C_2 e^{-2x} + C_3 xe^{-2x}$

3. The general solution of $y^{(4)} + 2y''' + 4y'' - 2y' - 5y = 0$ is: (Hint: $e^{-x}\cos 2x$ is a solution)

   Answer: $y = C_1 e^{-x}\cos\, 2x + C_2 e^{-x}\sin\, 2x + C_3 e^{x} + C_4 e^{-x}$

4. The homogeneous equation with constant coefficients that has

$$y = C_1 e^{-2x} + C_2 x e^{-2x} + C_3 \cos\, 2x + C_4 \sin\, 2x + C_5$$

   as its general solution is:

   Answer: $y^{(5)} + 4y^{(4)} + 8y''' + 16y'' + 16y' = 0$

5. The homogeneous equation with constant coefficients of least order that has

$$y = 2e^{3x} + 3\sin\, 2x + 2x + 7$$

   as a solution is:

   Answer: $y^{(5)} - 3y^{(4)} + 4y''' - 12y'' = 0$

6. A particular solution of $y''' - 2y'' - 3y' = 2e^{-x} + xe^{3x} + 2$ will have the form:

   Answer: $z = Axe^{-x} + (Bx^2 + Cx)e^{3x} + Dx$

7. A particular solution of $y^{(4)} - 16y = 2e^{-2x} + 3e^{4x} + \cos 2x + 5$ will have the form:

   Answer: $z = Axe^{-2x} + Be^{4x} + Cx\cos\, 2x + Dx\sin\, 2x + E$

8. The general solution of $y^{(4)} + 5y'' - 36y = -2\cos\, 3x + 3xe^{2x}$ will have the form:

   Answer: $y = C_1\cos\, 3x + C_2\sin\, 3x + C_3e^{2x} + C_4e^{-2x} + Ax\cos\, 3x + Bx\sin\, 3x + (Cx^2 + Dx)e^{2x}$

9. The general solution of $y''' + y'' + y' + y = 5\sin x + 2e^{x} - e^{-x} + 4x$ will have the form:

   Answer: $y = C_1\cos\, x + C_2\sin\, x + C_3e^{-x} + Ax\cos\, x + Bx\sin\, x + Ce^{x} + Dxe^{-x} + Ex + F$

10. A particular solution of $y^{(4)} - 5y''' + 7y'' - 3y' = 2e^{x} - 4xe^{3x} + 7$ will have the form:

   Answer: $z = Ax^2e^{x} + (Bx^2 + Cx)e^{3x} + Dx$

Laplace Transformations: Chapter 4

1. Find the Laplace transform of $f(x) = 2\,e^{-3x} + \cos\, 2x + 5x$.

   Answer: $F(s) = \dfrac{2}{s+3} + \dfrac{s}{s^2+4} + \dfrac{5}{s^2}.$

2. Find the Laplace transform of $f(x) = 3x\,e^{2x} + 5e^{x}\cos 3x + 4e^{x}\sin\, 3x$.

   Answer: $F(s) = \dfrac{3}{(s-2)^2} + \dfrac{5s+7}{s^2-2s+10}.$

3. If $F(s) = \dfrac{2}{s^2} + \dfrac{s-3}{s^2+4}$, then $\mathcal{L}^{-1}[F(s)]$ is:

Answer: $f(x) = 2x + \cos\, 2x - \frac{3}{2}\, \sin\, 2x.$

4. If $F(s) = \dfrac{4}{s^4 - 3s^3 + 2s^2}$, then $\mathcal{L}^{-1}[F(s)]$ is:

Answer: $f(x) = 3 + 2x + e^{2x} - 4e^x$

5. If $F(s) = \dfrac{3s^3 + 6s^2 + 36}{s^4 + 9s^2}$, then $\mathcal{L}^{-1}[F(s)]$ is:

Answer: $f(x) = 4x + 3\, \cos\, 3x + \frac{2}{3}\, \sin\, 3x.$

6. If $F(s) = \dfrac{3s+4}{(s-4)(s^2+16)}$, then $\mathcal{L}^{-1}[F(s)]$ is:

Answer: $f(x) = \frac{1}{2}\, e^{4x} - \frac{1}{2}\, \cos 4x + \frac{1}{4}\, \sin 4x$

7. If $F(s) = \dfrac{2s+1}{s^3 - s^2 - 8s + 12}$, then $\mathcal{L}^{-1}[F(s)]$ is: (Hint: 2 is a root)

Answer: $f(x) = \frac{1}{5}\, e^{2x} + xe^{2x} - \frac{1}{5}\, e^{-3x}$

8. If $F(s) = \dfrac{3s+1}{s^3 - 6s^2 + 13s - 20}$, then $\mathcal{L}^{-1}[F(s)]$ is: (Hint: 4 is a root)

Answer: $f(x) = e^{4x} - e^x \cos 2x$

9. Find the Laplace transform of the solution of the initial-value problem

$$y' + 4y = 3\, \cos\, 2x; \quad y(0) = 3.$$

Answer: $Y = \dfrac{3s}{(s^2+4)(s+4)} + \dfrac{3}{s+4}$

10. Find the Laplace transform of the solution of the initial-value problem

$$y'' + 3y' - 4y = 2xe^{-3x}; \quad y(0) = 2,\ y'(0) = -3.$$

Answer: $Y = \dfrac{2}{(s+3)^2(s^2+3s-4)} + \dfrac{2s+3}{s^2+-s-4}$

11. Find the value of δ such that the solution of the initial-value problem

$$y' - 3\, y = 2\, e^{-2x}; \quad y(0) = \delta$$

has limit 0 as $x \to \infty$.

Answer: $\delta = -\frac{2}{5}.$

12. If

$$f(x) = \begin{cases} x^2+1 & 0 \le x < 3 \\ 2x & x \ge 3 \end{cases}$$

then $\mathcal{L}[f(x)] =$

Answer: $F(s) = \dfrac{2}{s^3} + \dfrac{1}{s} - e^{-3s}\dfrac{2}{s^3} - 4\,e^{-3s}\dfrac{1}{s^2} - 4e^{-3s}\dfrac{1}{s}.$

13. If

$$f(x) = \begin{cases} \sin x & 0 \le x < \pi/2 \\ \cos 2x & x \ge \pi/2 \end{cases}$$

then $\mathcal{L}[f(x)] =$

Answer: $F(s) = \dfrac{1}{s^2+1} - e^{\pi s/2}\dfrac{s}{s^2+1} - e^{\pi s/2}\dfrac{s}{s^2+4}$

14. If

$$f(x) = \begin{cases} x^2+1 & 0 \le x < 2 \\ 4e^{3x} & x \ge 2 \end{cases}$$

then $\mathcal{L}[f(x)] =$

Answer: $F(s) = \dfrac{2}{s^3} + \dfrac{1}{s} - e^{-2s}\dfrac{2}{s^3} - 4\,e^{-2s}\dfrac{1}{s^2} - 5^{-2s}\dfrac{1}{s} + 4e^6\,e^{-2s}\dfrac{1}{s-3}.$

15. If

$$f(x) = \begin{cases} -2 & 0 \le x < 2 \\ x & 2 \le x < 5 \\ 3 & x \ge 5 \end{cases}$$

then $\mathcal{L}[f(x)] =$

Answer: $F(s) = -\dfrac{2}{s} - e^{-2s}\dfrac{1}{s^2} + 4\,e^{-2s}\dfrac{1}{s} - e^{-5s}\dfrac{1}{s^2} - 2\,e^{-5s}\dfrac{1}{s}.$

16. If $F(s) = \dfrac{4}{s^3} + \dfrac{2}{s^2} - \dfrac{3}{s} - e^{-2s}\dfrac{3}{s^2} + e^{-2s}\dfrac{4}{(s+3)^2}$, then $\mathcal{L}^{-1}[F(s)]$ is:

Answer: $f(x) = \begin{cases} 2x^2+2x-3 & 0 \le x < 2 \\ 2x^2 - x + 3 + 4(x-2)e^{3(x-2)}, & x \ge 2 \end{cases}$

17. If $F(s) = \dfrac{s+4e^{-3s}}{s^3-2s^2}$, then $\mathcal{L}^{-1}[F(s)]$ is:

Answer: $f(x) = \begin{cases} \frac{1}{2}\,e^{2x} - \frac{1}{2} & 0 \le x < 3 \\ \frac{9}{2} - 2x + \frac{1}{2}\,e^{2x} + e^{2(x-3)}, & x \ge 3 \end{cases}$

18. If $F(s) = \dfrac{1 - 2s + (s-4)e^{-\pi s}}{s^2+9}$, then $\mathcal{L}^{-1}[F(s)]$ is:

Answer: $f(x) = \begin{cases} \frac{1}{3}\sin 3x - 2\cos 3x, & 0 \le x < \pi \\ \frac{5}{3}\sin 3x - 3\cos 3x, & x \ge \pi \end{cases}$

19. If $F(s) = \dfrac{3s+1}{s^2-s-6} + \dfrac{(2s-9)e^{-4s}}{s^2-6s+13}$, then $\mathcal{L}^{-1}[F(s)]$ is:

Answer:

$$f(x) = \begin{cases} 2\,e^{3x} + e^{-2x}, & 0 \le x < 4 \\ 2\,e^{3x} + e^{-2x} + 2\,e^{3(x-4)}\cos\,2(x-4) - \frac{3}{2}\,e^{3(x-4)}\sin\,2(x-4), & x \ge 4 \end{cases}$$

20. If $F(s) = \dfrac{2}{s} + \dfrac{4}{s^3} - 2e^{-s}\dfrac{1}{s^2} + \dfrac{(2s+3)e^{-x}}{s^2-2s+10}$, then $\mathcal{L}^{-1}[F(s)]$ is:

Answer:

$$f(x) = \begin{cases} 2 + 2x^2 & 0 \le x < 1 \\ 4 - 2x + 2x^2 + 2e^{x-1}\cos 3(x-1) + \frac{5}{3}e^{x-1}\sin 3(x-1) & x \ge 1 \end{cases}$$