MATH 3321
Sample Questions for Exam 3

1. Find the solution set of the system of linear equations

$$\begin{aligned} 4x - y + 2z &= 3 \\ -4x + y - 3z &= -10 \\ 8x - 2y + 9z &= -1 \end{aligned}$$

Answer: No solution

2. Find the solution set of the system of linear equations

$$\begin{aligned} 2x - 5y - 3z &= 7 \\ -4x + 10y + 2z &= 6 \\ 6x - 15y - z &= -19 \end{aligned}$$

Answer: $x = \frac{5}{2}a - 4,\ y = a,\ z = -5,\ a$ any real number

3. Find the solution set of the system of linear equations

$$\begin{aligned} 5x - 3y + 2z &= 13 \\ 2x - y - 3z &= 1 \\ 4x - 2y + 4z &= 12 \end{aligned}$$

Answer: $x = 1,\ y = -2,\ z = 1$

4. Find the solution set of the system of linear equations

$$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 &= 0 \\ x_1 + 2x_2 + x_4 &= 4 \\ -x_1 - 2x_2 + 2x_3 + 4x_4 &= 5 \end{aligned}$$

Answer: $x_1 = 3 - 2a,\ x_2 = a,\ x_3 = 2,\ x_4 = 1,\ a$ any real number

5. Given the system of equations

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= -2 \\ -2x_1 + 5x_2 - x_3 + 4x_4 &= 1 \\ 3x_1 - 7x_2 + 2x_3 - 5x_4 &= 9 \\ 2x_2 + x_3 + 5x_4 &= -2 \end{aligned}$$

Find the rank of the matrix of coefficients and the rank of the augmented matrix. Does the system have a unique solution, infinitely many solutions, or no solutions.

Answer: $3, 4;$ no solutions

6. Find the row echelon form of $\begin{pmatrix} 1 & 2 & 3 & 2 \\ -1 & -2 & -2 & 1 \\ 2 & 4 & 8 & 12 \end{pmatrix}$

Answer: $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

7. Find the reduced row echelon form of $\begin{pmatrix} 1 & 2 & -3 & 4 & 2 \\ 2 & 5 & -2 & 1 & 1 \\ 5 & 12 & -7 & 6 & 7 \end{pmatrix}$

Answer: $\begin{pmatrix} 1 & 0 & -11 & 18 & 0 \\ 0 & 1 & 4 & -7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

8. Find the solution set of the system of linear equations

$$\begin{aligned} x_1 + 2x_2 - 3x_3 - 4x_4 &= 2 \\ 2x_1 + 4x_2 - 5x_3 - 7x_4 &= 7 \\ -3x_1 - 6x_2 + 11x_3 + 14x_4 &= 0 \end{aligned}$$

Answer: $x_1 = 11 - 2a + b,\ x_2 = a,\ x_3 = 3 - b,\ x_4 = b,\ a, b$ any real numbers

9. Find the solution set of the homogeneous system of linear equations

$$\begin{aligned} 3x_1 + x_2 - 5x_3 - x_4 &= 0 \\ 2x_1 + x_2 - 3x_3 - 2x_4 &= 0 \\ x_1 + x_2 - x_3 - 3x_4 &= 0 \end{aligned}$$

Answer: $x_1 = 2a - b,\ x_2 = -a + 4b,\ x_3 = a,\ x_4 = b,\ a, b$ any real numbers

10. For what values of k, does the system of equations

$$\begin{aligned} x - 2y &= 1 \\ x - y + kz &= -2\ . \\ ky + 4z &= 6 \end{aligned}$$

have (a) a unique solution? (b) infinitely many solutions? (c) no solutions?

Answer: (a) $k \neq \pm 2$ (b) $k = -2$ (c) $k = 2$

11. For what values of k, does the system of equations

$$\begin{aligned} x + 2y + 3z &= 4 \\ y + 5z &= 9 \\ 2x + 3y + (k^2 - 8)z &= k + 2 \end{aligned}$$

have (a) a unique solution? (b) infinitely many solutions? (c) no solutions?

Answer: (a) $k \neq \pm 3$ (b) $k = -3$ (c) $k = 3$

12. Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 1 \\ 2 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 3 & -2 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}$.

Perform the indicated operations, if possible: (a) AC (b) AB (c) $B + AC$ (d) CBA

Answer: $AC = \begin{pmatrix} 9 & 3 \\ -2 & -8 \end{pmatrix}$, AB no, $B{+}AC = \begin{pmatrix} 6 & 4 \\ 0 & -3 \end{pmatrix}$, $CBA = \begin{pmatrix} -26 & -15 & -25 \\ -4 & -18 & 4 \\ 2 & 43 & -19 \end{pmatrix}$

13. Find the inverse of $A = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$.

Answer: $A^{-1} = \begin{pmatrix} -1 & 3/2 \\ 2 & -5/2 \end{pmatrix}$

14. Determine whether or not $A = \begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix}$ has an inverse.

15. Find the inverse of $A = \begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix}$.

Answer: $A^{-1} = \begin{pmatrix} -16 & -11 & 3 \\ 7/2 & 5/2 & -1/2 \\ -5/2 & -3/2 & 1/2 \end{pmatrix}$

16. Show that the matrix of coefficients is non-singular and use Cramer's rule to find the values of y and z in the solution set,

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ 2x_2 - x_3 &= 1 \\ 2x_1 + 3x_2 &= 1 \end{aligned}$$

Answer: $y = -3,\ z = -7$

17. Find the values of λ for which the homogeneous system

$$\begin{aligned} (2-\lambda)x - 3y &= 0 \\ 4x + (2-\lambda)y &= 0 \end{aligned}$$

has only the trivial solution.

Answer: $\lambda \neq 5, -2$

18. Find the values of λ for which the system

$$\begin{aligned} \lambda x + 2y - 4z &= 0 \\ -x + y + \lambda z &= 0 \\ -y + 5z &= 0 \end{aligned}$$

has nontrivial solutions.

Answer: $\lambda = -2,\ \lambda = -3$

19. Given the set of vectors

$$\{\mathbf{u} = (\,1,\ -2,\ 1\,),\ \mathbf{v} = (\,2,\ 1,\ -1\,),\ \mathbf{w} = (\,7,\ -4,\ 1\,)\}$$

Is the set linearly dependent or linearly independent? If it is linearly dependent, express one of the vectors as a linear combination of the other two.

Answer: Dependent, $\mathbf{u} = -\frac{2}{3}\mathbf{v} + \frac{1}{3}\mathbf{w}$ **or** $\mathbf{v} = -\frac{3}{2}\mathbf{u} + \frac{1}{2}\mathbf{w}$ **or** $\mathbf{w} = 3\mathbf{u} + 2\mathbf{v}$

20. Given the set of vectors

$$\{\mathbf{v}_1 = (\,2,\ 0,\ -1\,),\ \mathbf{v}_2 = (\,-3,\ 1,\ 2\,),\ \mathbf{v}_3 = (\,8,\ -2,\ -5\,),\ \mathbf{v}_4 = (\,-9,\ 1,\ 5\,)\}$$

Is the set linearly dependent or linearly independent? If it is linearly dependent, how many independent vectors are there in the set?

Answer: Dependent, 2

21. For what values of a are the vectors

$$\mathbf{v}_1 = (\,a,\ 1,\ -1\,),\ \mathbf{v}_2 = (\,-1,\ 2a,\ 3\,),\ \mathbf{v}_3 = (\,-2,\ a,\ 2\,),\ \mathbf{v}_4 = (\,3a,\ -2,\ a\,)$$

linearly dependent?

Answer: All real numbers. (**Four** vectors in $\mathbb{R}^3$.)

22. For what values of a are the vectors

$$\mathbf{u} = (\,a,\ 1,\ -1\,),\ \mathbf{v} = (\,-1,\ 2a,\ 3\,),\ \mathbf{w} = (\,-2,\ a,\ 2\,)$$

linearly dependent?

Answer: $a = 4,\ -1$

23. The eigenvalues of a 3×3 matrix $A =$ are $\lambda_1 = 4,\ \lambda_2 = -2,\ \lambda_3 = 2$. What is the characteristic polynomial of A?

Answer: $p(\lambda) = (\lambda - 4)(\lambda + 2)(\lambda - 2)$ (which is $\lambda^3 - 4\lambda^2 - 4\lambda + 16$ if you multiply it out.)

24. The eigenvalues of a 3×3 matrix $A =$ are $\lambda_1 = -3,\ \lambda_2 = \lambda_3 = 2$. What is the characteristic polynomial of A?

Answer: $p(\lambda) = (\lambda + 3)(\lambda - 2)^2$ (which is $\lambda^3 - \lambda^2 - 8\lambda + 12$ if you multiply it out.)

25. Find the eigenvalues and the number of independent eigenvectors of $A = \begin{pmatrix} 3 & 5 & -2 \\ 0 & 2 & 2 \\ 0 & -2 & 6 \end{pmatrix}$.

Answer: $\lambda_1 = 3,\ \lambda_2 = \lambda_3 = 4$; two independent eigenvectors.

26. The eigenvalues and eigenvectors of $\begin{pmatrix} 6 & -2 & 0 \\ 4 & 0 & 0 \\ 4 & 1 & -1 \end{pmatrix}$ are:

Answer: $4, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad 2, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}; \quad -1, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

27. The eigenvalues and eigenvectors of $\begin{pmatrix} 3 & -2 & 1 \\ 2 & -2 & 2 \\ 2 & -3 & 3 \end{pmatrix}$ are: (Hint: 1 is an eigenvalue.)

Answer: $2, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad 1,\ 1, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

28. The eigenvalues and eigenvectors of $\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$ are: (Hint: 2 is an eigenvalue.)

Answer: $8, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad 2, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}; \quad 2, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

29. The eigenvalues and eigenvectors of $\begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}$ are:

Answer: $2+3i, \begin{pmatrix} -1 \\ 5 \end{pmatrix} + i\begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad 2-3i, \begin{pmatrix} -1 \\ 5 \end{pmatrix} - i\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

30. Find the general solution of $\mathbf{x}' = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}\mathbf{x}$.

Answer: $\mathbf{x} = C_1 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

31. Find the solution of the initial-value problem $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\mathbf{x}$; $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. HINT: 2 is a root of the characteristic polynomial.

Answer: General solution: $\mathbf{x} = C_1 e^{3t}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{t}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$;

Solution of the initial-value problem: $\mathbf{x} = 2e^{2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - e^{t}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

32. Find the general solution of $\mathbf{x}' = \begin{pmatrix} 5 & -2 \\ 2 & 1 \end{pmatrix}\mathbf{x}$.

Answer: $\mathbf{x}(t) = C_1 e^3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left[e^{3t}\begin{pmatrix} 3/2 \\ 1 \end{pmatrix} + te^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$

33. Find a fundamental set of solutions of $\mathbf{x}' = \begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix}\mathbf{x}$.

Answer: $\left\{e^{3t}\left[\cos 2t\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin 2t\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right],\ e^{3t}\left[\cos 2t\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sin 2t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]\right\}$

34. Find the general solution of $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\mathbf{x}$. HINT: 2 is a root of the characteristic polynomial.

Answer: $\mathbf{x} = C_1 e^{-t}\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + C_3\left[e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + te^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right]$

35. Find a fundamental set of solutions of $\mathbf{x}' = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}\mathbf{x}$. HINT: 10 is a root of the characteristic polynomial.

Answer: $\left\{e^{10t}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix},\ e^{t}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix},\ e^{t}\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}\right\}$