1. Find the general solution of $y' = x - 2y \cot 2x$. **Linear**

$y' + (2\cot 2x)y = x;$ multiply by $e^{\int 2\cot 2x\,dx} = e^{\ln \sin 2x} = \sin 2x.$

$(\sin 2x)y' + (2\cos 2x)y = x \sin 2x$

$\sin 2x)y = \int x \sin 2x\,dx = \frac{1}{2}\,x \cos 2x + \frac{1}{4} \sin 2x + C$

$y = -\frac{1}{2}\,x \cos 2x + \frac{1}{4} + C \csc 2x$

2. Find the general solution of $(xy + x)\frac{dy}{dx} = y - xy.$ **Separable**

$x(y+1)\,\frac{dy}{dx} = y(1-x)$

$\frac{y+1}{y}\,dy = \frac{1-x}{x}\,dx$

$\left(1 + \frac{1}{y}\right)dy = \left(\frac{1}{x} - 1\right)dx$

$y + \ln|y| = \ln|x| - x + C$

3. Find the general solution of $xy' = 2y - xy^2 \ln x.$ **Bernoulli**

$y' - \frac{2}{x}\,y = -(\ln x)\,y^2$

$y^{-2}y' - \frac{2}{x}\,y^{-1} = -\ln x$

Set $v = y^{-1}$. Then $v' = -y^{-2}y'$

$-v' - \frac{2}{x}\,v = -\ln x$

$v' + \frac{2}{x}\,v = \ln x.$ Multiply by $e^{\int 2/x\,dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

$x^2\,v' + 2x\,v = x^2 \ln x$

$x^2 v = \int x^2 \ln x\,dx = \frac{1}{3}\,x^3 \ln x - \frac{1}{9}\,x^3 + C$

$v = \frac{1}{3}\,x \ln x - \frac{1}{9}\,x + \frac{C}{x^2}$

$\frac{1}{y} = \frac{1}{3}\,x \ln x - \frac{1}{9}\,x + \frac{C}{x^2} = \frac{3x^3 \ln x - x^3 + C}{9x^2}$

$y = \frac{9x^2}{3x^3 \ln x - x^3 + C}$

4. Find the solution of the initial-value problem: $xy' = y + x^3\,e^x,\ y(1) = e.$ **Linear**

$y' - \frac{1}{x}\,y = x^2 e^x.$ Multiply by $e^{-\int 1/x\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$

$x^{-1}y' - x^{-2}y = x\,e^x$

$x^{-1}y = \int x\,e^x\,dx = x\,e^x - e^x + C$

$y = x^2\,e^x - x\,e^x + Cx$ general solution

$y(1) = e - e + C = e \Rightarrow C = e$

$y = x^2\,e^x - x\,e^x + ex$ solution of the initial-value problem.

5. Find the solution of the initial-value problem: $y' = xe^{(y-x^2)},\ y(0) = 0.$ **Separable**

$\frac{dy}{dx} = x\,e^y\,e^{-x^2}$

$e^{-y}\,dy = x\,e^{-x^2}\,dx$

$-e^{-y} = -\frac{1}{2}\,e^{-x^2} + C \quad \Rightarrow \quad e^{-y} = \frac{1}{2}\,e^{-x^2} + C$

$\ln\,e^{-y} = \ln\left[\frac{1}{2}\,e^{-x^2} + C\right] \quad \Rightarrow \quad -y = \ln\left[\frac{1}{2}\,e^{-x^2} + C\right] \quad \Rightarrow \quad y = -\ln\left[\frac{1}{2}\,e^{-x^2} + C\right]$

$y(0) = -\ln\left[\frac{1}{2} + C\right] = 0 \quad \Rightarrow \quad \frac{1}{2} + C = 1 \quad \Rightarrow \quad C = \frac{1}{2}$

$y = -\ln\left[\frac{1}{2}\,e^{-x^2} + \frac{1}{2}\right] = \ln\,2 - \ln\left[e^{-x^2} + 1\right]$

6. Find the general solution of $2x^3\frac{dy}{dx} = y(y^2 + 3x^2).$ **Homogeneous and Separable.**

Homogeneous because $f(tx, ty) = \frac{t^3y^3 + 3t^3x^2y}{2t^3x^3} = \frac{y^3 + 3x^2y}{2x^3} = f(x, y).$

Solution as a homogeneous equation:

$\frac{dy}{dx} = \frac{y^3 + 3x^2y}{2x^3}; \quad f(tx, ty) = \frac{t^3y^3 + 3t^3x^2y}{2t^3x^3} = \frac{y^3 + 3x^2y}{2x^3} = f(x, y).$

Set $y = vx$. Then $y' = v + xv'$.

$v + xv' = \frac{x^3v^3 + 3x^3v}{2x^3} = \frac{v^3 + 3v}{2}$

$xv' = \frac{v^3 + 3v}{2} - v = \frac{v^3 + v}{2}$

$\frac{2}{v(v^2 + 1)}\,dv = \frac{1}{x}\,dx$

$\int \frac{2}{v(v^2 + 1)}\,dv = \int \frac{1}{x}\,dx = \ln\,x + C = \ln\,Cx$

$\int \left(\frac{2}{v} - \frac{2v}{v^2 + 1}\right)\,dv = \ln\,Cx$

$2\ln\,v - \ln\,(v^2 + 1) = \ln\,Cx$

$\ln\left[\frac{v^2}{v^2 + 1}\right] = \ln\,Cx$

$\frac{v^2}{v^2 + 1} = Cx \quad \Rightarrow \quad \frac{y^2/x^2}{y^2/x^2 + 1} = Cx \quad \Rightarrow \quad y^2 = Cx(x^2 + y^2)$