NAME:

(write or print LEGIBLY)

1. Find the family of orthogonal trajectories of the family of circles that pass through the origin and have their center on the x-axis. Sketch the graphs of several members of each family.

   Equation: $(x-a)^2 + y^2 = a^2$ or $x^2 + y^2 = 2ax$.

   Differential equation for the family: $y' = \dfrac{y^2 - x^2}{2xy}$.

   Differential equation for the family of orthogonal trajectories:

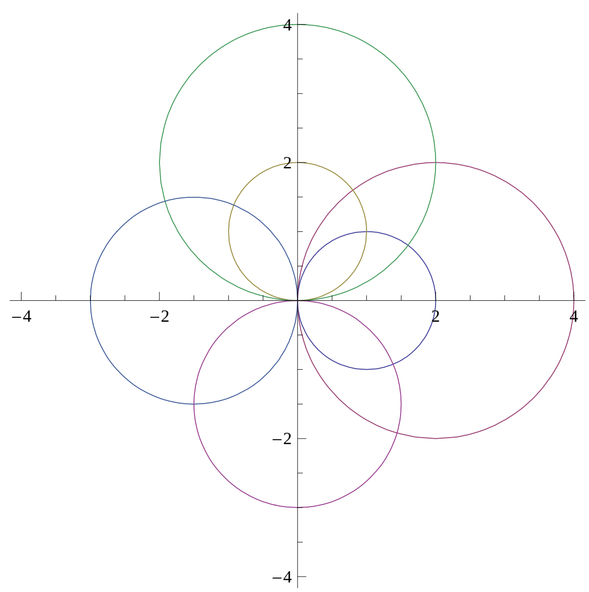
   $$y' = \frac{2xy}{x^2 - y^2},$$

   a homogeneous equation.

   General solution:

   $$x^2 + y^2 = Cy,$$

   the family of circles that pass through the origin and have their centers on the y-axis.



2. A biologist observes that a certain bacterial colony triples every 4 hours and after 12 hours occupies 1 square centimeter. Assume that the population is growing at a rate proportional to the amount present.

   (a) How much area did the colony occupy when first observed?

   (b) What is the doubling time for the colony?

$$\frac{dP}{dt} = kP \quad \Rightarrow \quad P(t) = P(0)e^{kt}.$$

$$P(4) = 3P(0) = P(0)e^{4k} \quad \Rightarrow \quad k = \frac{\ln 3}{4} \quad \Rightarrow \quad P(t) = P(0)e^{(t/4)\ln 3} = P(0)(3)^{t/4}.$$

$$P(12) = P(0)3^3 = 1 \quad \Rightarrow \quad P(0) = \frac{1}{27} \quad \Rightarrow \quad P(t) = \frac{1}{27}(3)^{t/4}.$$

(a) $P(0) = \dfrac{1}{27}$ square centimeters.

(b) $2P(0) = P(0)(3)^{t/4} \quad \Rightarrow \quad 3^{t/4} = 2 \quad \Rightarrow \quad t = \dfrac{4\ln 3}{\ln 2} \cong 2.52$ hours.

3. A metal ball at room temperature $20^o\,C$ is dropped into a container of boiling water $(100^o\,C)$. Given that the temperature of the ball increases 2^o in 2 seconds, find:

   (a) The temperature of the ball after 6 seconds in the boiling water.

   (b) How long it will take for the temperature of the ball to reach $90^o\,C$.

$$\frac{du}{dt} = -k(u-\sigma),\ u(0) = u_0 \quad \Rightarrow \quad u(t) = \sigma + (u_o - \sigma)e^{-kt}$$

$$u(t) = 100 + (20-100)e^{-kt} = 100 - 80e^{-kt}.$$

$$u(2) = 22 = 100 - 80e^{-2k} \quad \rightarrow \quad -80e^{-2k} = -78 \quad k = \frac{\ln(39/40)}{-2} \cong 0.01266.$$

$$u(t) = 100 - 80e^{-0.01266t}.$$

(a) $u(6) = 100 - 80e^{-0.01266(6)} \cong 25.851^o\,C.$

(b) $90 = 100 - 80e^{-0.01266t} \quad \Rightarrow \quad -80e^{-0.01266t} = -10 \quad \Rightarrow \quad t = \dfrac{\ln(1/8)}{-0.01266} \cong 164.25$ sec. $\cong 2.7$ min.

4. An object with mass 10 kg is dropped from a height of 200 m. Given that its drag coefficient is $k = 2.5$ N/(m/s), after how many seconds does the object hit the ground?

$mv' = -mg - kv$

$10v' = -10(9.8) - 2.5v \quad \Rightarrow \quad v' + \frac{1}{4}v = -9.8$

a first order linear equation.

$v(t) = -39.2 + Ce^{-t/4}; \quad v(0) = 0 \quad \Rightarrow C = 39.2.$

Therefore, $v(t) = 39.2e^{-t/4} - 39.2 \quad \rightarrow \quad y(t) = -156.8e^{-t/4} - 39.2t + C.$

$y(0) = 200 = -156.8 + C \quad \rightarrow \quad C = 356.8$ and $y(t) = 356.8 - 156.8e^{-t/4} - 39.2t.$

$y(t) = 356.8 - 156.8e^{-t/4} - 39.2t = 0 \quad \Rightarrow \quad t \cong 8.64$ sec.

5. A rumor spreads through a small town with a population of 5,000 at a rate proportional to the product of the number of people who have heard the rumor and the number who have not heard it. Suppose that 100 people initiated the rumor and that 500 people heard it after 3 days.

   (a) How many people willhave heard the rumor after 8 days?

   (b) How long will it take for half the population to hear the rumor?

$$\frac{dP}{dt} = kP(M-P), \; P(0) = R \quad \Rightarrow \quad P(t) = \frac{MR}{R + (M-R)e^{-kMt}}$$

$$P(t) = \frac{5000(100)}{100 + (4900)e^{-5000kt}} = \frac{5000}{1 + 49e^{-5000kt}}.$$

$$P(3) = \frac{5000}{1 + 49e^{-15000k}} = 500 \quad \Rightarrow \quad k = \frac{\ln(9/49)}{15000} \cong 0.000113.$$

$$P(t) = \frac{5000}{1 + 49e^{-5000(0.000113)t}} = \frac{5000}{1 + 49e^{-0.565t}}.$$

(a) $P(8) = \dfrac{5000}{1 + 49e^{-0.565(8)}} \cong 3260.38$ or 3260 people.

(b) $P(t) = \dfrac{5000}{1 + 49e^{-0.565t}} = 2500 \cong t = 6.88$ days.