1. Use the Laplace transform method to find the solution of the initial-value problem:

$$y'' + 2y' + y = x + e^x, \quad y(0) = 2,\ y'(0) = -1.$$

$$s^2Y - 2s + 1 + 2(sY - 2) + Y = \frac{1}{s^2} + \frac{1}{s-1}$$

$$\begin{aligned}
(s^2 + 2s + 1)Y &= \frac{1}{s^2} + \frac{1}{s-1} + 2s + 3 \\
Y &= \frac{1}{s^2(s+1)^2} + \frac{1}{(s-1)(s+1)^2} + \frac{2s+3}{(s+1)^2} \\
&= \frac{1}{s^2} - \frac{2}{s} + \frac{3/2}{(s+1)^2} + \frac{15/4}{s+1} + \frac{1/4}{s-1} \quad \text{(after partial fraction decomp)}
\end{aligned}$$

Therefore

$$y = x - 2 + \tfrac{3}{2}\, xe^{-x} + \tfrac{15}{4}e^{-x} + \tfrac{1}{4}e^x.$$

2. Given the initial-value problem

$$y'' - y' - 6y = 2e^{-x}; \quad y(0) = \alpha,\ y'(0) = -1.$$

What value(s) should be assigned to α so that the resulting solution will have limit 0 as $x \to \infty$?

The solution of the initial-value has the form

$$y = C_1e^{3x} + C_2e^{-2x} + Ae^{-x}$$

where C_1 and C_2 are determined by the initial conditions. If $C_1 \neq 0$ then $y(x) \to \infty$ as $x \to \infty$. Therefore, we want to choose α so that $C_1 = 0$.

Taking the Laplace transform

$$s^2Y - s\alpha + 1 - [sY - \alpha] - 6Y = \frac{2}{s+1}$$

$$\begin{aligned}
(s^2 - s - 6)Y &= \frac{2}{s+1} + \alpha s + \alpha + 1 \\
Y &= \frac{2}{(s-3)(s+2)(s+1)} + \frac{\alpha s + \alpha + 1}{(s-3)(s+2)} \\
&= \frac{1/10}{s-3} + \frac{(*)}{s+2} + \frac{(*)}{s+1} + \frac{4\alpha/5 + 1/5}{s-3} + \frac{(*)}{s+2} \quad \text{(we're only interested in the } s-3 \text{ term)} \\
&= \frac{4\alpha/5 + 3/10}{s-3} + \frac{(*)}{s+2} + \frac{(*)}{s+1}
\end{aligned}$$

Therefore, we choose α such that

$$\frac{4\alpha}{5} + \frac{3}{10} = 0 \Rightarrow \alpha = -\frac{3}{8}.$$

3. Find $\mathcal{L}[f]$ if

$$f(x) = \begin{cases} x & 0 \le x < 2 \\ 2 & 2 \le x < 4 \\ x^2+1 & x \ge 4 \end{cases}$$

$$\begin{aligned} f(x) &= x - xu(x-2) + 2u(x-2) - 2u(x-4) + (x^2+1)u(x-4) \\ &= x - [(x-2)+2]u(x-2) + 2u(x-2) - 2u(x-4) + \{([x-4]+4)^2+1\}u(x-4) \\ &= x - (x-2)u(x-2) - 2u(x-4) + (x-4)^2u(x-4) + 8(x-4)u(x-4) + 17u(x-4) \\ &= x - (x-2)u(x-2) + (x-4)^2u(x-4) + 8(x-4)u(x-4) + 15u(x-4) \end{aligned}$$

Therefore,

$$\mathcal{L}[f(x)] = \frac{1}{s^2} - e^{-2s}\frac{1}{s^2} + e^{-4s}\frac{2}{s^3} + 8e^{-4s}\frac{1}{s^2} + 15e^{-4s}\frac{1}{s}.$$

4. If $F(s) = \dfrac{s+(s-1)e^{-\pi s}}{s^2+1}$, what is f? Express f as a piecewise function.

$$F(s) = \frac{s}{s^2+1} + e^{-\pi s}\frac{s}{s^2+1} - e^{-\pi s}\frac{1}{s^2+1}.$$

$$\begin{aligned} f(x) &= \cos\, x + \cos\,(x-\pi)u(x-\pi) - \sin\,(x-\pi)u(x\pi) \\ &= \cos\, x - \cos\, xu(x-\pi + \sin\, xu(x-\pi) \end{aligned}$$

and

$$f(x) = \begin{cases} \cos\, x, & 0 \le x < \pi \\ \sin\, x, & x \ge \pi \end{cases}$$

5. Use the Laplace transform method to solve the initial-value problem

$$y'' - 3y' + 2y = f(x); \quad y(0) = 0,\ y'(0) = 0$$

where

$$f(x) = \begin{cases} 3 & 0 \le x < 2 \\ 2x-4 & x \ge 2 \end{cases}.$$

$f(x) = 3 - 3u(x-2) + (2x-4)u(x-2) = 3 - 3u(x-2) + 2(x-2)u(x-2).$

$$s^2Y - 3sY + 2Y = \frac{3}{s} - 3e^{-2s}\frac{1}{s} + 2e^{-2s}\frac{1}{s^2}$$

$$
\begin{aligned}
Y &= \frac{3}{s(s-2)(s-1)} + e^{-2s}\frac{3}{s(s-2)(s-1)} + e^{-2s}\frac{2}{s^2(s-2)(s-1)} \\
&= \frac{3/2}{s} - \frac{3}{s-1} + \frac{3/2}{s-2} + e^{-2s}\left[\frac{3/2}{s} - \frac{3}{s-1} + \frac{3/2}{s-2}\right] + e^{-2s}\left[\frac{1/2}{s-2} - \frac{2}{s-1} + \frac{1}{s^2} + \frac{3/2}{s}\right] \\
&= \frac{3/2}{s} - \frac{3}{s-1} + \frac{3/2}{s-2} + e^{-2s}\left[\frac{3}{s} - \frac{5}{s-1} + \frac{2}{s-2} + \frac{1}{s^2}\right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y(x) &= \tfrac{3}{2} - 3e^{x} + \tfrac{3}{2}e^{2x} + \left[3 - 5e^{x-2} + 2e^{2(x-2)} + (x-2)\right]u(x-2) \\
&= \tfrac{3}{2} - 3e^{x} + \tfrac{3}{2}e^{2x} + \left[x + 1 - 5e^{x-2} + 2e^{2(x-2)}\right]u(x-2)
\end{aligned}
$$

and

$$
y(x) = \begin{cases} \tfrac{3}{2} - 3e^{x} + \tfrac{3}{2}e^{2x}, & 0 \le x < 2 \\ x + \tfrac{5}{2} - 3e^{x} + \tfrac{3}{2}e^{2x} - 5e^{(x-2)} + 2e^{2(x-2)}, & x \ge 2 \end{cases}
$$