

POLYNOMIALS

This is a brief review of basic facts about polynomials with real coefficients.

DEFINITION 1. A *polynomial with real coefficients* is an expression of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and $a_0, a_1, a_2, \dots, a_n$, are real numbers with $a_n \neq 0$.

The nonnegative integer n is called the *degree* of p . The numbers $a_0, a_1, a_2, \dots, a_n$ are called the *coefficients* of p ; a_n is called the *leading coefficient*.

Example 1. 1. Polynomials of degree 0: Polynomials of degree 0 have the form

$$p(x) \equiv a, \quad a \text{ a nonzero constant}$$

2. Polynomials of degree 1: Polynomials of degree 1 have the form

$$p(x) = ax + b, \quad a \neq 0$$

Polynomials of degree 1 are called *linear polynomials*

3. Polynomials of degree 2: Polynomials of degree 2 have the form

$$p(x) = ax^2 + bx + c, \quad a \neq 0$$

Polynomials of degree 2 are called *quadratic polynomials* or *quadratics*.

4. Polynomials of degree 3: Polynomials of degree 3 have the form

$$p(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$$

Polynomials of degree 3 are called *cubic polynomials* or *cubics*.

5. Polynomials of degree 4: Polynomials of degree 4 have the form

$$p(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0, \quad a_4 \neq 0$$

Polynomials of degree 4 are called *quartic polynomials* or *quartics*.

6. Polynomials of degree 5: Polynomials of degree 5 have the form

$$p(x) = a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0, \quad a_5 \neq 0$$

Polynomials of degree 5 are called *quintic polynomials* or *quintics*.

DEFINITION 2. Let p be a polynomial. A number r such that $p(r) = 0$ is called a *root* or *zero* of p .

Example 2. 1. The number $r = \frac{-b}{a}$ is the unique root of the linear polynomial $p(x) = ax + b$.

2. Quadratic Formula: The numbers

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are the roots of the quadratic polynomial $p(x) = ax^2 + bx + c$.

3. There is a cubic formula and a quartic formula.

4. There are no formulas for the roots of a polynomial of degree 5 or greater.

THEOREM 1. A polynomial p of degree n has exactly n roots. The roots may be real numbers or complex numbers.

THEOREM 2. (Complex Root Theorem) If a complex number $r = a + bi$ is a root of the polynomial p , then the complex conjugate of r , $\bar{r} = a - bi$ is also a root of p ; complex roots of p occur in conjugate pairs.

COROLLARY A polynomial of odd degree must have at least one real root.

Example 3. 1. The two roots of $p(x) = x^2 - 2x - 15$ are $r_1 = 5$, $r_2 = -3$.

2. The two roots of $p(x) = x^2 + 4x + 4$ are $r_1 = -2$, $r_2 = -2$.

3. The three roots of $p(x) = x^3 - 5x^2 + 11x - 15$ are $r_1 = 3$, $r_2 = 1 + 2i$, $r_3 = 1 - 2i$.

THEOREM 3. (Factor Theorem) A number r is a root of a polynomial p if and only if $(x - r)$ is a factor of p . That is, r is a root of p if and only if

$$p(x) = (x - r)q(x)$$

where q is a polynomial of degree $n - 1$.

Example 4. 3 is a root of $p(x) = x^3 - 5x^2 + 11x - 15$;

$$p(x) = (x - 3)(x^2 - 2x + 5).$$

COROLLARY 1. If p is a polynomial of degree n with roots r_1, r_2, \dots, r_n , (roots can be repeated, some roots may be complex), then p can be factored into a product of linear factors:

$$p(x) = a_n [(x - r_1)(x - r_2) \cdots (x - r_n)].$$

COROLLARY 2. If p is a polynomial of degree n with roots r_1, r_2, \dots, r_n , then p can be factored into a product of linear factors and quadratic factors corresponding to its real and complex roots.

DEFINITION 3. A number r is a root of the polynomial p of *multiplicity* k if and only if $(x - r)^k$ is a factor of p and $(x - r)^{k+1}$ is not a factor of p . That is, r is a root of multiplicity k if and only if

$$p(x) = (x - r)^k q(x)$$

and $q(r) \neq 0$.

Example 5. The roots of

$$p(x) = x^4 - 5x^3 + 9x^2 - 7x + 2$$

are $r_1 = r_2 = r_3 = 1$, $r_4 = 2$. 1 is a root of multiplicity 3 since

$$p(x) = (x - 1)(x - 1)(x - 1)(x - 2) = (x - 1)^3(x - 2).$$