5.4A Vectors; Linear Dependence and Linear Independence

In this section we want to think of vectors as being ordered n-tuples of real numbers. Most of the time will write vectors as a row vectors, but occasionally it will be convenient to write them as columns. Our main interest is in the concept of linear dependence/linear independence of sets of vectors. At the end of the section we will extend this concept to linear dependence/independence of sets of functions.

We denote the set of ordered n-tuples of real numbers by the symbol $\mathbb{R}^n$. That is,

$$\mathbb{R}^n = \{(a_1,\ a_2,\ a_3, \ldots, a_n)\,|\, a_1,\ a_2,\ a_3,\ \ldots,\ a_n \ \text{ are real numbers}\}$$

In particular, $\mathbb{R}^2 = \{(a,\, b)\,|\, a,\ b \text{ real numbers}\}$, which we can identify with the set of all points in the plane, and $\mathbb{R}^3 = \{(a,\, b,\, c)\,|\, a,\ b,\ c \text{ real numbers}\}$, which we can identify with the set of points in space. We will use lower case boldface letters to denote vectors. The entries of a vector are called the *components* of the vector.

The operations of addition and multiplication by a number (scalar) were defined in Chapter 1. Addition is defined only for vectors with the same number of components. For any two vectors $\mathbf{u} = (a_1,\ a_2,\ a_3, \ldots, a_n)$ and $\mathbf{v} = (b_1,\ b_2,\ b_3, \ldots, b_n)$ in $\mathbb{R}^n$, we have

$$\mathbf{u} + \mathbf{v} = (a_1,\ a_2,\ a_3, \ldots, a_n) + (b_1,\ b_2,\ b_3, \ldots, b_n) = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3, \ldots,\ a_n + b_n)$$

and for any real number λ,

$$\lambda\,\mathbf{v} = \lambda\,(a_1,\ a_2,\ a_3, \ldots, a_n) = (\lambda a_1, \lambda a_2,\ \lambda a_3, \ldots, \lambda a_n).$$

Clearly, the sum of two vectors in $\mathbb{R}^n$ is another vector in $\mathbb{R}^n$ and a scalar multiple of a vector in $\mathbb{R}^n$ is a vector in $\mathbb{R}^n$. A sum of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

where $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ are vectors in $\mathbb{R}^n$ and $c_1,\ c_2,\ \ldots,\ c_k$ are real numbers is called a *linear combination* of $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$.

The set $\mathbb{R}^n$ together with the operations of addition and multiplication by a number is called a *vector space of dimension* n. The term "dimension n" will defined and explained in Section 5.5.

The zero vector in $\mathbb{R}^n$, which we'll denote by $\mathbf{0}$ is the vector

$$\mathbf{0} = (0,\ 0,\ 0, \ldots,\ 0).$$

For any vector $\mathbf{v} \in \mathbb{R}^n$, we have $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$; the zero vector is the additive identity in $\mathbb{R}^n$.

Linear Dependence and Linear Independence

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then $\mathbf{u}$ and $\mathbf{v}$ are *linearly dependent* if one of the vectors is a scalar multiple of the other (e.g., $\mathbf{u} = \lambda\mathbf{v}$ for some number λ); they are *linearly independent* if neither is a scalar multiple of the other.

Suppose that $\mathbf{u},\ \mathbf{v} \in \mathbb{R}^n$ are linearly dependent with $\mathbf{u} = \lambda\mathbf{v}$. Then

$$\mathbf{u} = \lambda\,\mathbf{v} \qquad \text{implies} \qquad \mathbf{u} - \lambda\,\mathbf{v} = \mathbf{0}.$$

This leads to an equivalent definition of linear dependence: $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if there exist two numbers c_1 and c_2, not both zero, such that

$$c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}.$$

(Note that if $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$ and $c_1 \neq 0$, then $\mathbf{u} = (c_2/c_1)\mathbf{v} = \lambda\mathbf{v}$.)

This is the idea that we'll use to define linear dependence/independence in general.

DEFINITION 1. The set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$ in $\mathbb{R}^n$ is *linearly dependent* if there exist k numbers $c_1,\ c_2,\ \ldots,\ c_k$, **not all zero**, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Otherwise, the set of vectors is *linearly independent.*

NOTE: If there exists one set of k numbers $c_1,\ c_2,\ \ldots,\ c_k$, not all zero, then there exist infinitely many such sets. For example, $2c_1,\ 2c_2,\ \ldots,\ 2c_k$ is another such set; and $\frac{1}{3}c_1,\ \frac{1}{3}c_2,\ \ldots,\ \frac{1}{3}c_k$ is another such set; and so on.

The definition of linear dependence can also be stated as: The vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ are linearly dependent if one of the vectors can be written as a linear combination of the others. For example if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

and $c_1 \neq 0$, then

$$\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2 - \frac{c_3}{c_1}\mathbf{v}_3 - \cdots - \frac{c_k}{c_1}\mathbf{v}_k = \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 + \cdots + \lambda_k\mathbf{v}_k,$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2,\ \mathbf{v}_3,\ \ldots,\ \mathbf{v}_k$. This form of the definition parallels the definition of the linear dependence of two vectors.

Stated in terms of linear independence, Definition 1 can be stated equivalently as:

> The set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$ is *linearly independent* if
>
> $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \quad \text{implies} \quad c_1 = c_2 = \cdots = c_k = 0.$$
>
> Otherwise, the set of vectors is *linearly dependent.*

Linearly dependent/independent sets in $\mathbb{R}^n$

Example 1. Vectors in $\mathbb{R}^2$.

(a) The vectors $\mathbf{u} = (3,\ 4)$ and $\mathbf{v} = (1,\ 3)$ are linearly independent because neither is a multiple of the other.

(b) The vectors $\mathbf{u} = (2,\ -3)$ and $\mathbf{v} = (-6,\ 9)$ are linearly dependent because $\mathbf{v} = -3\,\mathbf{u}$.

(c) Determine whether the vectors $\mathbf{v}_1 = (1,\ 2)$, $\mathbf{v}_2 = (-1,\ 3)$, $\mathbf{v}_3 = (5,\ 7)$ are linearly dependent or linearly independent.

SOLUTION In this case we need to determine whether or not there are three numbers $c_1,\ c_2,\ c_3,$ not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

Equating the components of the vector on the left and the vector on the right, we get the homogeneous system of equations

$$\begin{aligned} c_1 - c_2 + 5c_3 &= 0 \\ 2c_1 + 3c_2 + 7c_3 &= 0 \end{aligned}$$

We know that this system has nontrivial solutions since it is a homogeneous system with more unknowns than equations. Thus the vectors must be linearly dependent.

The three parts of Example 1 hold in general: any set of three or more vectors in $\mathbb{R}^2$ is linearly dependent. A set of two vectors in $\mathbb{R}^2$ is independent if neither vector is a multiple of the other; dependent if one vector is a multiple of the other

Example 2. Vectors in $\mathbb{R}^3$.

(a) The vectors $\mathbf{u} = (\,3,\ 4,\ -2\,)$ and $\mathbf{v} = (\,2,\ -6,\ 7\,)$ are linearly independent because neither is a multiple of the other.

(b) The vectors $\mathbf{u} = (\,-4,\ 6,\ -2\,)$ and $\mathbf{v} = (\,2,\ -3,\ 1\,)$ are linearly dependent because $\mathbf{v} = -2\,\mathbf{u}$.

(c) Determine whether the vectors $\mathbf{v}_1 = (\,1,\ -2,\ 1\,),\ \mathbf{v}_2 = (\,2,\ 1,\ -1\,),\ \mathbf{v}_3 = (\,7,\ -4,\ 1\,)$ are linearly dependent or linearly independent.

SOLUTION

Method 1. We need to determine whether or not there are three numbers $c_1,\ c_2,\ c_3,$ not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

Equating the components of the vector on the left and the vector on the right, we get the homogeneous system of equations

$$\begin{aligned} c_1 + 2c_2 + 7c_3 &= 0 \\ -2c_1 + c_2 - 4c_3 &= 0 \\ c_1 - c_2 + c_3 &= 0 \end{aligned} \tag{A}$$

Writing the augmented matrix and reducing to row echelon form, we get

$$\left(\begin{array}{rrr|r} 1 & 2 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 2 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The row echelon form implies that system (A) has infinitely nontrivial solutions. Thus, we can find three numbers $c_1,\ c_2,\ c_3,$ not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

In fact, we can find infinitely many such sets $c_1,\ c_2,\ c_3$. Note that the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ appear as the columns of the coefficient matrix in system (A).

Method 2. Form the matrix with rows $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ and reduce to echelon form:

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 7 & -4 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -3/5 \\ 0 & 0 & 0 \end{pmatrix}.$$

The row of zeros indicates that the zero vector is a nontrivial linear combination of $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$. Thus the vectors are linearly dependent.

Method 3. Calculate the determinant of the matrix of coefficients:

$$\begin{vmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -1 & 1 \end{vmatrix} = 0.$$

Therefore, as we saw in the preceding section, system (A) has infinitely many nontrivial solutions.

(d) Determine whether the vectors $\mathbf{v}_1 = (1,\ 2,\ -3),\ \mathbf{v}_2 = (1,\ -3,\ 2),\ \mathbf{v}_3 = (2,\ -1,\ 5)$ are linearly dependent or linearly independent.

SOLUTION In part (c) we illustrated three methods for determining whether or not a set of vectors is linearly dependent or linearly independent. We could use any one of the three methods here. The determinant method is probably the easiest. Since a determinant can be evaluated by expanding across any row or down any column, it does not make any difference whether we write the vectors as rows or columns. We'll write them as rows.

$$\begin{vmatrix} 1 & 2 & -3 \\ 1 & -3 & 2 \\ 2 & -1 & 5 \end{vmatrix} = -30.$$

Since the determinant is nonzero, the only solution to the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

is the trivial solution; the vectors are linearly independent.

Example 3. Determine whether the vectors $\mathbf{v}_1 = (1,\ 2,\ -4),\ \mathbf{v}_2 = (2,\ 0,\ 5),\ \mathbf{v}_3 = (1,\ -1,\ 7),\ \mathbf{v}_4 = (2,\ -2,\ -6)$ are linearly dependent or linearly independent.

SOLUTION In this case we need to determine whether or not there are four numbers $c_1,\ c_2,\ c_3,\ c_4$, not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}.$$

Equating the components of the vector on the left and the vector on the right, we get the homogeneous system of equations

$$\begin{aligned} c_1 + 2c_2 + c_3 + 2c_4 &= 0 \\ 2c_1 - c_3 - 2c_4 &= 0 \\ -4c_1 + 5c_2 + 7c_3 - 6c_4 &= 0 \end{aligned}$$

We know that this system has nontrivial solutions since it is a homogeneous system with more unknowns than equations. Thus the vectors must be linearly dependent.

Example 4. Vectors in $\mathbb{R}^4$

Let $\mathbf{v}_1 = (2,\ 0,\ -1\ 4)$, $\mathbf{v}_2 = (2,\ -1,\ 0\ 2)$, $\mathbf{v}_3 = (-2,\ 4,\ -3\ 4)$, $\mathbf{v}_4 = (1,\ -1,\ 3\ 0)$, $\mathbf{v}_5 = (0,\ 1,\ -5\ 3)$.

(a) Determine whether the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4,\ \mathbf{v}_5$ are linearly dependent of linearly independent.

SOLUTION The vectors are linearly dependent because the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{0}.$$

leads to a homogeneous system with more unknowns than equations.

(b) Determine whether the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4$ are linearly dependent or linearly independent.

SOLUTION To test for dependence/independence in this case, we have three options.

1. Solve the system of equations

   $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}.$$

   A nontrivial solution implies that the vectors are linearly dependent; if the trivial solution is the only solution, then the vectors are linearly independent.
2. Form the matrix A having $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4$ as the rows and reduce to row-echelon form. If the row-echelon form has one or more rows of zeros, the vectors are linearly dependent; four nonzero rows means the vectors are linearly independent.
3. Calculate $\det A$. $\det A = 0$ implies that the vectors are linearly dependent; $\det A \neq 0$ implies that the vectors are linearly independent.

Options 1 and 2 are essentially equivalent; the difference being that in option 1 the vectors appear as columns. Option 2 requires a little less writing so we'll use it.

$$A = \begin{pmatrix} 2 & 0 & -1 & 4 \\ 2 & -1 & 0 & 2 \\ -2 & 4 & -3 & 4 \\ 1 & -1 & 3 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -6 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad \text{verify this}$$

Therefore the vectors are linearly dependent. You can also check that $\det A = 0$.

(c) Determine whether $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ are linearly dependent or linearly independent.

SOLUTION Calculating a determinant is not an option here; three vectors with four components do not form a square matrix. We'll row reduce

$$A = \begin{pmatrix} 2 & 0 & -1 & 4 \\ 2 & -1 & 0 & 2 \\ -2 & 4 & -3 & 4 \end{pmatrix}$$

As you can verify,

$$\begin{pmatrix} 2 & 0 & -1 & 4 \\ 2 & -1 & 0 & 2 \\ -2 & 4 & -3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & -1 & 4 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore the vectors are linearly dependent.

(d) Determine whether the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_4,\ \mathbf{v}_5$ are linearly dependent or linearly independent.

SOLUTION You can verify that

$$\begin{vmatrix} 2 & 0 & -1 & 4 \\ 2 & -1 & 0 & 2 \\ 1 & -1 & 3 & 0 \\ 0 & 1 & -5 & 3 \end{vmatrix} = -5.$$

Therefore the vectors are linearly independent. ■

In general, suppose that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$ is a set of vectors in $\mathbb{R}^n$:

1. If $k > n$, the vectors are linearly dependent.
2. If $k = n$, write the matrix A having $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ as rows. Either reduce A to row echelon form, or calculate $\det A$. A row of zeros or $\det A = 0$ implies that the vectors are linearly dependent; all rows nonzero or $\det A \neq 0$ implies that the vectors are linearly independent.
3. If $k < n$, write the matrix A having $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ as rows and reduce to row echelon form. A row of zeros implies that the vectors are linearly dependent; all rows nonzero implies that the vectors are linearly independent.

Another look at systems of linear equations

Consider the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Note that we can write this system as the vector equation

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

which is

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{b}$$

where

$$\mathbf{v}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ \ldots, \ \mathbf{v}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Written in this form, the question of solving the system of equations can be interpreted as asking whether or not the vector $\mathbf{b}$ can be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

As we know, $\mathbf{b}$ may be written uniquely as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ (the system has a unique solution); $\mathbf{b}$ may not be expressible as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ (the system has no solution); or it may be possible to represent $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in infinitely many different ways (the system has infinitely many solutions).

Linear Dependence and Linear Independence of Functions

Two functions, f and g, are linearly dependent if one is a multiple of the other; otherwise they are linearly independent.

DEFINITION 2. Let $f_1, f_2, f_3, \ldots, f_n$ be functions defined on an interval I. The functions are *linearly dependent* if there exist n real numbers $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1f_1(x) + c_2f_2(x) + c_3f_3(x) + \cdots + c_nf_n(x) \equiv 0;$$

that is,

$$c_1f_1(x) + c_2f_2(x) + c_3f_3(x) + \cdots + c_nf_n(x) = 0 \quad \text{for all } x \in I.$$

Otherwise the functions are *linearly independent.*

Equivalently, the functions $f_1, f_2, f_3, \ldots, f_n$ are linearly independent if

$$c_1f_1(x) + c_2f_2(x) + c_3f_3(x) + \cdots + c_nf_n(x) \equiv 0$$

only when $c_1 = c_2 = \cdots = c_n = 0$.

Example 5. Let $f_1(x) = 1, \ f_2(x) = x, \ f_3(x) = x^2$ on $I = (-\infty, \infty)$. Show that f_1, f_2, f_3 are linearly independent.

SOLUTION Suppose that the functions are linearly dependent. Then there exist three numbers c_1, c_2, c_3, not all zero, such that

$$c_1\,1 + c_2\,x + c_3\,x^2 \equiv 0. \tag{B}$$

Method 1. Since (B) holds for all x, we'll let $x = 0$. Then we have

$$c_1 + c_2(0) + c_3(0) = 0 \quad \text{which implies} \quad c_1 = 0.$$

Since $c_1 = 0$, (B) becomes

$$c_2x + c_3x^2 \equiv 0 \quad \text{or} \quad x\,[c_2 + c_3x] \equiv 0.$$

Since x is not identically zero, we must have $c_2 + c_3 x \equiv 0$. Letting $x = 0$, we have $c_2 = 0$. Finally, $c_1 = c_2 = 0$ implies $c_3 = 0$. This contradicts our assumption that $f_1,\ f_2,\ f_3$ are linearly dependent. Thus, the functions are linearly independent.

Method 2. Since (B) holds for all x, we'll evaluate at $x = 0,\ x = 1,\ x = 2$. This yields the system of equations

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_3 &= 0 \\ c_1 + 2c_2 + 4c_3 &= 0. \end{aligned}$$

It is easy to verify that the only solution of this system is $c_1 = c_2 = c_3 = 0$. Thus, the functions are linearly independent.

Method 3. Our functions are differentiable, so we'll differentiate (B) twice to get

$$\begin{aligned} c_1\,1 + c_2\,x + c_3\,x^2 &\equiv 0 \\ c_2 + 2c_3 x &\equiv 0 \\ 2c_3 &\equiv 0 \end{aligned}$$

From the last equation, $c_3 = 0$. Substituting $c_3 = 0$ in the second equation gives $c_2 = 0$. Substituting $c_2 = 0$ and $c_3 = 0$ in the first equation gives $c_1 = 0$. Thus (B) holds only when $c_1 = c_2 = c_3 = 0$, which implies that the functions are linearly independent.

Example 6. Let $f_1(x) = \sin x,\ f_2(x) = \cos x,\ f_3(x) = \sin\left(x - \frac{1}{6}\pi\right),\ x \in (-\infty, \infty)$. Are these functions linearly dependent or linearly independent?

SOLUTION By the addition formula for the sine function

$$\sin\left(x - \tfrac{1}{6}\pi\right) = \sin x \cos \tfrac{1}{6}\pi - \cos x \sin \tfrac{1}{6}\pi = \tfrac{1}{2}\sqrt{3}\,\sin x - \tfrac{1}{2}\cos x,$$

Since f_3 is a linear combination of f_1 and f_2, we can conclude that $f_1,\ f_2,\ f_3$ are linearly dependent on $(-\infty, \infty)$.

A Test for Linear Independence of Functions; Wronskian

Our test for linear independence is an extension of Method 3 in Example 5.

THEOREM 1. Suppose that the functions $f_1,\ f_2,\ f_3,\ \ldots,\ f_n$ are $n-1$-times differentiable on an interval I. If the functions are linearly dependent on I, then

$$\begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ f_1''(x) & f_2''(x) & \ldots & f_n''(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix} \equiv 0 \quad \text{on} \quad I.$$

Proof Since the functions are linearly dependent on I, there exist n numbers $c_1,\ c_2,\ \ldots,\ c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) \equiv 0.$$

Differentiating this equation $n-1$ times, we get the system of equations

$$\begin{array}{rcl} c_1f_1(x)+c_2f_2(x)+\cdots+c_nf_n(x) & \equiv & 0 \\ c_1f_1'(x)+c_2f_2'(x)+\cdots+c_nf_n'(x) & \equiv & 0 \\ c_1f_1''(x)+c_2f_2''(x)+\cdots+c_nf_n''(x) & \equiv & 0 \\ \vdots & \vdots & \\ c_1f_1^{(n-1)}(x)+c_2f_2^{(n-1)}(x)+\cdots+c_nf_n^{(n-1)}(x) & \equiv & 0 \end{array} \tag{A}$$

Choose any point $a \in I$ and consider the system of n equations in n unknowns $z_1,\ z_2,\ \ldots, z_n$:

$$\begin{array}{rcl} f_1(a)z_1+f_2(a)z_2+\cdots+f_n(a)z_n & = & 0 \\ f_1'(a)z_1+f_2'(a)z_2+\cdots+f_n'(a)z_n & = & 0 \\ f_1''(a)z_1+f_2''(a)z_2+\cdots+f_n''(a)z_n & = & 0 \\ \vdots & \vdots & \\ f_1^{(n-1)}(a)z_1+f_2^{(n-1)}(a)z_2+\cdots+f_n^{(n-1)}(a)z_n & = & 0 \end{array}$$

This is a homogeneous system which, from (A), has a nontrivial solution $c_1,\ c_2,\ \ldots, c_n$. Therefore, as we showed in Section 5.4,

$$\begin{vmatrix} f_1(a) & f_2(a) & \ldots & f_n(a) \\ f_1'(a) & f_2'(a) & \ldots & f_n'(a) \\ f_1''(a) & f_2''(a) & \ldots & f_n''(a) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(a) & f_2^{(n-1)}(a) & \ldots & f_n^{(n-1)}(a) \end{vmatrix} = 0.$$

Since a was *any* point on I, we conclude that the determinant is zero for all points in I. ■

DEFINITION 3. Suppose that the functions $f_1,\ f_2,\ f_3,\ \ldots,\ f_n$ are $n-1$-times differentiable on an interval I. The determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ f_1''(x) & f_2''(x) & \ldots & f_n''(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the *Wronskian of* $f_1,\ f_2,\ f_3,\ \ldots,\ f_n$.

Theorem 1 can be stated equivalently as:

COROLLARY Suppose that the functions $f_1,\ f_2,\ f_3,\ \ldots,\ f_n$ are $(n-1)$-times differentiable on an interval I and let $W(x)$ be their Wronskian. If $W(x) \neq 0$ for at least one $x \in I$, then the functions are linearly independent on I.

This is a useful test for determining the linear independence of a set of functions.

Example 7. Show that the functions $f_1(x) \equiv 1,\ f_2(x) = x,\ f_3(x) = x^2,\ f_4(x) = x^3$ are linearly independent.

SOLUTION These functions are three-times differentiable on $(-\infty, \infty)$. Their Wronskian is

$$W(x) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12.$$

Since $W \neq 0$, the functions are linearly independent.

Note: You can use the Wronskian to show that any set of distinct powers of x is a linearly independent set.

Caution Theorem 1 says that if a set of (sufficiently differentiable) functions is linearly dependent on an interval I, then their Wronskian is identically zero on I. The theorem *does not say* that if the Wronskian of a set of functions is identically zero on some interval, then the functions are linearly dependent on that interval. Here is an example of a pair of functions which are linearly independent and whose Wronskian is identically zero.

Example 8. Let $f(x) = x^2$ and let

$$g(x) = \begin{cases} -x^2 & -2 < x < 0 \\ x^2 & 0 \leq x < 2 \end{cases}$$

on $(-2, 2)$. The only question is whether g is differentiable at 0. You can verify that it is. Thus we can form their Wronskian:

For $x \geq 0$,

$$W(x) = \left| \begin{pmatrix} x^2 & x^2 \\ 2x & 2x \end{pmatrix} \right| \equiv 0.$$

For $x < 0$,

$$W(x) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} \equiv 0.$$

Thus, $W(x) \equiv 0$ on $(-2, 2)$.

We can state that f and g are linearly independent because neither is a constant multiple of the other ($f = g$ on $[0, 2)$, $f = -g$ on $(-2, 0)$). Another way to see this is: Suppose that f and g are linearly dependent. Then there exist two numbers c_1, c_2, not both zero, such that

$$c_1 f(x) + c_2 g(x) \equiv 0 \quad \text{on} \quad (-2, 2).$$

If we evaluate this identity at $x = 1$ and $x = -1$, we get the pair of equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= 0 \end{aligned}$$

The only solution of this pair of equations is $c_1 = c_2 = 0$. Thus, f and g are linearly independent.

Exercises 5.4A

Exercises 1 – 8. Determine whether the set of vectors is linearly dependent or linearly independent. If it is linearly dependent, express one of the vectors as a linear combination of the others.

1. $\{(1,\ -2,\ 3),\ (-2,\ 4,\ 1),\ (-4,\ 8,\ 9)\}$.
2. $\{(1,\ 2,\ 5),\ (1,\ -2,\ 1),\ (2,\ 1,\ 4)\}$.
3. $\{(1,\ -1,\ 3),\ (0,\ 2,\ 3),\ (1,\ -1,\ 2),\ (-2,\ 6,\ 3)\}$.
4. $\{(1,\ 2,\ -3),\ (1,\ -3,\ 2),\ (2,\ -1,\ 5)\}$.
5. $\{(1,\ -2,\ 1),\ (2,\ 1,\ -1),\ (7,\ -4,\ 1)\}$.
6. $\{(1,\ 0,\ 2\ \ -2)\ (2,\ 1,\ 0\ \ 1),\ (2,\ -1,\ 0\ \ 1)\}$.
7. $\{(0,\ 0,\ 0\ \ 1)\ (4,\ -2,\ 0\ \ 2),\ (2,\ -1,\ 0\ \ 1),\ (1,\ 1,\ 0\ \ 1)\}$.
8. $\{(1,\ -1,\ 3),\ (0,\ 2,\ 3),\ (1,\ -1,\ 2),\ (-2,\ 6,\ 3)\}$.
9. For which values of b are the vectors $(3,\ b),\ (6,\ b-1)$ linearly independent?
10. For which values of b are the vectors $(1,\ b,\ 2b)\ (2,\ 1,\ 4)$ linearly independent?
11. Consider the matrix $A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, which is in row-echelon form. Show that the row vectors of A are a linear independent set. Are the nonzero row vectors of any matrix in row-echelon form linearly independent?
12. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be linearly independent. Prove that $\mathbf{v}_1+\mathbf{v}_2,\ \mathbf{v}_1-\mathbf{v}_2$ are linearly independent.

    Calculate the Wronskian of the set of functions. Then determine whether the functions are linearly dependent or linearly independent

13. $f_1(x) = e^{ax},\ f_2(x) = e^{bx},\quad a \neq b;\quad x \in (-\infty, \infty)$.
14. $f_1(x) = \sin\, ax,\ f_2(x) = \cos\, ax,\quad a \neq 0;\quad x \in (-\infty, \infty)$.
15. $f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = x^3;\quad x \in (-\infty, \infty)$.
16. $f_1(x) = 1,\ f_2(x) = x^{-1},\ f_3(x) = x^{-2};\quad x \in (0, \infty)$.
17. [a] If the Wronskian of a set of functions is identically zero on an interval I, then the functions are linearly dependent on I. True or false?

    [b] If a set of functions is linearly dependent on an interval I, then the Wronskian of the functions is identically zero on I. True or false?

    [c] If the Wronskian of a set of functions is nonzero at some points of an interval I and zero at other points, then the functions are linearly independent on I. True or false?