Section 4.7 The Initial Value Problem and Eigenvectors

2. The system is uncoupled, so we can solve each equation independently, using the initial value problem to obtain:

$$\begin{aligned} x(t) &= x_0 e^{3t} \\ y(t) &= y_0 e^{-2t}. \end{aligned}$$

All solutions are of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{3t} \\ y_0 e^{-2t} \end{pmatrix} = x_0 \left(e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + y_0 \left(e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

So all solutions are linear combinations of

$$U(t) = e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V(t) = e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

3. (a) In order to determine that $Y(t)$ is a solution to (4.7.10), substitute $Y(t)$ into both sides of the equation $\frac{dX}{dt} = CX$:

$$\frac{dY}{dt} = \frac{d}{dt}\left(e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \frac{d}{dt} \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 0 \end{pmatrix};$$

$$CY(t) = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 0 \end{pmatrix}.$$

Similarly, show that $Z(t)$ is a solution:

$$\frac{dZ}{dt} = \frac{d}{dt}\left(e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \frac{d}{dt} \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix};$$

$$CZ(t) = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix}.$$

(b) Again, verify that $X(t) = 2Y(t) - 14Z(t)$ is a solution to (4.7.10) by substituting into both sides of the equation and noting that the values are equal:

$$\frac{dX}{dt} = \frac{d}{dt}\left(2e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 14e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \frac{d}{dt} \begin{pmatrix} 2e^{2t} - 14e^{-t} \\ 14e^{-t} \end{pmatrix} = \begin{pmatrix} 4e^{2t} + 14e^{-t} \\ -14e^{-t} \end{pmatrix};$$

$$CX(t) = C\left(2Y(t) - 14Z(t)\right) = C\left(\begin{pmatrix} 2e^{2t} \\ 0 \end{pmatrix} - \begin{pmatrix} 14e^{-t} \\ -14e^{-t} \end{pmatrix} \right) = \begin{pmatrix} 4e^{2t} + 14e^{-t} \\ -14e^{-t} \end{pmatrix}.$$

(c) As demonstrated in Section 3.4, if $Y(t)$ and $Z(t)$ are both solutions to (4.7.10), then $X(t) = \alpha Y(t) + \beta Z(t)$ is also a solution to (4.7.10).

(d) **Answer:**

$$X(t) = 2e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution: Note that

$$X(t) = \alpha Y(t) + \beta Z(t) = \alpha e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a solution to (4.7.10). Substitute the value $X(0) = (3, -1)^t$ into the equation to find a solution with that initial condition:

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} = X(0) = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We now have the linear system:

$$\begin{array}{rcrcr} 3 & = & \alpha & + & \beta \\ -1 & = & & & -\beta \end{array}$$

which we can solve to find $\alpha = 2$ and $\beta = 1$.

4. **Answer:**

$$X(t) = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution: Note that if $Cv = \lambda v$, then $X(t) = e^{\lambda t}v$ is a solution to $\dot{X}(t) = CX(t)$. Let

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The eigenvalues corresponding to v_1 and v_2 are $\lambda_1 = 0$ and $\lambda_2 = 2$. This can be verified by calculating $Cv_1 = 0$ and $Cv_2 = 2v_2$. So,

$$X(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad X(t) = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are both solutions to $\dot{X}(t) = CX$. By the principle of superposition,

$$X(t) = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is also a solution. Substitute the given the initial condition into the equation to obtain

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = X(0) = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now, solve the linear system

$$\begin{array}{rcrcr} 2 & = & \alpha & + & \beta \\ 1 & = & \alpha & - & \beta \end{array}$$

to find that $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$.

5. **Answer:** Let

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The vector v_1 is an eigenvector of C with corresponding eigenvalue $a+b$, and v_2 is an eigenvector with eigenvalue $a-b$.

Solution: Calculate

$$Cv_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix} = (a+b) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$Cv_2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a-b \\ b-a \end{pmatrix} = (a-b) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

6. A vector (x, y) is an eigenvector of C if

$$C \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

that is, if

$$(C - \lambda I_2) \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

In this case,

$$\begin{pmatrix} 1-\lambda & 2 \\ -3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

This equation will have a nonzero solution (x, y) only if

$$\begin{pmatrix} 1-\lambda & 2 \\ -3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is not row equivalent to the identity matrix. Row reducing the matrix yields

$$\begin{pmatrix} 1 & \frac{2}{1-\lambda} \\ 0 & -1-\lambda+\frac{6}{1-\lambda} \end{pmatrix}$$

so C has an eigenvector when

$$-1-\lambda+\frac{6}{1-\lambda} = 0,$$

that is, when $\lambda^2 = -5$. Therefore, C has no real eigenvectors.

11. (a) **Answer:** If $\alpha = 1$ and $\beta = 1$, then

$$\alpha U(0) + \beta V(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Solution: Solve the linear system

$$\begin{array}{rcrcl} \alpha & - & \beta & = & 0 \\ & & \beta & = & 1. \end{array}$$

(b) Figure 11a shows y as a function of t. The figure was created by the MATLAB commands:

```
t = linspace(-8,2);
y = exp(t);
plot(t,y)
```

(c) Figure 11b shows the `pplane5` graph of the system, and Figure 11c shows the `y vs. t` graph.
(d) The two plots are identical, since the `pplane5` command `y vs. t` graphs the y component of the solution, which is precisely what we did by hand in (b).
(e) In this case, $\alpha = 2$ and $\beta = 1$. Since the y component of $U(t)$ is zero, the graphs of $y(t)$ are identical to those in (b) and (c).

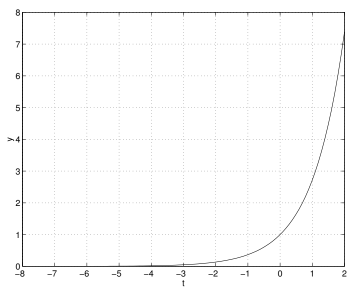

Figure 11a

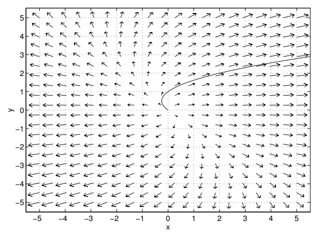

Figure 11b

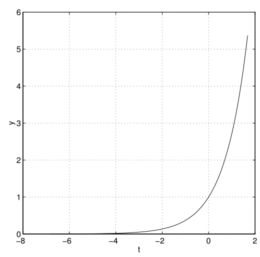

Figure 11c

Section 4.8 Eigenvalues of 2×2 Matrices

1. **Answer:** The matrix is not invertible when $\lambda = 5$ or $\lambda = -1$.
Solution: Corollary 3.8.3 states that a matrix is not invertible if and only if the determinant is zero; in this case, if

$$(1 - \lambda)(3 - \lambda) - (2)(4) = \lambda^2 - 4\lambda - 5 = 0.$$

3. The determinant of the matrix is 23, the trace is 7, and the characteristic polynomial is $p(\lambda) = \lambda^2 - 7\lambda + 23$.

4. The determinant of the matrix is -5, the trace is 0, and the characteristic polynomial is $p(\lambda) = \lambda^2 - 5$.

5. The determinant of the matrix is 0, the trace is 9, and the characteristic polynomial is $p(\lambda) = \lambda^2 - 9\lambda$.

6. **Answer:** The eigenvalues of the matrix are $\lambda_1 = -5$ and $\lambda_2 = 1$.
Solution: For any 2×2 matrix A, the eigenvalues are the roots of the characteristic polynomials, which can be found by solving the equation $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$. In this case, the characteristic polynomial of the matrix is $\lambda^2 + 4\lambda - 5$.

8. **Answer:** The matrix has a double eigenvalue at $\lambda = 1$.
Solution: The characteristic polynomial of the matrix is $\lambda^2 - 2\lambda + 1$. The eigenvalues are the roots of this polynomial.

10. The matrix has complex conjugate eigenvalues, since the mapping is a contracting rotation.

11. The mapping has real eigenvalues. Repeated mapping of any vector leads to a vector in one of two invariant directions. These directions are the eigenvectors.

Section 4.9 Initial Value Problems Revisited

2. **Answer:** The solution to $\dot{X} = CX$ satisfying this initial condition is

$$X(t) = 3e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3e^{2t} - 2e^{-t} \\ -2e^{-t} \end{pmatrix}.$$

Solution: First, find the eigenvalues of C, which are the roots of the characteristic polynomial

$$p_C(\lambda) = \lambda^2 - \mathrm{tr}(C)\lambda + \det(C) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

So the eigenvalues are: $\lambda_1 = 2$ and $\lambda_2 = -1$. To find the eigenvector associated to each eigenvalue, solve the equation $(C - \lambda_j I_2)v_j = 0$ for $j = 1$ and $j = 2$. Solve

$$\left(\begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) v_1 = \begin{pmatrix} 0 & -3 \\ 0 & -3 \end{pmatrix} v_1 = 0$$

to obtain $v_1 = (1,0)^t$ and solve

$$\left(\begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) v_2 = \begin{pmatrix} 3 & -3 \\ 0 & 0 \end{pmatrix} v_2 = 0$$

to obtain $v_2 = (1,1)^t$. We can then write the general solution

$$X(t) = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 = \alpha_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From this formula, find α_1 and α_2 by solving

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} = X(0) = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \alpha_2 \\ \alpha_2 \end{pmatrix}.$$

Solving the linear system

$$\begin{array}{rcrcr} \alpha_1 & + & \alpha_2 & = & 1 \\ & & \alpha_2 & = & -2 \end{array}$$

we obtain $\alpha_1 = 3$ and $\alpha_2 = -2$ and find the solution to the differential equation.

3. **Answer:** The solution to $\dot{X} = CX$ satisfying this initial condition is

$$X(t) = \frac{8}{3} e^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{5}{3} e^{-2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 8e^{t} - 10e^{-2t} \\ 16e^{t} - 5e^{-2t} \end{pmatrix}.$$

Solution: First, find the eigenvalues of C, which are the roots of the characteristic polynomial

$$p_C(\lambda) = \lambda^2 - \mathrm{tr}(C)\lambda + \det(C) = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2).$$

So the eigenvalues are: $\lambda_1 = 1$ and $\lambda_2 = -2$. To find the eigenvector associated to each eigenvalue, solve the equation $(C - \lambda_j I_2)v_j = 0$ for $j = 1$ and $j = 2$. Solve

$$\left(\begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) v_1 = \begin{pmatrix} -4 & 2 \\ -2 & 1 \end{pmatrix} v_1 = 0$$

to obtain $v_1 = (1,2)^t$ and solve

$$\left(\begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) v_2 = \begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix} v_2 = 0$$

to obtain $v_2 = (2,1)^t$. We can then write the general solution

$$X(t) = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 = \alpha_1 e^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 e^{-2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

From this formula, find α_1 and α_2 by solving

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix} = X(0) = \alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 + 2\alpha_2 \\ 2\alpha_1 + \alpha_2 \end{pmatrix}.$$

Solving the linear system

$$\begin{array}{rcrcr} \alpha_1 & + & 2\alpha_2 & = & -1 \\ 2\alpha_1 & + & \alpha_2 & = & 3 \end{array}$$

we obtain $\alpha_1 = \frac{8}{3}$ and $\alpha_2 = -\frac{5}{3}$ and find the solution to the differential equation.

4. **Answer:** The solution to $\dot{X} = CX$ satisfying this initial condition is

$$X(t) = \frac{3}{2}e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3e^{3t} - e^{t} \\ 3e^{3t} + e^{t} \end{pmatrix}.$$

Solution: First, find the eigenvalues of C, which are the roots of the characteristic polynomial

$$p_C(\lambda) = \lambda^2 - \mathrm{tr}(C)\lambda + \det(C) = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$$

So the eigenvalues are: $\lambda_1 = 3$ and $\lambda_2 = 1$. To find the eigenvector associated to each eigenvalue, solve the equation $(C - \lambda_j I_2)v_j = 0$ for $j = 1$ and $j = 2$. Solve

$$\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right) v_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} v_1 = 0$$

to obtain $v_1 = (1, 1)^t$ and solve

$$\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) v_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v_2 = 0$$

to obtain $v_2 = (1, -1)^t$. We can then write the general solution

$$X(t) = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 = \alpha_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

From this formula, find α_1 and α_2 by solving

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = X(0) = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha_1 + 2\alpha_2 \\ \alpha_1 - \alpha_2 \end{pmatrix}.$$

Solving the linear system

$$\begin{array}{rcrcr} \alpha_1 & + & 2\alpha_2 & = & 1 \\ \alpha_1 & - & \alpha_2 & = & 2 \end{array}$$

we obtain $\alpha_1 = \frac{3}{2}$ and $\alpha_2 = -\frac{1}{2}$ and find the solution to the differential equation.

5. **Answer:** The solution to the differential equation $\dot{X} = CX$ with the given restrictions is

$$X(t) = \frac{1}{5}e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{2}{5}e^{4t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} e^{-t} + 4e^{4t} \\ 2e^{-t} - 2e^{4t} \end{pmatrix}.$$

Solution: First, find the matrix C using the given information: First, since C is symmetric, we can write

$$C = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

Then, we are given $\mathrm{tr}(C) = a + d = 3$, so we can rewrite C as

$$C = \begin{pmatrix} a & b \\ b & 3-a \end{pmatrix}.$$

Since $X(t) = e^{-t}(1,2)^t$ is a solution, $\lambda_1 = -1$ must be an eigenvalue of C with associated eigenvector $v_1 = (1,2)^t$. Thus $Cv_1 = \lambda_1 v_1$, or

$$\begin{pmatrix} a & b \\ b & 3-a \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a+2b \\ b+2(3-a) \end{pmatrix} = \begin{pmatrix} a+2b \\ -2a+b+6 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

This equation yields the linear system

$$\begin{array}{rcrcr} a & + & 2b & = & -1 \\ -2a & + & b & = & -8 \end{array}$$

which we can solve to obtain $a = 3$ and $b = -2$. So

$$C = \begin{pmatrix} 3 & -2 \\ -2 & 0 \end{pmatrix}.$$

Now, find λ_2, the other root of

$$p_C(\lambda) = \lambda^2 - \mathrm{tr}(C) + \det(C) = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4).$$

Thus, the second eigenvalue of C is $\lambda_2 = 4$, and we can solve

$$(C - \lambda_2 I_2)v_2 = \left(\begin{pmatrix} 3 & -2 \\ -2 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right)v_2 = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}v_2 = 0$$

to obtain $v_2 = (2,-1)^t$, the eigenvector associated to λ_2. The general solution to $\dot{X} = CX$ is

$$X(t) = \alpha_1 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 e^{4t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Find α_1 and α_2 by substituting the initial condition $X(0) = X_0$ into this formula:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = X(0) = \alpha_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha_1 + 2\alpha_2 \\ 2\alpha_1 - \alpha_2 \end{pmatrix}.$$

Thus, $\alpha_1 = \frac{1}{5}$ and $\alpha_2 = \frac{2}{5}$, so we find the general solution.

7. **Answer:** The solution to the differential equation $\dot{X} = CX$ with the given initial condition is

$$X(t) \approx 1.0860e^{-1.8035t} \begin{pmatrix} -0.9005 \\ 0.4348 \end{pmatrix} - 2.4527e^{4.4835t} \begin{pmatrix} -0.8880 \\ -0.4598 \end{pmatrix}.$$

Solution: In MATLAB, enter the matrix `C` and the vector `X0`. Then, type

```
lambda = eig(C)
```

to obtain the eigenvalues of C, which are $\lambda_1 \approx -1.0835$ and $\lambda_2 \approx 4.4835$. Find the eigenvectors v_1 and v_2 associated to λ_1 and λ_2 by typing

```
v1 = null(C - lambda(1)*eye(2))
v2 = null(C - lambda(2)*eye(2))
```

Thus, the general solution is

$$X(t) = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 \approx \alpha_1 e^{-1.8035t} \begin{pmatrix} -0.9005 \\ 0.4348 \end{pmatrix} + \alpha_2 e^{4.4835t} \begin{pmatrix} -0.8880 \\ -0.4598 \end{pmatrix}.$$

The initial condition is

$$X_0 = X(0) = \alpha_1 v_1 + \alpha_2 v_2.$$

We can solve this linear system by creating the matrix $A = (v_1|v_2)$, and computing $A^{-1}X_0$. In MATLAB, type

```
A = [v1 v2]
alpha = inv(A)*X0
```

obtaining $\alpha_1 \approx 1.0860$ and $\alpha_2 \approx -2.4527$.

8. **Answer:** $X(0.5) = (0.155, 0.386)^t$ and the two methods agree to three decimal places.
Solution: (a) The result of the `pplane5` integration is given in Figure 8a. After zooming several times we arrive at Figure 8b. By inspection $X(0.5) = (0.155, 0.386)$.
(b) Enter the matrix C into MATLAB by typing

```
C = [2.65 -2.34; -1.5 -1.2];
```

Find the eigenvalues and eigenvectors of this matrix by typing `[V,D] = eig(C)` and obtaining

```
V =
    0.9510    0.4525
   -0.3093    0.8918
D =
    3.4112         0
         0   -1.9612
```

Therefore the general solution to this differential equation is:

$$X(t) = \alpha e^{3.4112t} \begin{pmatrix} 0.9510 \\ -0.3093 \end{pmatrix} + \beta e^{-1.9612t} \begin{pmatrix} 0.4525 \\ 0.8918 \end{pmatrix}.$$

It follows that

$$X(0) = V \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = V^{-1}X_0 = \begin{pmatrix} 0.9510 & 0.4525 \\ -0.3093 & 0.8918 \end{pmatrix} \begin{pmatrix} 0.5 \\ 1.0 \end{pmatrix} = \begin{pmatrix} -0.0067 \\ 1.1191 \end{pmatrix}$$

The last calculation is done by typing `coeff = inv(V)*[0.5;1.0]`. Therefore, the solution to the initial value problem is:

$$X(t) = -0.0067e^{3.4112t} \begin{pmatrix} 0.9510 \\ -0.3093 \end{pmatrix} + 1.1191e^{-1.9612t} \begin{pmatrix} 0.4525 \\ 0.8918 \end{pmatrix}.$$

We can evaluate $X(0.5)$ in MATLAB by typing

```
X5 = coeff(1)*exp(D(1,1)*0.5)*V(:,1) + coeff(2)*exp(D(2,2)*0.5)*V(:,2)
```

and obtaining

```
X5 =
    0.1547
    0.3858
```

(c) The two answers agree to three decimal places.

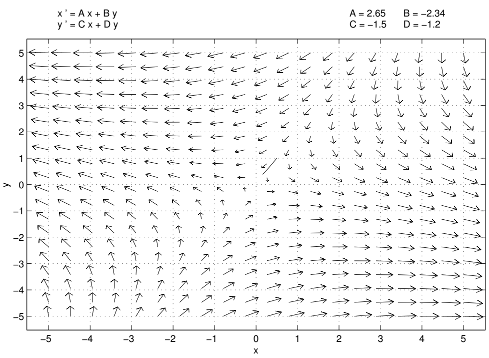

Figure 8a

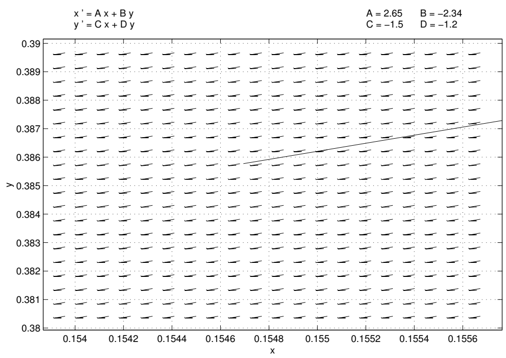

Figure 8b