SUMMARY: CHAPTERS 14 and 15

1. Functions of Several Variables: Let $z = f(x, y)$ or $w = f(x, y, z)$.

Domain D**:** If the domain of f is not given explicitly, or implicitly by an application, then, by convention, the domain is the set of all points $\mathbf{x}$ [$\mathbf{x} = (x, y)$ or $\mathbf{x} = (x, y, z)$] such that $f(\mathbf{x})$ is a real number.

Range: The range of f is the set of values $f(\mathbf{x})$, $\mathbf{x} \in D$.

Graph: Let $z = f(x, y)$, $(x, y) \in D$. The *graph* of f is the set of all points $(x, y, z) = (x, y, f(x, y))$ in space. For our purposes, the graph of $z = f(x, y)$ is a surface in space.

The graph of $w = f(x, y, z)$ is a "hypersurface" in 4-dimensional space.

Level curves and level surfaces: Given $z = f(x, y)$. The plane curves $f(x, y) = C$, C constant are called the *level curves* of f. Let $w = F(x, y, z)$. The surfaces in 3-space $F(x, y, z) = C$ are called the *level surfaces* of F.

2. Limits: Given $z = f(x, y)$ or $w = f(x, y, z)$.

Let $\mathbf{x}_0 = (x_0, y_0)$ or (x_0, y_0, z_0) and $\mathbf{x} = (x, y)$ or (x, y, z). Assume that f is defined in some neighborhood of the point $\mathbf{x}_0$ except, possibly, at $\mathbf{x}_0$ itself.

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L$$

if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(\mathbf{x}) - L| < \epsilon \quad \text{whenever} \quad \|\mathbf{x} - \mathbf{x}_0\| < \delta.$$

3. Partial Derivatives: Given a function $z = f(x, y)$. Let $\mathbf{x} = (x, y)$.

The *partial derivative of* f *with respect to* x *at the point* $\mathbf{x}$ is given by

$$f_x = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} = \lim_{h \to 0} \frac{f(\mathbf{x} + h\,\mathbf{i}) - f(\mathbf{x})}{h} \quad \text{(provided the limit exists)}\ .$$

The *partial derivative of* f *with respect to* y *at the point* $\mathbf{x}$ is given by

$$f_y = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h} = \lim_{h \to 0} \frac{f(\mathbf{x} + h\,\mathbf{j}) - f(\mathbf{x})}{h} \quad \text{(provided the limit exists)}\ .$$

Corresponding definitions hold for the function $w = F(x, y, z)$. For example:

The *partial derivative of* F *with respect to* z *at the point* $\mathbf{x} = (x, y, z)$ is given by

$$F_z = \lim_{h \to 0} \frac{F(x, y, z+h) - F(x, y, z)}{h} = \lim_{h \to 0} \frac{F(\mathbf{x} + h\,\mathbf{k}) - F(\mathbf{x})}{h} \quad \text{(provided the limit exists)} .$$

Notations: $f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial z}{\partial x}$, $\quad f_y = \dfrac{\partial f}{\partial y} = \dfrac{\partial z}{\partial y}$, etc.

Higher Derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial\,(\partial f/\partial x)}{\partial x}, \quad f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial\,(\partial f/\partial x)}{\partial y},$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial\,(\partial f/\partial y)}{\partial y}, \quad f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial\,(\partial f/\partial y)}{\partial x}.$$

THEOREM: If the first partials and the "mixed" second partials are continuous on D, then $f_{xy} = f_{yx}$.

4. Gradient: Given $z = f(x, y)$; $w = F(x, y, z)$

Gradient of f: $\nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j}$ $\qquad$ gradient of F: $\nabla F = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$

∇f is normal to the level curves of f; ∇F is normal to the level surfaces of F.

Directional derivatives: Let $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ be a unit vector and let $\mathbf{x} = (x, y)$. The directional derivative of f at $\mathbf{x}$ in the direction $\mathbf{u}$ is given by:

$$f'_{\mathbf{u}} = \lim_{h \to 0} \frac{f(\mathbf{x} + h\,\mathbf{u}) - f(\mathbf{x})}{h} \quad \text{provided the limit exists}$$

Similarly, if $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$ is a unit vector and $\mathbf{x} = (x, y, z)$, then the directional derivative of F at $\mathbf{x}$ in the direction $\mathbf{u}$ is

$$F'_{\mathbf{u}} = \lim_{h \to 0} \frac{F(\mathbf{x} + h\,\mathbf{u}) - F(\mathbf{x})}{h} \quad \text{provided the limit exists}$$

Calculation of directional derivatives:

$$f'_{\mathbf{u}} = \nabla f(\mathbf{x}) \cdot \mathbf{u}; \qquad F'_{\mathbf{u}} = \nabla F(\mathbf{x}) \cdot \mathbf{u}.$$

Tangent planes and normal lines: Equations for the tangent plane and normal line to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0), $[z_0 = f(x_0, y_0)]$ are given by:

tangent plane: $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0;$

normal line: $x = x_0 + f_x(x_0, y_0)\,t,\ y = y_0 + f_y(x_0, y_0)\,t,\ z = z_0 - t$

Equations for the tangent plane and normal line to the level surface $F(x, y, z) = C$ at the point $P_0\,(x_0, y_0, z_0)$ are given by:

tangent plane: $F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0;$

normal line: $x = x_0 + F_x(P_0)\,t,\ y = y_0 + F_y(P_0)t,\ z = z_0 + F_z(P_0)\,t.$

5. Chain Rules: Given $z = f(x, y);\ w = F(x, y, z)$

a. If $x = x(t)$ and $y = y(t)$, then: $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}.$

If $x = x(t), y = y(t)$ and $z = z(t)$, then: $\dfrac{dF}{dt} = \dfrac{\partial F}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial F}{\partial z}\dfrac{dz}{dt}.$

b. If $x = x(s, t)$ and $y = y(s, t)$, then: $\dfrac{\partial f}{ds} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}.$

If $x = x(s, t), y = y(s, t)$ and $z = z(s, t)$, then: $\dfrac{\partial F}{\partial t} = \dfrac{\partial F}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial F}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial t}.$

Similarly for $\dfrac{\partial f}{\partial s}, \dfrac{\partial F}{\partial s}.$

6. Extreme Values: Let f be a function of several variables $[z = f(x, y),\ w = f(x, y, z)]$, and let $\mathbf{x}_0$ be an interior point of the domain of f.

a. f has a *local maximum* at $\mathbf{x}_0$ if $f(\mathbf{x}_0) \geq f(\mathbf{x})$ for all $\mathbf{x}$ in some neighborhood of $\mathbf{x}_0$.

b. f has a *local minimum* at $\mathbf{x}_0$ if $f(\mathbf{x}_0) \leq f(\mathbf{x})$ for all $\mathbf{x}$ in some neighborhood of $\mathbf{x}_0$.

c. **Theorem:** If f has a local extreme value at $\mathbf{x}_0$, then either

$$\nabla f(\mathbf{x}_0) = 0 \quad \text{or} \quad \nabla f(\mathbf{x}_0) \text{ does not exist.}$$

A point $\mathbf{x}_0$ at which either of these conditions holds is called a *critical point* of f; a point $\mathbf{x}_0$ at $\nabla f(\mathbf{x}_0) = 0$ is called a *stationary point* of f.

Second Partials Test: Let $z = f(x, y)$ have continuous first and second partial derivatives in a neighborhood of a stationary point $\mathbf{x}_0$. Set

$$A = \frac{\partial^2 f}{\partial x^2}(\mathbf{x}_0), \quad B = \frac{\partial^2 f}{\partial x\,\partial y}(\mathbf{x}_0), \quad C = \frac{\partial^2 f}{\partial y^2}(\mathbf{x}_0)$$

and set $D = AC - B^2$.

1. If $D < 0$, then $\mathbf{x}_0$ is a saddle point of f.
2. If $D > 0$, then f has:

   a local minimum at $\mathbf{x}_0$ if $A > 0$,

   a local maximum at $\mathbf{x}_0$ if $A < 0$.
3. ???? if $D = 0$.

7. Lagrange Multipliers:

If $\mathbf{x}_0$ maximizes or minimizes f subject to the side condition $g(\mathbf{x}) = 0$, then there exists a scalar λ such that

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0) \qquad \text{(provided } \nabla g(\mathbf{x}_0) \neq 0\text{)}.$$

To find the maxima/minima of $z = f(x, y)$ subject to the side condition $g(x, y) = 0$, solve the system of equations:

$$\begin{aligned} f_x(x,y) &= \lambda\, g_x(x,y) \\ f_y(x,y) &= \lambda\, g_y(x,y) \\ g(x,y) &= 0 \end{aligned}$$

To find the maxima/minima of $w = F(x, y, z)$ subject to the side condition $G(x, y, z) = 0$, solve the system of equations:

$$\begin{aligned} F_x(x,y,z) &= \lambda\, G_x(x,y,z) \\ F_y(x,y,z) &= \lambda\, G_y(x,y,z) \\ F_z(x,y,z) &= \lambda\, G_z(x,y,z) \\ G(x,y) &= 0 \end{aligned}$$

8. Reconstructing a Function from its Gradient:

Let $P = P(x, y)$ and $Q = Q(x, y)$ be continuously differentiable functions on a simply connected open region Ω. The vector function $\mathbf{R}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is the gradient of some function $z = f(x, y)$ on Ω if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

If $\mathbf{R}$ is the gradient of a function f, then

$$f(x,y) = \int P(x,y)\,dx + \phi(y)$$

where ϕ is a function of y which is to be determined so that $\partial f/\partial y = Q(x, y)$.

Alternatively,

$$f(x,y) = \int Q(x,y)\,dy + \psi(y)$$

where ψ is a function of x which is to be determined so that $\partial f/\partial x = P(x, y)$.