THE MAIN INEQUALITY OF 3D VECTOR ANALYSIS

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ABSTRACT. This paper proves some simple inequalities for Sobolev vector fields on nice bounded 3-dimensional regions, subject to homogeneous mixed normal and tangential boundary data. The fields just have divergence and curl in L^2. For the limit cases of prescribed normal, respectively tangential, on the whole boundary, the inequalities were proved by K.O. Friedrichs who called the result the *main inequality of vector analysis*. For this mixed case, the optimal constants in the inequality are described, together with the fields for which equality holds. The detailed results depend on a special orthogonal decomposition and the analysis of associated eigenvalue problems.

1. INTRODUCTION

The analysis of many problems in classical continuum and electromagnetic field theories often depends on inequalities involving the divergence and curl of a vector field. Thus Friedrichs in [13], equation 5, calls the inequality

$$\int_\Omega [|\operatorname{curl} v|^2 + |\operatorname{div} v|^2]\, d^3x \;\geq\; c \int_\Omega |v|^2\, d^3x, \tag{1.1}$$

the *main inequality* of vector analysis. Here Ω is a nice bounded region in space and v is a vector field. He proved (1.1) held provided v satisfied either zero normal or zero tangential boundary conditions and also is L^2- orthogonal to the associated class of harmonic fields.

This and related inequalities have been further refined and described by others including Duvaut and Lions [9], Foias and Temam [12], Leis [15], and Saranen [16]. Surveys of such results are given in Cessenat [7], Girault and Raviart [14], chapter 1, section 3 and in chapter 9, section 1 of [8] written by Cessenat. The existing literature generally treats situations where the same homogeneous boundary condition is imposed everywhere of the boundary $\partial\Omega$.

Here we shall prove that (1.1) holds when v satisfies mixed zero normal and tangential boundary conditions and is L^2- orthogonal to the associated class of harmonic fields. These mixed boundary conditions arise in many electromagnetic problems and the inequality is needed for the proof of theorem 14.4 of Auchmuty and Alexander [5]. As discussed in Fernandes and Gilardi [11], the case of mixed type boundary data is quite common in engineering applications and numerical simulations.

Let $H_{DC\Sigma}(\Omega)$ be the space of vector fields which are L^2 on Ω, whose curl and divergence of the fields also are L^2 and which satisfy the mixed boundary conditions (2.8). This space has a natural inner product and is a Hilbert space. In section 3, the class of H^1-fields with the same boundary conditions is shown to be equal to $H_{DC\Sigma}(\Omega)$ with the DC-norm equivalent to the usual H^1- norm. The equivalence of the norms on these spaces was

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called the *auxiliary inequality* by Friedrichs and is proved by direct estimation in Section 3.

To prove (1.1) for fields in $H_{DC\Sigma}(\Omega)$, a special orthogonal decomposition into gradient, curl and harmonic components is developed. This result is described in theorem 9.4 and is a variation on the L^2-decomposition theory for the case of homogeneous normal or tangential data described in Auchmuty [3]. There are many possible Hodge-Weyl decompositions of vector fields into sums of a gradient, a curl and a harmonic component. The analysis here depends on ensuring that the corresponding operators are orthogonal projections with respect to the DC-inner product. For a recent geometric discussion of similar decompositions of very smooth fields see Cantarella, De Turck and Gluck [6].

Inequalities on the individual components of this decomposition are obtained by analyzing associated eigenvalue problems and obtaining coercivity results for the curl and div operators on subspaces of $H_{DC\Sigma}(\Omega)$. This is done in sections 5 and 8 and result in the inequalities described in theorems 6.1 and 10.1. These individual inequalities have many applications since, in many field theories, these operators are treated individually. These results are combined to provide the proof of (1.1). The optimal constant in the inequality is characterized and those fields for which equality holds are described. The two-dimensional version of (1.1) is implicit in the analysis of [4] and is much simpler as the vector fields can be represented using two scalar functions, a gradient potential and a stream function.

2. Definitions and Notation.

Throughout this paper, Ω is a non-empty bounded, connected, open subset of $\mathbb{R}^3$. Such sets will be called regions. Its closure is denoted by $\overline{\Omega}$ and its boundary is $\partial\Omega := \overline{\Omega} \setminus \Omega$. Points in Ω are denoted by $x = (x_1, x_2, x_3)$ and Cartesian coordinates will be used exclusively. When u, v are vectors in $\mathbb{R}^3$, their scalar product, Euclidean norm and vector product are denoted $u \cdot v$, $|u|$, and $u \wedge v$, respectively.

Further conditions on Ω will be required; namely

(B1): *Ω is a bounded region in $\mathbb{R}^3$ and $\partial\Omega$ is the union of a finite number of disjoint closed C^2 surfaces; each surface having finite surface area.*

A closed surface Σ in space is said to be C^2 if it has a unique unit outward normal ν at each point and ν is a continuously differential vector field on Σ. See [14], Section 1.1. for more details on this definition.

When (B1) holds and $\partial\Omega$ consists of $J+1$ disjoint, closed surfaces, then J is the *second Betti number* of Ω, or the dimension of the second de Rham cohomology group of Ω. Geometrically it counts the number of *holes* in the region Ω. A subset is called a component if it is a maximal (with respect to inclusion) subset.

Let $H^1(\Omega)$ be the usual Sobolev space of real-valued functions on Ω with the H^1- inner product

$$\langle \varphi, \psi \rangle_1 := \int_\Omega [\varphi(x).\psi(x) \ + \ \nabla\varphi(x) \cdot \nabla\psi(x)] \, d^3x. \tag{2.1}$$

Here $\nabla\varphi$ is the *gradient* of the function φ and is defined by

$$\nabla\varphi(x) := \big(\varphi_{,1}(x), \varphi_{,2}(x), \varphi_{,3}(x)\big) \tag{2.2}$$

with $\varphi_{,j}(x) := \partial\phi(x)/\partial x_j$ being the j^{th} weak derivative of φ on Ω. Analytically the assumption that (B1) holds is suffficient to guarantee the validity of Rellich's theorem that the imbedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact. See Amick [1].

When $\varphi \in W^{1,p}(\Omega)$ for some $p \geq 1$ and Ω obeys (B1), then the *trace of* φ *on* $\partial\Omega$ is well-defined and is a Lebesgue integrable function, see [10], Section 4.2 for details. If φ, $\psi \in W^{1,p}(\Omega)$ for some $p \geq 3/2$, then the *Gauss-Green theorem* holds in the form

$$\int_\Omega \varphi\,\psi_{,i}\,d^3x \;=\; \int_{\partial\Omega} \varphi\,\psi\,\nu_j\,d\sigma - \int_\Omega \psi\,\varphi_{,j}\,d^3x \tag{2.3}$$

for $1 \leq j \leq 3$. Here surface measure is denoted $d\sigma$ and is 2-dimensional Hausdorff measure. Two vector-valued consequences are that, when each of the following integrals is finite,

$$\int_\Omega u\cdot\nabla\varphi\,d^3x \;=\; \int_{\partial\Omega} \varphi\,(u\cdot\nu)\,d\sigma - \int_\Omega \varphi\,\mathrm{div}\,u\,d^3x, \quad \text{and} \tag{2.4}$$

$$\int_\Omega u\cdot\mathrm{curl}\,v\,d^3x \;=\; \int_{\partial\Omega} v\cdot(u\wedge\nu)\,d\sigma + \int_\Omega v\cdot\mathrm{curl}\,u\,d^3x. \tag{2.5}$$

Let Σ be a open subset of $\partial\Omega$ and define $\tilde{\Sigma} := \partial\Omega\setminus\overline{\Sigma}$. Define $C_{\Sigma 0}(\overline{\Omega})$ to be the space of continuous functions on $\overline{\Omega}$ which are zero on Σ and let $H^1_{\Sigma 0}(\Omega)$ to be the closure of $C_{\Sigma 0}(\overline{\Omega}) \cap H^1(\Omega)$ with respect to the inner product (2.1).

When $v : \Omega \to \mathbb{R}^3$ is a vector field, its Cartesian components are denoted v_i, so that $v(x) = (v_1(x), v_2(x), v_3(x))$. Its derivative matrix is $Dv(x) := (v_{j,k}(x))$. v is said to be in $L^2(\Omega;\mathbb{R}^3)$, or $H^1(\Omega;\mathbb{R}^3)$ when each component v_j is in $L^2(\Omega)$ or $H^1(\Omega)$ respectively. These are Hilbert spaces with respect to the inner products

$$\langle u, v\rangle \;:=\; \int_\Omega u(x)\cdot v(x)\,d^3x, \qquad \text{and} \tag{2.6}$$

$$\langle u, v\rangle_1 \;:=\; \int_\Omega \Big[\,u(x)\cdot v(x) \;+\; \sum_{j,k=1}^{3} u_{j,k}(x)\;v_{j,k}(x)\,\Big]\,d^3x. \tag{2.7}$$

The corresponding norms are denoted $\|u\|, \|u\|_1$ respectively. When no subscript is indicated, the corresponding norm is an L^2- norm. Let $C_{\Sigma 0}(\overline{\Omega};\mathbb{R}^3)$ be the space of continuous vector fields on Ω which satisfy

$$v\wedge\nu \;=\; 0 \;\text{ on }\; \Sigma \qquad \text{and} \;\; v\cdot\nu \;=\; 0 \text{ on } \tilde{\Sigma}. \tag{2.8}$$

Define $H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$ to be the closure of $C_{\Sigma 0}(\overline{\Omega};\mathbb{R}^3) \cap H^1(\Omega;\mathbb{R}^3)$ with respect to the inner product (2.7).

The classical case of no flux, or zero tangential component correspond to the cases $\Sigma = \emptyset$ or $\Sigma = \partial\Omega$ respectively. The corresponding Sobolev spaces will be denoted $H^1_{\nu 0}(\Omega;\mathbb{R}^3), H^1_{\tau 0}(\Omega;\mathbb{R}^3)$ and many of their properties are described in chapter 9 of [8].

When $v \in L^2(\Omega;\mathbb{R}^3)$ then a function $\rho \in L^1_{\text{loc}}(\Omega)$ is said to be the (weak) *divergence of* v *on* Ω provided

$$\int_\Omega [\varphi\,\rho + \nabla\varphi\cdot v]\;d^3x \;=\; 0. \tag{2.9}$$

for all φ in the space $C_c^\infty(\Omega)$ of all C^∞ functions on Ω whose support is a compact subset of Ω. The field u is *solenoidal* on Ω provided $\rho \equiv 0$ on Ω.

Similarly when $v \in L^2(\Omega;\mathbb{R}^3)$ then a field $\omega \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$ is the (weak) *curl of v on Ω* provided

$$\int_\Omega [v \cdot \operatorname{curl} u - \omega \cdot u]\ d^3x \ = \ 0 \qquad \text{for all } u \in C_c^\infty(\Omega;\mathbb{R}^3). \tag{2.10}$$

v is *irrotational on* Ω provided (2.10) holds with $\omega \equiv 0$.

A field $v \in L^2(\Omega;\mathbb{R}^3)$ is *harmonic* on Ω if it is irrotational and solenoidal on Ω. Define $\mathcal{H}_\Sigma(\Omega)$ to be the subspace of harmonic vector fields in $H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$. This is a closed subspace - which will be called the space of mixed harmonic fields on Ω obeying (2.8). It may be the zero subspace; in general its dimension depends on on the topology of Ω, Σ and $\tilde{\Sigma}$.

Define $H_{DC}(\Omega)$ to be the space of all fields v in $L^2(\Omega;\mathbb{R}^3)$ such that $\operatorname{div} v$ and $\operatorname{curl} v$ are also in $L^2(\Omega;\mathbb{R}^3)$. This is a Hilbert space under the inner product

$$\langle u, v\rangle_{DC} \ := \ \int_\Omega [u \cdot v \ + \ \operatorname{curl} u \cdot \operatorname{curl} v \ + \ \operatorname{div} u . \operatorname{div} v]\ d^3x. \tag{2.11}$$

This will be called the *DC-inner product* and $H_{DC\Sigma}(\Omega)$ will be the closure of $C_{\Sigma 0}(\overline{\Omega};\mathbb{R}^3) \cap H^1(\Omega;\mathbb{R}^3)$ with respect to this inner product.

In this paper our primary interest is in the space $H_{DC\Sigma}(\Omega)$ and in showing that the main inequality holds for fields in this space. To do this we require some further conditions on the region Ω and the sets Σ and $\tilde{\Sigma}$. The assumptions on the subsets where the mixed boundary conditions (2.8) hold are

(B2): *Σ and $\tilde{\Sigma}$ are nonempty and each have a finite number of disjoint open components. The Euclidean distance between disjoint components of Σ, and of $\tilde{\Sigma}$, is bounded below by a positive number.*

Note that when Σ and $\tilde{\Sigma}$ satisfy (B2), then it remains true with Σ and $\tilde{\Sigma}$ interchanged.

Our proof of the main inequality requires the analysis of certain eigenproblems. The relevant results about these eigenproblems depend on variational principles involving the maximization of quadratic functionals on closed convex subsets of a real Hilbert space. To describe these results some elementary concepts and results from convex analysis will be used. Let H be a real Hilbert space with inner product denoted by $[.,.]$ and $f : H \to (-\infty, \infty]$ be a given functional. If $f(u)$ is finite, an element $w \in H$ is a *subgradient* of f at u provided

$$f(v) \ \geq \ f(u) + [w, v-u] \qquad \text{for all} \quad v \in H.$$

The *subdifferential $\partial f(u)$ of f at u* is the set of all such subgradients. When f is convex and G-differentiable at u, then $\partial f(u) = \{Df(u)\}$ is a singleton. For more information on these issues, see Aubin [2]. In particular the *indicator functional* of a closed convex set C in H is the functional $I_C : H \to [0,\infty]$ defined by

$$I_C(u) \ = \ \begin{cases} 0 & \text{for } \ u \in C, \\ \infty & \quad \text{for } u \notin C. \end{cases} \tag{2.12}$$

When C is the closed unit ball of radius 1 in a closed subspace V of H, then its subdifferential is given, when $u \in C$, by

$$\partial I_C(u) \;=\; \begin{cases} V^{\perp} & \text{when } \|u\| < 1, \\ \{\lambda u + w : \lambda \geq 0 \ \& \ w \in V^{\perp}\} & \text{when } \|u\| = 1. \end{cases} \tag{2.13}$$

Here $V^{\perp}$ is the orthogonal complement of V in H. The proof of this a nice exercise using the sharp form of Schwarz' inequality. The extremality result that will be used is the following.

Theorem 2.1. *Let C be a closed convex subset of a real Hilbert space H and $\mathcal{F} : H \to \mathbb{R}$ be a G-differentable functional on H. If $\hat{u}$ maximizes $\mathcal{F}$ on C, then $\hat{u}$ satisfies*

$$D\mathcal{F}(u) \in \quad \partial I_C(u) \tag{2.14}$$

When C is a closed ball, centered at the origin, in a closed subspace V of H, and $\hat{u}$ maximizes $\mathcal{F}$ on C, then $\hat{u}$ satisfies

$$[D\mathcal{F}(u), h] \;=\; [\lambda u + w, h] \quad \textit{for some} \quad \lambda \geq 0, \;\; w \in V^{\perp} \textit{ and all } h \in H. \tag{2.15}$$

This first part of this result is easy to prove using convex analysis methods and the second part just uses the expression for the subdifferential given above. The following corollary is useful in many eigenvalue problems and simplifies otherwise lengthy proofs that certain multipliers are zero and that attention may be restricted to test functions in specific subspaces.

Corollary 2.2. *Let C be a closed ball, centered at the origin, of a closed subspace V of H and $\mathcal{F}$ as in theorem 2.1. Suppose $\hat{u}$ maximizes $\mathcal{F}$ on C and $D\mathcal{F}(\hat{u}) \in V$, then $\hat{u}$ satisfies*

$$[D\mathcal{F}(u), v] \;=\; \lambda \, [u, v] \quad \textit{for some} \quad \lambda \geq 0 \textit{ and all } v \in V. \tag{2.16}$$

Proof. Substitute $h \;=\; v_1 + v_2$ in (2.15), where $v_1 \in V$ and $v_2 \in V^{\perp}$. Then $[w, v_2] = 0$ for all $v_2 \in V^{\perp}$. Thus $w \;=\; 0$ and (2.16) follows. □

3. Div-curl Estimates

In this section we shall show that the DC-inner product (2.11) defines an equivalent inner product to the usual inner product (2.7) on $H^1_{\Sigma 0}(\Omega; \mathbb{R}^3)$. The first result is the following which shows that the DC-norm is weaker than the usual H^1- norm on vector fields.

Proposition 3.1. *Assume Ω is an open set in $\mathbb{R}^3$ and $u \in H^1(\Omega; \mathbb{R}^3)$, then*

$$\int_{\Omega} [\, |\operatorname{curl} u\,|^2 \;+\; |\operatorname{div} u|^2 \,] \, d^3x \;\leq\; 3 \int_{\Omega} \sum_{j,k=1}^{3} |u_{j,k}(x)|^2 \; d^3x. \tag{3.1}$$

Proof. Let $A = (a_{jk})$ be a 3x3 real matrix and define $vec(A) := (a_{23} - a_{32}, a_{13} - a_{31}, a_{12} - a_{21})$. Elementary algebra yields

$$0 \;\leq\; |\; tr\ A\;|^2 \;+\; |vec(A)\;|_2^{\,2} \;\leq\; 3 \sum_{j=1}^{3} a_{jj}^{\;2} \;+\; 2 \sum_{j \neq k} a_{jk}^{\;2}. \tag{3.2}$$

Substitute $Du(x)$ for A and integrate over Ω, then (3.1) follows. $\square$

These norms are not equivalent on $H^1(\Omega;\mathbb{R}^3)$ but under further assumptions on Ω, Σ, the following inequality holds.

Theorem 3.2. *Assume that Ω, Σ satisfy (B1) and (B2). Then there is a constant C_Σ such that*

$$\int_\Omega \sum_{j,k=1}^{3} |u_{j,k}(x)|^2 \, d^3x \;\le\; \int_\Omega [\, |\operatorname{curl} u\,|^2 \;+\; |\operatorname{div} u|^2 \,]\, d^3x \;+\; C_\Sigma \int_{\partial\Omega} |u|^2 d\sigma. \tag{3.3}$$

Proof. When u is C^1 on $\overline{\Omega}$, a standard application of the Gauss-Green theorem yields

$$\begin{aligned} \int_\Omega \sum_{j,k=1}^{3} |u_{j,k}(x)|^2 \, d^3x \;&=\; \int_\Omega [\, |\operatorname{curl} u\,|^2 \;+\; |\operatorname{div} u|^2 \,]\, d^3x \\ &+ \int_{\partial\Omega} [\, u \cdot \frac{\partial u}{\partial \nu} \;-\; (u\cdot\nu)\operatorname{div} u \;+\; (u \wedge \nu)\cdot \operatorname{curl} u \,]\, d\sigma. \end{aligned} \tag{3.4}$$

To prove (3.3), it is sufficient to treat the boundary integral. On Σ, $u \wedge \nu \;=\; 0$ and

$$(u\cdot\nu)\operatorname{div}\; u(x) \;=\; 2|u(x)|^2\, H(x) \;+\; u \cdot \frac{\partial u}{\partial \nu}(x)$$

from equation (1.27) in [8], chapter IX. Here $H(x)$ is the mean curvature of $\partial\Omega$ at x. The surface integral over Σ becomes

$$-2 \int_\Sigma H(x)\; |u(x)|^2 d\sigma$$

On $\tilde{\Sigma}$, $u \cdot \nu \;=\; 0$ so the surface integral over $\tilde{\Sigma}$ becomes

$$\int_{\tilde{\Sigma}} \nu_j u_k u_{j,k} d\sigma \;=\; \int_{\tilde{\Sigma}} [u_k[u_j\nu_j]_{,k} \;-\; u_j u_k \nu_{j,k}\,]\; d\sigma$$

where now ν is taken to be a C^1 extension of the normal field to a neighborhood of $\tilde{\Sigma}$. This may be done in consequence of our regularity assumptions $(B1)$ and $(B2)$. Choose C_Σ to be the maximum of $|H|$ on $\overline{\Sigma}$ and $|\nu_{j,k}(x)|$ on $\overline{\tilde{\Sigma}}$ then (3.3) follows. $\square$

Our results depend on the following result which is a Hilbert space version of a result often known as Peetre's lemma. See Girault and Raviart [14], chapter 1, section 3 for a discussion of a general version of this result.

Theorem 3.3. *Let H_1, H_2 be Hilbert spaces and X be a reflexive Banach space. Assume $L : H_1 \to H_2$ is continuous and $K : H_1 \to X$ is compact and there is a C such that*

$$\|v\|_1 \;\le\; C\; [\; \|Lv\|_2 \;+\; \|Kv\|_X \;] \qquad \textit{for all} \quad v \in H_1. \tag{3.5}$$

Then (i): the null space N(L) of L is finite dimensional and the range of L is a closed subspace of H_2, and

(ii): if H_0 is the orthogonal complement of N(L) in H_1 and P_0 is the projection of H_1 onto H_0, then there is a C_0 such that

$$\|P_0 v\|_1 \;\le\; C_0\; \|Lv\|_2 \quad \textit{for all} \quad v \in H_1. \tag{3.6}$$

In this theorem take $H_1 = H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$, $H_2 = L^2(\Omega;\mathbb{R}^4)$ and $Lv := (\operatorname{curl} v, \operatorname{div} v)$. Also $X := L^2(\partial\Omega;\mathbb{R}^3)$ and K to be the trace map. Then L is continuous and condition (B1) is sufficient for the trace map to be compact. The last two theorems combine to yield the following

Theorem 3.4. *Assume that Ω, Σ satisfy (B1) and (B2). Then $\mathcal{H}_\Sigma(\Omega)$ is finite dimensional and $H^1_{\Sigma 0}(\Omega;\mathbb{R}^3) = H_{DC\Sigma}(\Omega)$. The DC- inner product is an equivalent inner product on $H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$.*

Proof. With the above choices of spaces and operators, the inequality (3.3) yields (3.5), so the assumptions of theorem 3.3 hold. The null space of L is $\mathcal{H}_\Sigma(\Omega)$ so (i) of theorem 3.3 implies this space is finite dimensional. Let P_h be the projection from $H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$ onto $\mathcal{H}_\Sigma(\Omega)$. Then $Q_h := I - P_h$ plays the role of P_0 in (ii) of theorem 3.3.

When $v \in H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$ then from H^1- orthogonality and (3.6),

$$\|v\|_1^2 = \|P_h v\|_1^2 + \|Q_h v\|_1^2 \le \|P_h v\|_1^2 + {C_0}^2 \|v\|_{DC}^2$$

The H^1- norm and the DC- norm are equivalent on $\mathcal{H}_\Sigma(\Omega)$, since this space is finite dimensional. Thus there is a C_1 such that

$$\|v\|_1 \le C_1 \, \|v\|_{DC} \quad \text{for all} \quad v \in H^1_{\Sigma 0}(\Omega;\mathbb{R}^3). \tag{3.7}$$

This shows that the H^1- norm and the DC- norm are equivalent on the subspace $C_{\Sigma 0}(\overline{\Omega};\mathbb{R}^3) \cap H^1(\Omega;\mathbb{R}^3)$, so the completion of this space with respect to the two inner products will be the same space and the corresponding inner products and norms also are equivalent. □

For the case of zero normal, or zero tangential, components of the field on the boundary, Friedrichs [13] called the equivalence of these norms, the *auxiliary inequality* for 3d vector fields.

This result extends parts of theorem 3, Chapter IX, section 1 of [9] to this case of mixed boundary data. Henceforth we will usually use the DC- inner product and norm.

4. The Gradient Projection

Our interest is in proving the main inequality (1.1) for fields in $H_{DC\Sigma}(\Omega)$. To do this a *Hodge-Weyl decomposition* will be used. That is, given $v \in H_{DC\Sigma}(\Omega)$, an orthogonal representation of the form

$$v(x) = \nabla\varphi(x) + \operatorname{curl} A(x) + h(x) \tag{4.1}$$

will be described. Each of these terms on the right hand side will be specified by a Hilbert space projection using Riesz' characterization applied to $H_{DC\Sigma}(\Omega)$ and specific closed subspaces. This will be done in a manner similar to the variational approach described for L^2- fields in Auchmuty [3].

In this section the gradient projection in (4.1) will be described and characterized. Given $v \in H_{DC\Sigma}(\Omega)$, consider the functional $\mathcal{D}_v : H^1_{\Sigma 0}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{D}_v(\varphi) := \int_\Omega [|\nabla\varphi|^2 - 2v \cdot \nabla\varphi] \, d^3x. \tag{4.2}$$

We shall first show that the minimizer of this functional exists, and provides an appropriate scalar potential in the decomposition (4.1). The resulting field satisfies the same boundary conditions. Note that this functional $\mathcal{D}_v$ differs from the L^2- norm of $(\nabla\varphi - v)$ by a constant, so the solutions of this variational prioblem define the L^2- gradient projection of v when the scalar potential is required to be in $H^1_{\Sigma 0}(\Omega)$.

Theorem 4.1. *Assume Ω, Σ satisfy (B1) and (B2) and $v \in H_{DC\Sigma}(\Omega)$. Then there is a unique minimizer φ_v of $\mathcal{D}_v$ on $H^1_{\Sigma 0}(\Omega)$. A function $\varphi \in H^1_{\Sigma 0}(\Omega)$ minimizes $\mathcal{D}_v$ if and only if it is a solution of*

$$\int_\Omega (\nabla\varphi - v)\cdot\nabla\psi \; d^3x \;=\; 0 \quad \textit{for all} \quad \psi \in H^1_{\Sigma 0}(\Omega). \tag{4.3}$$

Proof. The functional $\mathcal{D}_v$ defined by (4.2) is continuous and convex by standard arguments. Hence it is weakly lower semi-continuous on $H^1(\Omega)$ and on $H^1_{\Sigma 0}(\Omega)$. Theorem 5.1 of the next section implies that there is a $\lambda_1(\Sigma) > 0$ such that

$$\int_\Omega |\nabla\varphi|^2 \; d^3x \;\geq\; \lambda_1(\Sigma) \int_\Omega |\varphi(x)|^2 \; d^3x \tag{4.4}$$

for all $\varphi \in H^1_{\Sigma 0}(\Omega)$. Thus

$$\mathcal{D}_v(\varphi) \;\geq\; (1/2) \int_\Omega \left[|\nabla\varphi|^2 + \lambda_1(\Sigma)\varphi^2\right] \; d^3x - 2\,\|u\|\,\|\nabla\varphi\|\,.$$

Thus $\mathcal{D}_v$ is coercive and strictly convex on $H^1_{\Sigma 0}(\Omega)$, so there is a unique minimizer φ_v of $\mathcal{D}_v$ on $H^1_{\Sigma 0}(\Omega)$. See [17], Chapter 42, or [5], Chapter 6, for such existence theorems. The functional $\mathcal{D}_v$ is Gateaux differentiable on $H^1_{\Sigma 0}(\Omega)$ and its derivative $\mathcal{D}'_v(\varphi)$ satisfies

$$\langle \mathcal{D}'_u(\varphi), \psi\rangle \;=\; 2\int_\Omega (\nabla\varphi - u)\cdot\nabla\psi \; d^3x \tag{4.5}$$

for all $\psi \in H^1_{\Sigma 0}(\Omega)$. From the convexity of $\mathcal{D}_v, \varphi$ minimizes this functional if and only if $\langle \mathcal{D}'_u(\varphi), \psi\rangle = 0$ for all $\psi \in H^1_{\Sigma 0}(\Omega)$. Thus (4.3) holds. □

Substitute φ_v for ψ in (4.3) then Schwarz inequality yields

$$\|\nabla\varphi_v\|^2 \;=\; \int_\Omega |\nabla\phi_v|^2 \, d^3x \;\leq\; \|v\|^2. \tag{4.6}$$

When φ is smooth enough, an integration by parts of (4.3) gives

$$\int_\Omega (\Delta\varphi - \operatorname{div} v)\;\psi\; d^3x \;+\; \int_{\partial\Omega} (\frac{\partial\varphi}{\partial\nu} - v\cdot\nu)\;\psi\; d\sigma \;=\; 0 \quad \text{for all} \quad \psi \in H^1_{\Sigma 0}(\Omega). \tag{4.7}$$

This is the weak form of the mixed boundary value problem

$$\Delta\varphi \;=\; \operatorname{div} v \quad \text{on} \quad \Omega, \qquad \text{subject to} \tag{4.8}$$

$$\varphi \;=\; 0 \;\; \text{on } \Sigma \quad \text{and} \quad \frac{\partial\varphi}{\partial\nu} \;=\; 0 \text{ on } \tilde{\Sigma}. \tag{4.9}$$

Define the map $P_{G\Sigma} : H_{DC\Sigma}(\Omega) \to L^2(\Omega;\mathbb{R}^3)$ by

$$P_{G\Sigma} v \;:=\; \nabla\varphi_v \tag{4.10}$$

where φ_v as above. This map can be shown to be linear and we will now prove that it is a projection on $H_{DC\Sigma}(\Omega)$. Define $H_{\Sigma 0}(\Delta;\Omega) := \{\varphi \in H^1_{\Sigma 0}(\Omega) : \Delta\varphi \in L^2(\Omega)\}$. This is a Hilbert space under the inner product

$$\langle \varphi, \psi \rangle_\Delta := \langle \varphi, \psi \rangle_1 + \int_\Omega \Delta\varphi . \Delta\psi \, d^3x. \tag{4.11}$$

Observe that when $\varphi \in H_{\Sigma 0}(\Delta;\Omega)$ then $\nabla\varphi$ is in $H_{DC}(\Omega)$ and, in a trace sense,

$$\nabla\varphi \wedge \nu = 0 \qquad \text{on} \quad \Sigma. \tag{4.12}$$

The gradient projection in $H_{DC\Sigma}(\Omega)$ is described by the solution of the problem of finding a function in $H_{\Sigma 0}(\Delta;\Omega)$ which minimizes $\mathcal{F}_v : H_{\Sigma 0}(\Delta;\Omega) \to \mathbb{R}$ defined by

$$\mathcal{F}_v(\varphi) := \|v - \nabla\varphi\|^2_{DC} \tag{4.13}$$

$$= \mathcal{D}_v(\varphi) + \int_\Omega \left[|v|^2 + |\operatorname{curl} v|^2 + |\Delta\varphi - \operatorname{div} v|^2\right] d^3x. \tag{4.14}$$

These solutions may be described quite simply.

Theorem 4.2. *Assume Ω, Σ satisfy (B1) and (B2) and $v \in H_{DC\Sigma}(\Omega)$. There is a unique minimizer of $\mathcal{F}_v$ on $H_{\Sigma 0}(\Delta;\Omega)$. It is φ_v, the minimizer of $\mathcal{D}_v$ and moreover, $\nabla\varphi_v$ is in $H_{DC\Sigma}(\Omega)$.*

Proof. Suppose φ_v minimizes $\mathcal{D}_v$ on $H^1_{\Sigma 0}(\Omega)$. Then it satisfies (4.8), so $\operatorname{div} v \in L^2(\Omega)$ and thus φ_v is in $H_{\Sigma 0}(\Delta;\Omega)$ as . By inspection of (4.14) it must minimize $\mathcal{F}_v$ on $H_{\Sigma 0}(\Delta;\Omega)$. The boundary conditions (4.9) and (4.12) are satisfied by φ_v so $\nabla\varphi_v$ satisfies (2.8) and is in $H_{DC\Sigma}(\Omega)$. □

This result shows that the range of $P_{G\Sigma}$ is actually the subspace

$$G_{\Sigma 0}(\Omega) := \{\nabla\varphi : \varphi \in H_{\Sigma 0}(\Delta;\Omega)\} \cap H_{DC\Sigma}(\Omega).$$

Theorem 4.3. *Assume Ω, Σ satisfy (B1) and (B2). Then $G_{\Sigma 0}(\Omega)$ is a closed subspace of $H_{DC\Sigma}(\Omega)$ and $P_{G\Sigma}$ defined by (4.10) is the linear projection onto $G_{\Sigma 0}(\Omega)$.*

Proof. The map $P_{G\Sigma}$ is linear by standard arguments. From (4.6) and (4.8), one sees that

$$\|\nabla\varphi_v\|^2_{DC} = \int_\Omega \left[|\nabla\varphi_v|^2 + |\Delta\varphi_v|^2\right] d^3x \tag{4.15}$$

$$\leq \int_\Omega \left[|v|^2 + |\operatorname{div} v|^2\right] d^3x \leq \|v\|^2_{DC}. \tag{4.16}$$

Hence $P_{G\Sigma}$ is a bounded linear map of $H_{DC\Sigma}(\Omega)$ to $H_{DC}(\Omega)$. From theorem 4.2, its range is a subspace of $H_{DC\Sigma}(\Omega)$. Since there is a solution φ_v for each $v \in H_{DC\Sigma}(\Omega)$, $G_{\Sigma 0}(\Omega)$ will be a closed subspace of $H_{DC\Sigma}(\Omega)$ and this is the projection onto this subspace from corollary 3.3 of [3]. □

5. The Mixed Laplacian Eigenproblem

The proof to be given of the main inequality (1.1) requires some specific results about the properties of solutions of two eigenproblems. Although these problems are close to some standard eigenproblems, the author was not able to find all the necessary results in the literature. In consequence, this section describes the theory of these eigenvalue problems *ab initio* and uses a formulation arising from a convex analysis version of the usual Rayleigh principle. This approach generalizes in a straightforward manner to the vector-valued eigenproblem described in section 8.

The classical eigenproblem to be considered here is to find non-trivial solutions of

$$-\Delta\varphi = \lambda\varphi \quad \text{on} \quad \Omega, \qquad \text{subject to} \tag{5.1}$$

$$\varphi = 0 \ \text{ on } \Sigma \quad \text{and} \quad \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \tilde{\Sigma}. \tag{5.2}$$

The weak version of this is to find those values of λ for which there are non-zero solutions in $H^1_{\Sigma 0}(\Omega)$ of

$$\int_\Omega (\nabla\varphi\cdot\nabla\psi - \lambda\varphi\psi)\, d^3x = 0 \quad \text{for all} \quad \psi \in H^1_{\Sigma 0}(\Omega). \tag{5.3}$$

This will be called the *mixed Laplacian eigenproblem on* Ω. An integration by parts shows that, if φ is smooth and satisfies (5.3), then it also satisfies

$$\int_\Omega (\Delta\varphi - \lambda\varphi)\psi\, d^3x + \int_{\partial\Omega} \psi\frac{\partial\varphi}{\partial\nu}\, d\sigma = 0 \quad \text{for all} \quad \psi \in H^1_{\Sigma 0}(\Omega). \tag{5.4}$$

Thus the eigenfunctions of this problem satisfy (5.1)-(5.2) in a weak sense and are in $H_{\Sigma 0}(\Delta;\Omega)$.

Consider the functional $\mathcal{Q} : H^1_{\Sigma 0}(\Omega) \to [0,\infty)$ defined by

$$\mathcal{Q}(\varphi) := \int_\Omega \varphi(x)^2\, d^3x, \tag{5.5}$$

and the variational principle of maximizing $\mathcal{Q}$ on the unit ball

$$B_1 := \{\varphi \in H^1_{\Sigma 0}(\Omega) : \|\varphi\|_1^2 \leq 1\}. \tag{5.6}$$

$$\text{Define} \quad \alpha_1(\Sigma) := sup_{\varphi\in B_1}\, \mathcal{Q}(\varphi). \tag{5.7}$$

The following theorem shows that the solutions of this variational principle provide the least eigenvalue $\lambda_1(\Sigma)$ and a corresponding normalized eigenfunction χ_1 of (5.3).

Theorem 5.1. *Assume Ω, Σ satisfy (B1) and (B2). Then there are functions $\pm\chi_1$ in B_1 which maximize $\mathcal{Q}$ on B_1. These functions are solutions of (5.3) corresponding to the least eigenvalue $\lambda_1(\Sigma)$ of the mixed Laplacian eigenproblem. Moreover $\lambda_1(\Sigma) > 0$ and $\alpha_1(\Sigma) = (1+\lambda_1(\Sigma))^{-1}$.*

Proof. When Ω satisfies (B1), the imbedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact from Rellich's theorem, so the functional $\mathcal{Q}$ is weakly continuous. The space $H^1_{\Sigma 0}(\Omega)$ is a closed subspace of $H^1(\Omega)$ so B_1 is weakly compact in the H^1-norm. Thus there are maximizers of $\mathcal{Q}$ on B_1 from the basic existence theorem. See [17], chapter 42 or [5], chapter 6 for details. Since $\mathcal{Q}$ and B_1 are symmetric, when χ_1 is a maximizer, so is $-\chi_1$.

A Lagrangian for this inequality constrained problem is to find critical points of the functional $\mathcal{L} : H^1_{\Sigma 0}(\Omega) \times [0,\infty) \to \mathbb{R}$ defined by

$$\mathcal{L}(\varphi,\mu) \ := \ \mu \ \|\varphi\|_1^2 - \mathcal{Q}(\varphi). \tag{5.8}$$

The critical points of this functional satisfy

$$\int_\Omega (\mu\nabla\varphi\cdot\nabla\psi \ + \ (\mu-1)\varphi\psi)\, d^3x \ = \ 0 \quad \text{for all} \quad \psi \in H^1_{\Sigma 0}(\Omega). \tag{5.9}$$

The maximizer χ_1 satisfies this from the theory of inequality constrained optimization, and this equation has the form (5.3) with $\lambda = (1-\mu)/\mu$. Let $\lambda_1 := \lambda_1(\Sigma)$ be the corresponding value of λ, put $\psi = \chi_1$ in this equation then

$$\int_\Omega |\nabla\chi_1|^2 \, d^3x \ = \ \lambda_1 \int_\Omega {\chi_1}^2 \, d^3x. \tag{5.10}$$

This implies $\lambda_1(\Sigma) \geq 0$. The boundary conditions and (B2) do not permit $\lambda_1 = 0$, so $\lambda_1(\Sigma) > 0$. Thus $\alpha_1(\Sigma) = (1+\lambda_1(\Sigma))^{-1}$ and this quantity is less than 1. If there is an eigenvalue λ of (5.3) with $\lambda < \lambda_1(\Sigma)$ then there is a corresponding eigenfunction $\chi \in B_1$ with

$$\mathcal{Q}(\chi) \ = \ (1+\lambda)^{-1} > \alpha_1(\Sigma).$$

This contradicts the definition of $\alpha_1(\Sigma)$, so $\lambda_1(\Sigma)$ is the least eigenvalue of this problem. □

When χ_j, χ_k are two eigenfunctions of (5.3) corresponding to distinct eigenvalues λ_j, λ_k, then they are L^2-orthogonal. Choose ψ in (5.3) to be these eigenfunctions then

$$\int_\Omega \nabla\chi_j\cdot\nabla\chi_k \, d^3x \ = \ \lambda_j \int_\Omega \chi_j \ \chi_k \, d^3x \ = \ \lambda_k \int_\Omega \chi_j \ \chi_k \, d^3x. \tag{5.11}$$

so χ_j, χ_k will also be H^1- orthogonal on Ω.

Higher eigenvalues and eigenfunctions of this problem may be found by induction in the usual manner. Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ be the first m eigenvalues of (5.3) and $\chi_1, \chi_2, \ldots, \chi_m$ be a corresponding family of H^1- orthonormal eigenfunctions of this problem. Consider the variational problem of maximizing $\mathcal{Q}$ as above on the set

$$B_{1m} \ := \ \{\varphi \in B_1 : \langle \varphi, \chi_j\rangle_1 \ = \ 0 \text{ for } 1 \leq j \leq m\}. \tag{5.12}$$

$$\text{Define} \quad \alpha_{m+1}(\Sigma) \ := \ sup_{\varphi\in B_{1m}} \ \mathcal{Q}(\varphi). \tag{5.13}$$

The solutions of this variational principle provide the next eigenvalue $\lambda_{m+1}(\Sigma)$ and a corresponding normalized eigenfunction χ_{m+1} of (5.3). Specifically one has,

Theorem 5.2. *Assume Ω, Σ satisfy (B1) and (B2). Then there are functions $\pm\chi_{m+1}$ in B_{1m} which maximize $\mathcal{Q}$ on B_{1m}. These functions are solutions of (5.3) corresponding to the eigenvalue $\lambda_{m+1}(\Sigma) \geq \lambda_m(\Sigma)$ of the mixed Laplacian eigenproblem, $\|\chi_{m+1}\|_1 = 1$ and $\alpha_{m+1}(\Sigma) \ = \ (1+\lambda_{m+1}(\Sigma))^{-1}$.*

Proof. The existence proof follows as above in Theorem 5.1 and by homogeniety $\|\chi_{m+1}\|_1 = 1$.

In view of the extra m equality constraints, the appropriate Lagrangian for this problem now is $\mathcal{L} : H^1_{\Sigma 0}(\Omega) \times [0,\infty) \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$\mathcal{L}(\varphi,\mu,\nu) \ := \ \mu \ \|\varphi\|_1^2 - \mathcal{Q}(\varphi) + \sum_{j=1}^m \nu_j \langle \varphi, \chi_j \rangle. \tag{5.14}$$

The critical points of this functional satisfy

$$\int_\Omega [\mu \nabla\varphi \cdot \nabla\psi \ + \ (\mu - 1)\varphi\psi \ + \ (1/2)\sum_{j=1}^m \nu_j \ \chi_j \psi] \ d^3x \ = \ 0 \quad \text{for all} \quad \psi \in H^1_{\Sigma 0}(\Omega). \tag{5.15}$$

The maximizer χ_{m+1} satisfies this equation. Take $\psi = \chi_k$ for some $k \in \{1,\ldots,m\}$, then χ_{m+1} obeys

$$\int_\Omega \ \mu \nabla\chi \cdot \nabla\chi_k \ + \ [(\mu - 1) \ \chi \ + \ (1/2)\sum_{j=1}^m \nu_j \ \chi_j] \ \chi_k \ d^3x \ = \ 0 \tag{5.16}$$

The first term here is zero upon an integration by parts and the fact that χ_k is an eigenfunction, the second term is zero since $\chi_{m+1} \in B_{1m}$ and all except one term in the last sum vanishes. Hence $\nu_k = 0$ for each $1 \le k \le m$. Thus (5.15) implies that χ_{m+1} satisfies (5.3) or it is an eigenfunction of this problem corresponding to an eigenvalue λ_{m+1}. Just as in the proof of theorem 5.1, this is the next smallest eigenvalue of the problem. □

This construction provides a straightforward proof of the following result. We will say that the sequence $\{\lambda_m : m \ge 1\}$ is increasing when $\lambda_{m+1} \ge \lambda_m$ for all m. It is strictly increasing when strict inequality holds here for each m.

Theorem 5.3. *Assume Ω, Σ satisfy (B1) and (B2). Let $\{\lambda_m : m \ge 1\}$ be the increasing sequence of eigenvalues defined above and $\{\chi_m : m \ge 1\}$ be a corresponding family of H^1-orthonormal eigenfunctions. Then (i): $\lim_{m\to\infty} \lambda_m = \infty$, and*
(ii): the set $\{\chi_m : m \ge 1\}$ is a maximal H^1- orthonormal set in $H^1_{\Sigma 0}(\Omega)$.

Proof. Assume (i) is false, and the increasing sequence $\{\lambda_m\}$ is bounded above by a number L. The corresponding sequence of eigenfunctions is orthonormal in $H^1_{\Sigma 0}(\Omega)$ by construction and is orthogonal in $L^2(\Omega)$ with

$$\|\chi_m\|^2 \ = \ (1+\lambda_m)^{-1} \ \ge (1+L)^{-1} \tag{5.17}$$

Since the sequence of eigenfunctions is orthonormal in $H^1_{\Sigma 0}(\Omega)$, it converges weakly to 0. Thus it converges strongly to 0 in $L^2(\Omega)$ as Rellich's theorem holds. This contradicts (5.17), so L must be infinite.

If the set $\{\chi_m : m \ge 1\}$ is not a maximal orthonormal set in $H^1_{\Sigma 0}(\Omega)$, there will be a $\psi \in B_1$ with $\langle \psi, \chi_m \rangle \ = \ 0$ for all m. Evaluate $\mathcal{Q}(\psi)$. This is positive, and if λ_M is chosen so that

$$\mathcal{Q}(\psi) \ > \ (1+\lambda_M)^{-1}$$

then we obtain a contradiction to the definition of χ_M. Thus (ii) holds. □

The sequence $\{\chi_m : m \geq 1\}$ is in $H_{\Sigma 0}(\Delta;\Omega)$, so (4.11) implies that

$$\langle \chi_k, \chi_l \rangle_\Delta \;=\; \delta_{kl}\; [1 \;+\; \frac{{\lambda_k}^2}{1+\lambda_k}\;]$$

In view of this there is an orthonormal set $\{\tilde{\chi}_m : m \geq 1\}$ in $H_{\Sigma 0}(\Delta;\Omega)$, with each $\tilde{\chi}_m = C_m\, \chi_m$ and the following holds

Corollary 5.4. *The set $\{\tilde{\chi}_m : m \geq 1\}$ is a maximal orthonormal set in $H_{\Sigma 0}(\Delta;\Omega)$.*

Proof. If this set is not maximal then there is a ψ in $H_{\Sigma 0}(\Delta;\Omega)$ which is Δ- orthogonal to each element. In this case it would also be H^1- orthogonal to each χ_m. This is impossible as these functions are maximal in $H^1_{\Sigma 0}(\Omega)$, so the result follows. □

6. The Mixed Divergence Estimate

The results of the last section enable the proof of the main inequality for irrotational fields.

Theorem 6.1. *Assume Ω, Σ satisfy (B1) and (B2) and $\lambda_1(\Sigma)$ is the least eigenvalue of (5.3). Then*

$$\|\mathrm{div}\, v\|^2 \;\geq\; \lambda_1(\Sigma)\; \|v\|^2 \qquad \textit{for all} \quad v \in G_{\Sigma 0}(\Omega). \tag{6.1}$$

Equality holds here when v is a multiple of $\nabla\chi_1$ and χ_1 is an eigenfunction corresponding to $\lambda_1(\Sigma)$.

Proof. Suppose $v \in G_{\Sigma 0}(\Omega)$ and $P_{G\Sigma}$ is the projection defined by (4.10). Then $v \;=\; \nabla\varphi_v$ where φ_v is defined as above and is in $H_{\Sigma 0}(\Delta;\Omega)$. Assume

$$\varphi_v \;=\; \sum_{m=1}^{\infty} c_m\; \tilde{\chi}_m,$$

where $\{\tilde{\chi}_m : m \geq 1\}$ is the orthonormal basis of $H_{\Sigma 0}(\Delta;\Omega)$ defined in the last section. From Parseval's equality, in view of theorem 5.3 and Corollary 5.4,

$$\|v\|^2 \;=\; \sum_{m=1}^{\infty} {c_m}^2\; \|\nabla\tilde{\chi}_m\|^2, \quad \text{and} \tag{6.2}$$

$$\|\mathrm{div}\, v\|^2 \;=\; \sum_{m=1}^{\infty} {c_m}^2\; \|\Delta\tilde{\chi}_m\|^2. \tag{6.3}$$

However, for each m,

$$\|\Delta\tilde{\chi}_m\|^2 \;=\; {\lambda_m}^2\|\tilde{\chi}_m\|^2 \;=\; \lambda_m\; \|\nabla\tilde{\chi}_m\|^2 \tag{6.4}$$

Substitute this in (6.2) and (6.3), then (6.1) follows. When $c_m = 0$ for all $m \geq 2$, then equality holds in (6.1). □

7. Vector Potentials and the Curl Subspace

To describe an orthogonal decomposition of the form (4.1), the following characterization of the orthogonal complement of $G_{\Sigma 0}(\Omega)$ will be used.

Proposition 7.1. *Suppose (B1)-(B2) hold, then a field* $v \in H_{DC}(\Omega)$ *is* L^2*-orthogonal to* $G_{\Sigma 0}(\Omega)$ *if and only if*

$$\operatorname{div} v \;=\; 0 \quad \text{on } \Omega \quad \text{and} \qquad v \cdot \nu \;=\; 0 \quad \text{on } \tilde{\Sigma}. \tag{7.1}$$

Suppose $v \;=\; \operatorname{curl} A$ *for some* $A \in H_{DC}(\Omega)$*, then* v *is* L^2*-orthogonal to* $G_{\Sigma 0}(\Omega)$*, if and only if*

$$A \wedge \nu \;=\; 0 \qquad \text{on } \tilde{\Sigma}. \tag{7.2}$$

Proof. Assume v is L^2-orthogonal to $G_{\Sigma 0}(\Omega)$, then

$$\int_\Omega v \cdot \nabla\varphi \; d^3x \;=\; \int_{\partial\Omega} (v \cdot \nu)\; \varphi \; d\sigma \;-\; \int_\Omega \varphi \operatorname{div} v \; d^3x \;=\; 0.$$

for all $\varphi \in H_{\Sigma 0}(\Delta;\Omega)$ from the Gauss-Green theorem. There are sufficiently many φ to yield (7.1). Conversely when (7.1) holds and $\varphi \in H_{\Sigma 0}(\Delta;\Omega)$, then this equation shows that v will be L^2-orthogonal to $G_{\Sigma 0}(\Omega)$. When $v \;=\; \operatorname{curl} A$, (2.5) yields

$$\int_\Omega \operatorname{curl} A \cdot \nabla\varphi \; d^3x \;=\; \int_{\partial\Omega} A \cdot (\nabla\varphi \wedge \nu) d\sigma \;=\; \int_{\tilde{\Sigma}} \nabla\varphi \cdot (\nu \wedge A) d\sigma.$$

Here φ is assumed to be C^1- and in $H_{\Sigma 0}(\Delta;\Omega)$. The second sentence of the proposition follows. □

There is considerable non-uniqueness in most definitions of a vector potential. The following result specifies the class of equivalent potentials satisfying a specific boundary condition of the form (7.2).

Proposition 7.2. *Suppose (B1), (B2) hold,* $A \in H^1(\Omega;\mathbb{R}^3)$ *and* $A \wedge \nu = 0$ *on* Σ*. Then there is a vector field* $B \in H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$ *which satisfies* $\operatorname{curl} B = \operatorname{curl} A$ *on* Ω*. Two fields* $A, B \in H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$ *satisfy*

$$\operatorname{curl} B \;=\; \operatorname{curl} A \quad \text{on } \Omega \quad \text{if and only if} \qquad B \;=\; A \;+\; \nabla\varphi + h \tag{7.3}$$

for some φ *in* $H^1_{\Sigma 0}(\Omega)$ *and* $h \in \mathcal{H}_\Sigma(\Omega)$*. Moreover there is a unique* $\tilde{A}$ *in* $H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$ *which satisfies* $\operatorname{curl} \tilde{A} = \operatorname{curl} A$ *on* Ω *and also is solenoidal and* L^2*-orthogonal to* $\mathcal{H}_\Sigma(\Omega)$*.*

Proof. Given such an A, let $\varphi \in H^1_{\Sigma 0}(\Omega)$ be the solution of

$$\Delta\varphi \;=\; \operatorname{div} A \quad \text{on} \quad \Omega, \qquad \text{subject to} \tag{7.4}$$

$$\varphi \;=\; 0 \;\text{ on } \Sigma \quad \text{and} \quad \frac{\partial\varphi}{\partial\nu} \;=\; A \cdot \nu \text{ on } \tilde{\Sigma}. \tag{7.5}$$

Then $B \;:=\; A - \nabla\varphi$ is in $H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$ and satisfies $\operatorname{div} B = 0$ and $\operatorname{curl} B = \operatorname{curl} A$ on Ω as claimed.

When $A, B \in H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$, let $\varphi_1 \in H^1_{\Sigma 0}(\Omega)$ be the solution of

$$\Delta\varphi \;=\; \operatorname{div}(B - A) \quad \text{on} \quad \Omega, \qquad \text{subject to} \tag{7.6}$$

$$\varphi \;=\; 0 \;\text{ on } \Sigma \quad \text{and} \quad \frac{\partial\varphi}{\partial\nu} \;=\; 0 \text{ on } \tilde{\Sigma}. \tag{7.7}$$

This system has a unique solution and then $h := B - A - \nabla\varphi_1$ is in $\mathcal{H}_\Sigma(\Omega)$. Thus (7.3) holds.

Given $A \in H^1(\Omega;\mathbb{R}^3)$ such that $A \wedge \nu = 0$ on Σ, let $B \in H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$ be as in the first part of this proof. When $B^{(1)}, B^{(2)}$ are two such fields then their difference must be in $\mathcal{H}_\Sigma(\Omega)$, so the last statement holds. □

In view of this proposition define $V_\Sigma(\Omega)$ to be the subspace of $H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$ of fields which are solenoidal and $Z_\Sigma(\Omega)$ to be the subspace of $H^1_{\Sigma 0}(\Omega;\mathbb{R}^3)$ of fields which are solenoidal and L^2-orthogonal to $\mathcal{H}_\Sigma(\Omega)$. Both subspaces are closed subspaces and

$$V_\Sigma(\Omega) \;=\; Z_\Sigma(\Omega) \oplus \mathcal{H}_\Sigma(\Omega)$$

where this decomposition is both L^2- and DC- orthogonal. Note that from the definition the fields in the spaces $V_\Sigma(\Omega), Z_\Sigma(\Omega)$ satisfy the boundary conditions (2.8) in a trace sense. Define

$$Curl_\Sigma(\Omega) \;:=\; \{\operatorname{curl} A : A \in H^1(\Omega;\mathbb{R}^3),\;\; A \wedge \nu = 0 \quad \text{on} \quad \Sigma\}. \tag{7.8}$$

Then proposition 7.2 says there will be a unique vector potential $\tilde{A} \in Z_\Sigma(\Omega)$ for each field $v \in Curl_\Sigma(\Omega)$. From proposition 7.1 the space $Curl_{\tilde{\Sigma}}(\Omega)$ will be L^2-orthogonal to $G_{\Sigma 0}(\Omega)$.

8. The Mixed $Curl^2$ Eigenproblem

In this section an eigenvalue problem for the curl^2 operator subject to mixed normal and tangential boundary conditions will be studied. The classical form of this eigenproblem is to find those μ such that there are non-zero vector fields A satisfying

$$\operatorname{curl}^2 A \;=\; \mu A \qquad \text{and} \qquad \operatorname{div} A \;=\; 0 \quad \text{in } \Omega, \tag{8.1}$$

$$A \wedge \nu \;=\; 0 \quad \text{on} \quad \Sigma, \text{ and} \quad A \cdot \nu \;=\; 0 \quad \text{on} \quad \tilde{\Sigma}. \tag{8.2}$$

A classical solution of this satisfies the following.

Proposition 8.1. *Suppose (B1), (B2) hold, $\mu \neq 0$ and $A \in H^1(\Omega;\mathbb{R}^3)$ is a solution of (8.1)- (8.2) which is C^1 on $\overline{\Omega}$. Then A is L^2-orthogonal to $G_{\Sigma 0}(\Omega)$ and*

$$\operatorname{curl} A \wedge \nu \;=\; 0 \quad \textit{on} \quad \tilde{\Sigma}. \tag{8.3}$$

Proof. The L^2- orthogonality holds from (8.1), (8.2) and proposition 7.1. From (2.5) and the fact that φ is constant on Σ,

$$\langle \nabla\varphi, \operatorname{curl}^2 A\rangle \;=\; \int_{\tilde{\Sigma}} \nabla\varphi \cdot (\nu \wedge \operatorname{curl} A)\, d\sigma \tag{8.4}$$

so (8.3) holds as there are enough allowable such φ. □

The weak version of this eigenproblem is to find those values of μ for which there are non-zero solutions $A \in V_\Sigma(\Omega)$ of

$$\int_\Omega (\operatorname{curl} A \cdot \operatorname{curl} B \;-\; \mu A \cdot B)\, d^3x \;=\; 0 \quad \text{for all} \quad B \in V_\Sigma(\Omega). \tag{8.5}$$

This will be called the Σ-*mixed* curl^2 *eigenproblem on* Ω - and henceforth we will concentrate on solutions of this system (8.5). When A is H^2- and (8.5) holds then the Gauss-Green theorem (2.5) shows that the eigenfields satisfy

$$(8.6) \quad \int_{\partial\Omega} B \cdot (\operatorname{curl} A \wedge \nu)\, d\sigma \;+\; \int_{\Omega} B \cdot (\operatorname{curl}^2 A - \mu A)\, d^3x \;=\; 0 \quad \text{for all} \quad B \in V_\Sigma(\Omega).$$

This, and the definition of $V_\Sigma(\Omega)$, show that (8.1) and (8.3) hold in a weak sense when (μ, A) satisfy (8.5).

Suppose $C := \operatorname{curl} A$ and A is an eigenfield of (8.5). Then, from (2.4) and (2.5) respectively,

$$(8.7) \qquad \int_\Omega C \cdot \nabla\varphi \;\, d^3x \;=\; \int_{\partial\Omega} \varphi(C\cdot\nu)\, d\sigma \;=\; -\int_{\partial\Omega} \nabla\varphi\cdot(A\wedge\nu)\, d\sigma$$

From the first part of (8.2), the last integral is zero for all $\varphi \in H^1_{\tilde{\Sigma}0}(\Omega)$. Thus for all such φ,

$$(8.8) \qquad \int_\Sigma \varphi(C\cdot\nu)\, d\sigma \;=\; 0 \quad \text{or} \qquad \operatorname{curl} A\cdot\nu \;=\; 0 \quad \text{on} \quad \Sigma.$$

This and (8.6) implies that the eigenfields of this problem are such that $\operatorname{curl} A$ satisfies, in a trace sense, the *dual* boundary conditions

$$(8.9) \qquad C\wedge\nu \;=\; 0 \;\text{ on }\; \tilde{\Sigma} \qquad \text{and} \;\; C\cdot\nu \;=\; 0 \text{ on } \;\Sigma.$$

A special role is played by the null-eigenfields of this problem, which may be described as follows.

Proposition 8.2. *Suppose (B1), (B2) hold, then 0 is an eigenvalue of (8.5) if and only if $\mathcal{H}_\Sigma(\Omega)$ is non-zero. When 0 is an eigenvalue, the corresponding eigenspace is $\mathcal{H}_\Sigma(\Omega)$ and is finite dimensional.*

Proof. If 0 is an eigenvalue of (8.5), then a corresponding eigenfield $\hat{A}$ in $V_\Sigma(\Omega)$ obeys $\operatorname{curl}\hat{A} \equiv 0$. It also satisfies the other criteria to be in $\mathcal{H}_\Sigma(\Omega)$. Conversely each field in $\mathcal{H}_\Sigma(\Omega)$ will be an eigenfield of (8.5) corresponding to the eigenvalue 0. The finite dimensionality of this space follows from theorem 3.4. Alternatively this may be proved directly as follows. Let $\{h^{(m)} : m \in \mathcal{M}\}$ be a maximal DC-orthonormal set in $\mathcal{H}_\Sigma(\Omega)$. Then $\|h^{(m)}\|_{DC} \;=\; \|h^{(m)}\| \;=\; 1$ for all $m \in \mathcal{M}$. If $\mathcal{M}$ is infinite then this sequence converges weakly to 0 in $H^1_{\Sigma0}(\Omega;\mathbb{R}^3)$. From Rellich's theorem it must converge strongly to 0 in $L^2(\Omega;\mathbb{R}^3)$. This contradicts the fact that their L^2-norm is 1, so $\mathcal{M}$ is a finite set. $\square$

In Auchmuty and Alexander [5], sections 15 and 16, a geometric interpretation of this space is provided, and explicit bases are described. These constructions, and the dimension of the space depend on the differential topology of Ω, Σ and $\tilde{\Sigma}$ and this is described there.

When $A^{(j)}, A^{(k)}$ are two eigenfields of (8.5) corresponding to distinct eigenvalues μ_j, μ_k, then

$$(8.10) \quad \int_\Omega \operatorname{curl} A^{(j)}\cdot\operatorname{curl} A^{(k)}\, d^3x \;=\; \mu_j \int_\Omega A^{(j)}\cdot A^{(k)}\, d^3x \;=\; \mu_k \int_\Omega A^{(j)}\cdot A^{(k)}\, d^3x.$$

Hence $A^{(j)}, A^{(k)}$ will be both L^2- and DC- orthogonal on Ω.

In view of this, and the results described at the end of section 7, the non-zero eigenfields of this mixed curl^2 eigenproblem will be the solutions in $Z_\Sigma(\Omega)$ of (8.5) and this equation need only hold for all $B \in Z_\Sigma(\Omega)$.

The eigenfields of this problem may be found using variational principles in a similar manner to the analysis of the mixed Laplacian eigenproblem. Consider the functional $\mathcal{Q}_1 : Z_\Sigma(\Omega) \to [0, \infty)$ defined by

$$\mathcal{Q}_1(A) \ := \ \int_\Omega |A(x)|^2 \ d^3x, \tag{8.11}$$

and the variational principle of maximizing $\mathcal{Q}_1$ on the unit ball

$$C_1 \ := \ \{A \in Z_\Sigma(\Omega) : \|A\|_{DC} \ \le \ 1\}. \tag{8.12}$$

$$\text{Define} \quad \gamma_1(\Sigma) \ := \ sup_{A \in C_1} \ \mathcal{Q}_1(A). \tag{8.13}$$

The following theorem shows that the solutions of this variational principle provide the least positive eigenvalue $\mu_1(\Sigma)$ and a corresponding normalized eigenfield $A^{(1)}$ of (8.5).

Theorem 8.3. *Assume Ω, Σ satisfy (B1) and (B2). Then there are fields $\pm A^{(1)}$ in C_1 which maximize $\mathcal{Q}_1$ on C_1. These fields are solutions of (8.5) corresponding to the least positive eigenvalue $\mu_1(\Sigma)$ of the Σ-mixed* curl^2 *eigenproblem. Moreover $\mu_1(\Sigma) > 0$ and $\gamma_1(\Sigma) \ = \ (1 + \mu_1(\Sigma))^{-1}$.*

Proof. The set C_1 and the space $Z_\Sigma(\Omega)$ are closed subsets of $H^1_{\Sigma 0}(\Omega; \mathbb{R}^3)$ and thus of $H^1(\Omega; \mathbb{R}^3)$. Thus C_1 is a weakly compact in $H^1_{\Sigma 0}(\Omega; \mathbb{R}^3)$, as it is convex and bounded. The imbedding of $H^1(\Omega; \mathbb{R}^3)$ into $L^2(\Omega; \mathbb{R}^3)$ is compact from Rellich's theorem as (B1) holds. So $\mathcal{Q}$ is weakly continuous and attains its supremum on C_1. Hence there are maximizers $\pm A^{(1)}$ of $\mathcal{Q}$ on C_1.

To find the conditions satisfied at a maximizer, take H to be $H^1_{\Sigma 0}(\Omega; \mathbb{R}^3)$ with the DC inner product and V to be $Z_\Sigma(\Omega)$. From corollary 2.2, equation (2.16) the maximizers are solutions of

$$\int_\Omega [(2 - \lambda) A \cdot B - \lambda \, \text{curl}\, A \cdot \text{curl}\, B] \ d^3x \ = \ 0. \tag{8.14}$$

for some $\lambda \ge 0$ and all $B \in Z_\Sigma(\Omega)$. If $\lambda = 0$, then $A \ = \ 0$ and this is not a maximizer. Thus $\lambda > 0$ and (8.5) holds with $\mu \ = \ \mu_1(\Sigma) \ = \ (2 - \lambda)/\lambda > 0$. The relationship between $\gamma_1(\Sigma)$ and $\mu_1(\Sigma)$ hold as in theorem 5.1. □

In particular this theorem shows that

$$\int_\Omega |\text{curl}\, A|^2 \ d^3x \ \ge \ \mu_1(\Sigma) \int_\Omega |A|^2 \ d^3x \qquad \text{for all} \quad A \in Z_\Sigma(\Omega). \tag{8.15}$$

The number $\mu_1(\Sigma)$ is called the principal eigenvalue of this Σ-mixed curl^2 eigenproblem and is the least non-zero eigenvalue of the problem.

Higher eigenvalues and corresponding eigenfields of this problem may also be found using an induction procedure. Let $0 < \mu_1 \le \mu_2 \le \ldots \le \mu_m$ be the first m positive eigenvalues of (8.5) and $\{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset Z_\Sigma(\Omega)$ be a corresponding family of DC-orthonormal eigenfields of this problem. Consider the variational problem of maximizing $\mathcal{Q}_1$ as above on the closed ball

$$C_{1m} \quad := \quad \{A \in C_1 : \langle A, A^{(j)} \rangle \; = \; 0, \text{ for } 1 \leq j \leq m\}. \tag{8.16}$$

(8.17) Define $\gamma_{m+1}(\Sigma) \quad := \quad sup_{A \in C_{1m}} \; \mathcal{Q}_1(A).$

The solutions of this variational principle provide the next eigenvalue μ_{m+1} and a corresponding normalized eigenfield $A^{(m+1)}$ of (8.5). Specifically one has,

Theorem 8.4. *Assume Ω, Σ satisfy (B1) and (B2). Then there are fields $\pm A^{(m+1)}$ in C_{1m} which maximize $\mathcal{Q}_1$ on C_{1m}. These functions are solutions of (8.5) corresponding to the eigenvalue $\mu_{m+1} \geq \mu_m$ of the Σ-mixed* curl^2 *eigenproblem. Moreover $\left\|A^{(m+1)}\right\|_{DC} = 1$ and $\gamma_{m+1}(\Sigma) \; = \; (1 + \mu_{m+1})^{-1}$.*

Proof. The existence of solutions follows as in theorem 8.3. The conditions satisfied at a maximizer are also obtained as there but now take $V := \{B \in Z_\Sigma(\Omega) : \langle A, A^{(j)} \rangle \; = \; 0, \text{ for } 1 \leq j \leq m\}$. The G-derivative $D\mathcal{Q}_1(A) = 2A$ so it is in V when $A \in C_{1m}$. Thus the maximizers satisfy (8.14) for some $\lambda > 0$ and all $B \in Z_\Sigma(\Omega)$. The theorem now follows as previously. □

Just as before, this construction leads to the following result about the eigenvalues and eigenfields of this problem and the construction of an orthonormal basis of the subspace $Z_\Sigma(\Omega)$.

Theorem 8.5. *Assume Ω, Σ satisfy (B1) and (B2). Let $\{\mu_m : m \geq 1\}$ be the increasing sequence of positive eigenvalues of the Σ-mixed* curl^2 *eigenproblem defined above and $\{A^{(m)} : m \geq 1\}$ be a corresponding family of DC- orthonormal eigenfields. Then*
(i): $\lim_{m \to \infty} \mu_m = \infty$, and
(ii): the set $\{A^{(m)} : m \geq 1\}$ is a maximal DC- orthonormal set in $Z_\Sigma(\Omega)$.

Proof. Assume (i) is false, and the increasing sequence $\{\mu_m\}$ is bounded above by L. The corresponding sequence of eigenfield is orthonormal in $Z_\Sigma(\Omega)$ by construction and is orthogonal in $L^2(\Omega; \mathbb{R}^3)$ with

$$\left\|A^{(m)}\right\|^2 \; = \; (1 + \mu_m)^{-1} \; \geq (1 + L)^{-1} \tag{8.18}$$

Since the sequence of eigenfields is orthonormal in $Z_\Sigma(\Omega)$, it converges weakly to 0. Thus it converges strongly to 0 in $L^2(\Omega; \mathbb{R}^3)$ as Rellich's theorem holds. L finite contradicts (8.18), so L must be infinite.

If the set $\{A^{(m)} : m \geq 1\}$ is not a maximal orthonormal set in $Z_\Sigma(\Omega)$, there will be a $B \in C_1$ with $\langle B, A^{(m)} \rangle_{DC} \; = \; 0$ for all m. Suppose $\mathcal{Q}_1(B) > 0$. Choose μ_M so that

$$\mathcal{Q}_1(B) \; > \; (1 + \mu_M)^{-1}$$

then we have a contradiction to the definition of $A^{(M)}$. Thus $\mathcal{Q}_1(B) = 0$, so $B = 0$ and (ii) holds. □

9. The Curl Projection

The results of the last two sections will now be used to describe the curl component in the orthogonal decomposition (4.1).

Let $Curl_\Sigma(\Omega)$ be the space defined in (7.8). From Riesz' theorem and the discussion of section 7, the L^2- projection of $H_{DC}(\Omega)$ onto $Curl_\Sigma(\Omega)$ will be defined by minimizing $\|v - \operatorname{curl} A\|^2$ over all $A \in Z_\Sigma(\Omega)$. Given $v \in H_{DC}(\Omega)$, consider the variational problem of minimizing $\mathcal{C}_v : Z_\Sigma(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{C}_v(A) \ := \ \int_\Omega [|\operatorname{curl} A(x)|^2 - 2v(x) \cdot \operatorname{curl} A(x)] \ d^3x. \tag{9.1}$$

The results about this minimization problem may be summarized as follows.

Theorem 9.1. *Assume Ω, Σ satisfy (B1) and (B2) and $v \in L^2(\Omega; \mathbb{R}^3)$. Then there is a unique minimizer A_v of $\mathcal{C}_v$ on $Z_\Sigma(\Omega)$. A field $A \in Z_\Sigma(\Omega)$ minimizes $\mathcal{C}_v$ if and only if it is a solution of*

$$\int_\Omega (\operatorname{curl} A - v) \cdot \operatorname{curl} B \ d^3x \ = \ 0 \quad \textit{for all} \quad A \in Z_\Sigma(\Omega). \tag{9.2}$$

Proof. The functional $\mathcal{C}_v$ is convex and continuous on $H^1(\Omega; \mathbb{R}^3)$ and thus on $Z_\Sigma(\Omega)$. Hence it is weakly lower semi-continuous. From (8.15) and Schwarz' inequality, one sees that

$$\mathcal{C}_v(A) \ \geq \ \int_\Omega (1/2)[|\operatorname{curl} A(x)|^2 + \mu_1(\Sigma) \ |A(x)|^2] \ d^3x \ - \ 2\,\|v\|\,\|\operatorname{curl} A\|\,.$$

This is coercive and strictly convex on $Z_\Sigma(\Omega)$ since $\mu_1(\Sigma) > 0$. Thus there is a unique minimizer of $\mathcal{C}_v$ on $Z_\Sigma(\Omega)$. The extremality condition (9.2) is found by evaluating the G-derivative of the quadratic functional $\mathcal{C}_v$. □

When A is smooth enough, using (2.4) with (9.2) yields

$$\int_{\partial\Omega} B \cdot ((\operatorname{curl} A - v) \wedge \nu) \ d\sigma \ + \ \int_\Omega B \cdot (\operatorname{curl}^2 A - \operatorname{curl} v) \ d^3x \ = \ 0 \tag{9.3}$$

for all B in $Z_\Sigma(\Omega)$. Thus A_v is a weak solution of the system

$$\operatorname{curl}^2 A \ = \ \operatorname{curl} v, \qquad \operatorname{div} A \ = \ 0 \quad \text{on} \quad \Omega, \tag{9.4}$$

$$A \wedge \nu \ = \ 0 \ \text{ on } \Sigma, \quad A \cdot \nu \ = \ 0 \quad \text{and} \quad \operatorname{curl} A \wedge \nu \ = \ v \wedge \nu \quad \text{on } \tilde{\Sigma}. \tag{9.5}$$

Consider the linear mapping $P_{C\Sigma} : L^2(\Omega; \mathbb{R}^3) \to Curl_\Sigma(\Omega)$ defined by

$$P_{C\Sigma}\ v \ := \ \operatorname{curl} A_v \tag{9.6}$$

Here A_v will be called the $\Sigma -$ curl potential of v, $P_{C\Sigma}$ is the $\Sigma -$ curl projection and $\operatorname{curl} A_v$ is the Σ-mixed curl component of v. This theorem can be restated as follows.

Corollary 9.2. *$Curl_\Sigma(\Omega)$ is a closed subspace of $L^2(\Omega; \mathbb{R}^3)$ and $P_{C\Sigma}$ is the projection of $L^2(\Omega; \mathbb{R}^3)$ onto $Curl_\Sigma(\Omega)$. When $v \in H_{DC}(\Omega)$, so is $P_{C\Sigma} v$.*

Proof. The result that $Curl_\Sigma(\Omega)$ is a closed subspace of $L^2(\Omega; \mathbb{R}^3)$ follows from corollary 3.3 of [3] as for each v in $L^2(\Omega; \mathbb{R}^3)$ there is a minimizer of $\mathcal{C}_v$. Moreover the projection is defined by the solution of this norm-minimization problem. When $v \in H_{DC}(\Omega)$, then $\operatorname{curl} A_v$ is in $H_{DC}(\Omega)$ from (9.4). □

From these results and proposition 7.1, one sees that it is the $\tilde{\Sigma}-$curl projection which will map $H_{DC\Sigma}(\Omega)$ into itself and into a subspace that is orthogonal to $G_{\Sigma 0}(\Omega)$. That is the allowable potentials should satisfy the dual boundary conditions (8.9). Define

$$Cu_\Sigma(\Omega) \;:=\; H_{DC\Sigma}(\Omega) \cap Curl_{\tilde{\Sigma}}(\Omega) \tag{9.7}$$

This is the "subspace of non-harmonic curls" in $H_{DC\Sigma}(\Omega)$ and $P_{C\tilde{\Sigma}}$ maps $H_{DC\Sigma}(\Omega)$ into this subspace. The following analysis shows that $P_{C\tilde{\Sigma}}$ is a DC-orthogonal projection on $H_{DC\Sigma}(\Omega)$.

Define $H_{\tilde{\Sigma}}(\mathrm{curl}^2,\Omega) \;:=\; \{A \in Z_{\tilde{\Sigma}}(\Omega) : \mathrm{curl}^2 A \in L^2(\Omega;\mathbb{R}^3)\}$. This is a real Hilbert space under the inner product

$$\langle A, B\rangle_C \;:=\; \int_\Omega [A\cdot B \;+\; \mathrm{curl}\, A\cdot \mathrm{curl}\, B \;+\; \mathrm{curl}^2 A\cdot \mathrm{curl}^2 B]\; d^3x. \tag{9.8}$$

Now consider the question of describing the DC-curl projection on $H_{DC\Sigma}(\Omega)$. Define the functional $\mathcal{G}_v : H_{\tilde{\Sigma}}(\mathrm{curl}^2,\Omega) \to [0,\infty)$ by

$$\mathcal{G}_v(A) \;:=\; \|v - \mathrm{curl}\, A\|_{DC}{}^2 \tag{9.9}$$

$$\;=\; \mathcal{C}_v(A) \;+\; \int_\Omega [|v|^2 + |\mathrm{div}\, v|^2 + \left|\mathrm{curl}^2 A - \mathrm{curl}\, v\right|^2]\; d^3x. \tag{9.10}$$

The solution of this problem is given by the following result.

Theorem 9.3. *Assume Ω, Σ satisfy (B1) and (B2) and $v \in H_{DC\Sigma}(\Omega)$. There is a unique minimizer $\tilde{A}_v$ of $\mathcal{G}_v$ on $H_{\tilde{\Sigma}}(\mathrm{curl}^2,\Omega)$ and $\tilde{A}_v$ minimizes $\mathcal{C}_v$ on $Z_{\tilde{\Sigma}}(\Omega)$. Moreover $Cu_\Sigma(\Omega)$ is a closed subspace of $H_{DC\Sigma}(\Omega)$ and $P_{C\tilde{\Sigma}}v := \mathrm{curl}\, \tilde{A}_v$ is the DC-projection of $H_{DC\Sigma}(\Omega)$ onto $Cu_\Sigma(\Omega)$.*

Proof. Suppose $\tilde{A}_v$ minimizes $\mathcal{C}_v$ on $Z_{\tilde{\Sigma}}(\Omega)$. Then it satisfies (9.4) so $\mathrm{curl}\, A_v$ is in $H_{DC}(\Omega)$. It satisfies the boundary conditions (2.8) from proposition 7.1) and from equation 9.5. By inspection of (9.10), $\tilde{A}_v$ minimizes $\mathcal{G}_v$ on $H_{\tilde{\Sigma}}(\mathrm{curl}^2,\Omega)$ as it minimizes $\mathcal{C}_v$ and (9.4) holds. For each v in $H_{DC\Sigma}(\Omega)$, there is a minimizer of this problem so $Cu_\Sigma(\Omega)$ is a closed subspace of $H_{DC\Sigma}(\Omega)$ from corollary 3.3 of [3]. The operator $P_{C\tilde{\Sigma}}v$ defines the projection from Riesz theorem as $H_{DC\Sigma}(\Omega)$ is a Hilbert space. □

These results show that $\tilde{A}_v$ is the appropriate choice of a vector potential in (4.1) so that the decomposition is orthogonal. This may be strengthened to the following *Hodge-Weyl* type decomposition theorm for the space $H_{DC\Sigma}(\Omega)$.

Theorem 9.4. *When (B1) and (B2) hold then*

$$H_{DC\Sigma}(\Omega) \;=\; G_{\Sigma 0}(\Omega) \oplus Cu_\Sigma(\Omega) \oplus \mathcal{H}_\Sigma(\Omega) \tag{9.11}$$

and this decomposition is both L^2- and DC- orthogonal.

Proof. If v is L^2-orthogonal to $Cu_\Sigma(\Omega)$, then

$$\int_\Omega \mathrm{curl}\, A\cdot v\; d^3x \;=\; 0 \quad \text{for all} \quad A \in Z_{\tilde{\Sigma}}(\Omega). \tag{9.12}$$

Use (2.5) and the definition of $Z_{\tilde{\Sigma}}(\Omega)$, then this implies

$$\mathrm{curl}\, v \;=\; 0 \quad \text{on } \Omega \quad \text{and} \qquad v\wedge\nu \;=\; 0 \quad \text{on } \Sigma. \tag{9.13}$$

When v is L^2-orthogonal to $G_{\Sigma 0}(\Omega)$, then (7.1) holds. Hence the fields which are L^2-orthogonal to $G_{\Sigma 0}(\Omega) \oplus Cu_\Sigma(\Omega)$ are precisely the fields in $\mathcal{H}_\Sigma(\Omega)$. Thus (9.11) holds in an L^2-sense. This is also a DC-orthogonal decomposition so the result follows. □

An interesting, and physically important, consequence of this definition of the vector potentials for fields in $H_{DC\Sigma}(\Omega)$ is the following.

Theorem 9.5. *Suppose (B1) and (B2) hold, and (μ, A) is an eigenpair of the Σ-mixed* curl^2 *eigenproblem on Ω with $\mu > 0$. Then $(\mu, \operatorname{curl} A)$ is an eigenpair of the $\tilde{\Sigma}$-mixed* curl^2 *eigenproblem on Ω.*

Proof. The assumption is that $A \in Z_\Sigma(\Omega)$ is a solution of

$$\int_\Omega (\operatorname{curl} A \cdot \operatorname{curl} B \; - \; \mu A \cdot B)\, d^3x \; = \; 0 \quad \text{for all} \quad B \in Z_\Sigma(\Omega). \tag{9.14}$$

from (8.5) and the comments after (8.10). Given $B \in Z_\Sigma(\Omega)$, there is a unique $D \in Z_{\tilde{\Sigma}}(\Omega)$ with $B \; = \; \operatorname{curl} D$. Let $C := \operatorname{curl} A$, use the boundary conditions satisfied by A, B, D, then (2.5) implies that

$$\int_\Omega A \cdot \operatorname{curl} D \;\; d^3x = \int_\Omega D \cdot C \;\; d^3x, \quad \int_\Omega C \cdot \operatorname{curl} B \;\; d^3x = \int_\Omega \operatorname{curl} D \cdot \operatorname{curl} C \;\; d^3x$$

so

$$\int_\Omega (\operatorname{curl} C \cdot \operatorname{curl} D \; - \; \mu C \cdot D)\, d^3x \; = \; 0 \quad \text{for all} \quad D \in Z_{\tilde{\Sigma}}(\Omega). \tag{9.15}$$

Thus (μ, C) is an eigenpair of the $\tilde{\Sigma}$- eigenproblem. □

This shows that the Σ-mixed and $\tilde{\Sigma}$-mixed curl^2 eigenproblems have the same non-zero eigenvalues, each of the same multiplicity. In particular $\mu_1(\Sigma) \; = \; \mu_1(\tilde{\Sigma})$ and the problems are isospectral - except perhaps for the null eigenvalue. This may be stated as follows, with $C^{(m)} \; := \; \operatorname{curl} A^{(m)}$.

Theorem 9.6. *Assume Ω, Σ satisfy (B1) and (B2). Let $\{\mu_m : m \geq 1\}$ be the increasing sequence of positive eigenvalues of the Σ-mixed* curl^2 *eigenproblem on Ω defined above and $\{A^{(m)} : m \geq 1\}$ be a corresponding family of DC- orthonormal eigenfields. Then*
(i): the $\tilde{\Sigma}$-mixed curl^2 *eigenproblem on Ω has precisely the same non-zero eigenvalues, each with the same multiplicity, and*
(ii): the set $\{{\mu_m}^{-1/2} C^{(m)} : m \geq 1\}$ is a maximal DC-orthonormal set in $Z_{\tilde{\Sigma}}(\Omega)$.

10. The Curl Estimate and the Main Inequality

We are now in a position to prove the main inequality for fields in $H_{DC\Sigma}(\Omega)$. First there is a curl estimate that holds for the solenoidal fields in $Cu_\Sigma(\Omega)$.

Theorem 10.1. *Assume Ω, Σ satisfy (B1) and (B2) and $\mu_1(\Sigma)$ is the principal eigenvalue of (8.15). Then*

$$\|\operatorname{curl} v\|^2 \;\; \geq \;\; \mu_1(\Sigma) \;\|v\|^2 \qquad \textit{for all} \quad v \in Cu_\Sigma(\Omega). \tag{10.1}$$

Equality holds here when v is a multiple of $\operatorname{curl} A$ *with A an eigenfield of the $\tilde{\Sigma}$-mixed curl^2 eigenproblem corresponding to the eigenvalue $\mu_1(\Sigma)$.*

Proof. If $v \in Cu_\Sigma(\Omega)$, then $v = \operatorname{curl} \tilde{A}_v$ with $\tilde{A}_v \in H_{\tilde{\Sigma}}(\operatorname{curl}^2, \Omega)$. Suppose

$$A_v(x) = \sum_{m=1}^{\infty} c_m \, A^{(m)}(x)$$

where $\{A^{(m)} : m \geq 1\}$ is the orthonormal basis of $Z_{\tilde{\Sigma}}(\Omega)$ defined in section 8. Then

$$\|v\|^2 = \sum_{m=1}^{\infty} {c_m}^2 \left\|\operatorname{curl} A^{(m)}\right\|^2, \quad \text{and} \tag{10.2}$$

$$\|\operatorname{curl} v\|^2 = \sum_{m=1}^{\infty} {c_m}^2 \left\|\operatorname{curl}^2 A^{(m)}\right\|^2. \tag{10.3}$$

using the orthogonality of these fields. However, for each m,

$$\left\|\operatorname{curl}^2 A^{(m)}\right\|^2 = {\mu_m}^2 \left\|A^{(m)}\right\|^2 = \mu_m \left\|\operatorname{curl} A^{(m)}\right\|^2 \tag{10.4}$$

Substitute this in (10.2) and (10.3), then (10.1) and the criterion for equality follows. □

This result, together with theorem 9.4, leads to the specific form of the main inequality.

Theorem 10.2. *Assume (B1), (B2) hold and $v \in H_{DC\Sigma}(\Omega)$ is L^2- orthogonal to $\mathcal{H}_\Sigma(\Omega)$. Then the main inequality (1.1) holds with $c := \min(\lambda_1(\Sigma), \mu_1(\Sigma)) > 0$.*

Proof. When v as in this theorem then, from theorems 4.3, 9.3 and 9.4,

$$v = \nabla \varphi_v + \operatorname{curl} \tilde{A}_v$$

with $\nabla \varphi_v$ being the scalar potential defined as in section 4 and $\tilde{A}_v \in Z_{\tilde{\Sigma}}(\Omega)$ being the vector potential defined as in the last section.

From the inequalities of 6.1 and 10.1,

$$\|\operatorname{div} v\|^2 \geq \lambda_1(\Sigma) \, \|\nabla \varphi_v\|^2 \quad \text{and} \quad \|\operatorname{curl} v\|^2 \geq \mu_1(\Sigma) \left\|\operatorname{curl} \tilde{A}_v\right\|^2 \tag{10.5}$$

Add these inequalities, then since this is an L^2-orthogonal decomposition,

$$\|v\|^2 = \|\nabla \varphi_v\|^2 + \left\|\operatorname{curl} \tilde{A}_v\right\|^2.$$

so (1.1) holds with the given value of c. □

Since each of the inequalities (6.1) and (10.1) are equalities for specific fields in the respective subspaces, then equality will be attained in (1.1) for a subspace of fields and the constant c given here is best possible.

An important question for numerical analysis is whether this inequality holds when the requirement in (B1) that the surfaces of the region be C^2 is replaced by a weaker regularity condition? One might expect the characterization of the constant c in theorem 10.2 holds when the boundary is, say, Lipschitz continuous. In this paper, the C^2 regularity of the boundary was explicitly used in the proof of theorem 3.2 and this result was necessary for theorem 3.4. The question becomes whether these results can be modified to cases of less smooth boundaries?

REFERENCES

[1] C. J. Amick, "Some remarks on Rellich's theorem and the Poincaré inequality," J. London Math. Soc. (2) **18** (1973), 81-93.

[2] J-P Aubin, *Optima and Equilibria*, Springer Verlag New York, 1993.

[3] G. Auchmuty, "Orthogonal decompositions and bases for three-dimensional vector fields", Numer. Funct. Anal. and Optimiz. **15** (1994), 445-488.

[4] G. Auchmuty and J.C. Alexander, "L^2-well-posedness of planar div-curl systems", Arch Rat Mech & Anal, **160** (2001), 91-134.

[5] G. Auchmuty and J.C. Alexander, "L^2-well-posedness of div-curl systems on bounded regions in space", submitted.

[5] P. Blanchard and E. Brüning, *Variational Methods in Mathematical Physics*, Springer Verlag, Berlin (1992).

[6] J. Cantarella, D. De Turck and H. Gluck, "Vector Calculus and the Topology of Domains in 3-Space", Am. Math. Monthly **109** (2002), 409-442.

[7] M. Cessenat, *Mathematical Methods in Electromagnetism: linear theory and applications*, World Scientific, Singapore (1996).

[8] R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, 6 volumes, Springer Verlag, Berlin (1990).

[9] G. Duvaut and J.L. Lions, *Inequalities in Mechanics and Physics*, Springer, Berlin (1976).

[10] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton (1992).

[11] P. Fernandes and G. Gilardi, "Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions," Math. Models and Methods in Applied Sciences, **7** (1997), 957-991.

[12] C. Foias and R. Temam, "Remarques sur les *é*quations de Navier-Stokes stationnaires et les ph*é*nomenes successifs de bifurcation," Annali Scuola Normale Superiore - Pisa, (1978), 29-63.

[13] K.O. Friedrichs, "Differential Forms on Riemannian Manifolds", Communications in Pure and Applied Mathematics, **8** (1955), 551-590.

[14] V. Girault and P.A. Raviart, *Finite Element Methods for the Navier-Stokes Equations*, Springer Verlag, Berlin (1986).

[15] R. Leis, "Zur Theorie elektromagnetischer Schwingungen in anisotropen Medien", Math. Z. **106** (1968), 213-224.

[16] J. Saranen, "On an inequality of Friedrichs", Math. Scand. **51** (1982), 310-322.

[17] E. Zeidler, *Nonlinear Functional Analysis and its Applications, III: Variational Methods and Optimization*, Springer Verlag, New York (1985).

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