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Steklov Eigenproblems and the Representation of Solutions of Elliptic Boundary Value Problems

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ABSTRACT

This paper describes some properties and applications of Steklov eigenproblems for prototypical second-order elliptic operators on bounded regions in $\mathbb{R}^n$. Results are described for Schroedinger and weighted harmonic equations. A variational description of the least eigenvalue leads to optimal L^2-trace inequalities. It is shown that the eigenfunctions provide complete orthonormal bases of certain closed subspaces of $H^1(\Omega)$ and also of $L^2(\partial\Omega, d\sigma)$. This allows the description, and representation, of solution operators for homogeneous elliptic equations subject to inhomogeneous Dirichlet, Neumann or Robin boundary data. They are also used to describe Robin to Dirichlet and Neumann to Dirichlet operators for these equations, and to describe the spectrum of these operators. The allowable regions are quite general; in particular classes of bounded regions with a finite number of disjoint Lipschitz components for the boundary are allowed.

Key Words: Steklov eigenproblems; A-harmonic functions; Schroedinger operators; Neumann to Dirichlet operator; Robin to Dirichlet operator.

Mathematics Subject Classification: Primary 35P10; Secondary 35J20, 35J25, 49R50.

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1. INTRODUCTION

This paper will describe some results about, and applications of, Steklov eigenproblems for prototypical second order elliptic partial differential operators on bounded regions in $\mathbb{R}^n$. These eigenproblems are described and analyzed for Schroedinger type operators in Secs. 3–5 and for weighted harmonic operators in Secs. 6–9.

For both classes of eigenproblems, under mild regularity assumptions, the existence of an unbounded, infinite, discrete spectrum is demonstrated. The least positive eigenvalue of these problems is shown to be the optimal constant in certain trace inequalities. Moreover a corresponding family of Steklov eigenfunctions will be constructed which is an orthonormal basis of the subspace of $H^1(\Omega)$ orthogonal to $H_0^1(\Omega)$ with respect to specific inner products.

These results lead to orthogonal series expansions, in terms of the Steklov eigenfunctions, for the solutions of homogeneous elliptic equations with non-homogeneous boundary conditions. These series are described in Secs. 9 and 10 and will be shown to converge strongly in $H^1(\Omega)$. The expansions provide a spectral-type representation for the solution operators of linear boundary value problems of the form

$$Lu(x) = 0 \quad \text{in } \Omega, \ \text{ subject to } \ Bu(x) = g(x) \quad \text{on } \partial\Omega. \tag{1.1}$$

Here the boundary conditions may be of Dirichlet, Robin or Neumann type. The solution operators classically have been defined using Poisson, Robin or Neumann boundary integral kernels as in part B of Bergman and Schiffer (1953). Here they are shown to be strong (H^1-)limits of certain finite rank boundary integral operators. The approach is quite different to that based on the use of single and double layer potentials as described in DiBenedetto (1995, Chap. 3), or Kress (1989, Sec. 6.4).

These results depend on proofs that certain families of Steklov eigenfunctions are maximal orthonormal sets in certain closed subspaces of $H^1(\Omega)$ and also in $L^2(\partial\Omega, d\sigma)$. These completeness results are described in Theorems 5.3, 7.3, 9.4 and 10.3 and are based on variational arguments. Then elementary Hilbert space theory is used to describe the solutions of these boundary value problems. These results also provide spectral characterizations of the trace space $H^{1/2}(\partial\Omega)$ for the different equations.

The methods used to obtain the results described here may be generalized in a variety of ways. No effort has been made to describe the most general operators to which this approach applies. We have, however, tried to identify simple boundary regularity requirements; they are that (B1) and (B2) of section 2 hold. In particular, the boundary is not required to be C^1 so this approach applies to many regions used in computational simulations.

Many of the results described here are related to issues of interest in the theory of inverse problems. In particular, Sec. 11 describes results about the Robin to Dirichlet and Neumann to Dirichlet maps. There the restrictions of the Steklov eigenfunctions to the boundary are shown to be eigenfunctions of these operators and the eigenvalues of these maps are related to the Steklov eigenvalues.

2. DEFINITIONS AND NOTATION

This paper will treat issues arising in the study of boundary value problems on regions Ω in $\mathbb{R}^n, n \geq 2$. A region is a non-empty, connected, open subset of $\mathbb{R}^n$. Its closure is denoted $\overline{\Omega}$ and its boundary is $\partial\Omega := \overline{\Omega} \setminus \Omega$. Points in Ω are denoted by $x = (x_1, x_2, \ldots, x_n)$ and Cartesian coordinates will be used exclusively.

Further conditions on Ω will be required. In the following, we will use the definitions and terminology of Evans and Gariepy (1992), save that $\sigma, d\sigma$ will represent Hausdorff $(n-1)$-dimensional measure and integration with respect to this measure, respectively. This measure is called surface area and our basic assumption will be:

(B1). Ω is a bounded region in $\mathbb{R}^n$ and its boundary $\partial\Omega$ is the union of a finite number of disjoint closed Lipschitz surfaces; each surface having finite surface area.

When this holds there is an outward unit normal ν defined at σ *a.e.* point of $\partial\Omega$. The real Lebesgue spaces $L^p(\Omega)$ and $L^p(\partial\Omega, d\sigma), 1 \leq p \leq \infty$ will be defined in the standard manner and have the usual p-norm denoted by $\|u\|_p$ and $\|u\|_{p,\partial\Omega}$, respectively. The L^2-inner products are denoted

$$\langle u, v\rangle := \int_\Omega u(x)v(x)\,dx \quad \text{and} \quad \langle u, v\rangle_\partial := \int_{\partial\Omega} uv\,d\sigma.$$

All functions in this paper will take values in $\overline{\mathbb{R}} := [-\infty, \infty]$ and derivatives should be taken in a weak sense. A real sequence $\{x_m : m \geq 1\}$ is said to be (strictly) increasing if $x_{m+1}(>) \geq x_m$ for all m. Similarly a function u is said to be (strictly) positive on a set E, if $u(x) \geq (>)0$ on E. The gradient of a function u will be denoted ∇u.

Let $H^1(\Omega)$ be the usual real Sobolev space of functions on Ω. It is a real Hilbert space under the standard H^1-inner product

$$[u, v]_1 := \int_\Omega [u(x) \cdot v(x) + \nabla u(x) \cdot \nabla v(x)]\,dx. \tag{2.1}$$

The corresponding norm will be denoted by $\|u\|_{1,2}$

The region Ω is said to satisfy *Rellich's theorem* provided the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ is compact for $1 \leq p < p_S$ where $p_S(n) := 2n/(n-2)$ when $n \geq 3$, or $p_S(2) = \infty$ when $n = 2$.

There are a number of different criteria on Ω and $\partial\Omega$ that imply this result. When (B1) holds it is Theorem 1 in Sec. 4.6 of Evans and Gariepy (1992). See also Amick (1973). DiBenedetto (2001), in Theorem 14.1 of Chap. 9 shows that the result holds when Ω is bounded and satisfies a "cone property." Adams and Fournier give a thorough treatment of conditions for this result in Chap. 6 of Adams and Fournier (2003) and show that it also holds for some classes of unbounded regions.

When (B1) holds and $u \in W^{1,1}(\Omega)$ then the trace of u on $\partial\Omega$ is well-defined and is a Lebesgue integrable function with respect to σ, see Evans and Gariepy (1992), Sec. 4.2 for details. The region Ω is said to satisfy a *compact trace theorem* provided the trace mapping $\Gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ is compact. The trace map is the linear extension of the map restricting Lipschitz continuous functions on $\overline{\Omega}$ to $\partial\Omega$. Sometimes we will just use u in place of Γu when considering the trace of a function on $\partial\Omega$.

Evans and Gariepy (1992, Sec. 4.3), shows that Γ is continuous when $\partial\Omega$ satisfies (B1). Theorem 1.5.1.10 of Grisvard (1985) proves an inequality that implies the compact trace theorem when $\partial\Omega$ satisfies (B1). This inequality is also proved in DiBenedetto (2001, Chap. 9, Sec. 18) under stronger regularity conditions on the boundary. Most descriptions of trace theorems in the current literature involve the space $H^{1/2}(\partial\Omega)$ but here we shall only use a simpler analysis involving Lebesgue spaces.

In general, we shall require that the region satisfy

(B2). Ω and $\partial\Omega$ satisfy (B1), the Rellich theorem and the compact trace theorem.

In this paper, we shall use various standard results from the calculus of variations and convex analysis. Background material on such methods may be found in Blanchard and Brüning (1992) or Zeidler (1985), both of which have discussions of the variational principles for the Dirichlet eigenvalues and eigenfunctions of second order elliptic operators. The variational principles used here are variants of the principles described there and are analogous to those for the Laplacian described in Sec. 5 of Auchmuty (2004). Some quite different unconstrained variational principles for eigenvalue problems are described in Auchmuty (2001).

In this paper, all the variational principles, and functionals will be defined on (closed convex subsets of) $H^1(\Omega)$. When $\mathscr{F} : H^1(\Omega) \to (-\infty, \infty]$ is a functional, then $\mathscr{F}$ is said to be G-differentiable at a point $u \in H^1(\Omega)$ if there is a $\mathscr{F}'(u)$ such that

$$\lim_{t\to 0} t^{-1}\,[\mathscr{F}(u+tv) - \mathscr{F}(u)] = \mathscr{F}'(u)(v) \quad \text{for all } v \in H^1(\Omega),$$

with $\mathscr{F}'(u)$ a continuous linear functional on $H^1(\Omega)$. In this case, $\mathscr{F}'(u)$ is called the G-derivative of $\mathscr{F}$ at u.

3. THE SCHROEDINGER–STEKLOV EIGENPROBLEM

Assume Ω is a region in $\mathbb{R}^n$ which satisfies (B1). The classical form of the Steklov eigenproblem for a Schroedinger-type operator is to find those values of μ for which there is a non-trivial classical solution $\hat{u}$ of the system

$$Lu(x) := c(x)u(x) - \Delta u(x) = 0 \quad \text{on } \Omega \tag{3.1}$$

$$\text{subject to} \quad \frac{\partial u}{\partial \nu}(x) = \mu\rho(x)u(x) \quad \text{on } \partial\Omega \tag{3.2}$$

The functions c, ρ should satisfy

(A1). c is positive on Ω, in $L^p(\Omega)$ for $p \geq n/2$ when $n \geq 3$, ($p > 1$ when $n = 2$) and $\int_\Omega c\, dx > 0$.

(A2). ρ is in $L^\infty(\partial\Omega, d\sigma)$, positive on $\partial\Omega$, and

$$\int_{\partial\Omega} \rho\, d\sigma = 1. \tag{3.3}$$

The weak form of (3.1)–(3.2) is to find the real values of μ such that there is a non-zero solution u in $H^1(\Omega)$ of

$$\int_\Omega [\nabla u \cdot \nabla v + cuv]\, dx - \mu \int_{\partial\Omega} \rho uv\, d\sigma = 0 \quad \text{for all } v \in H^1(\Omega). \tag{3.4}$$

This will be called the Steklov eigenproblem for (L, ρ).

There is some literature on problems of this type; see Bandle (1980, Chap. 3) for instance. She describes a standard variational principle of Rayleigh type for the first eigenvalue of this problem. From (3.4) with $v = u$, one sees that any eigenvalue must be positive. Here we shall describe a different variational principle for the least positive eigenvalue and corresponding eigenfunction of (3.4).

Let K be the subset of $H^1(\Omega)$ of functions satisfying

$$\mathscr{D}_c(u) := \int_\Omega [|\nabla u|^2 + cu^2]\, dx \;\leq\; 1 \tag{3.5}$$

Define $\mathscr{B} : H^1(\Omega) \to [0, \infty)$ and $\langle ., . \rangle_\rho$ by

$$\mathscr{B}(u) := \int_{\partial\Omega} \rho u^2\, d\sigma \quad \text{and} \quad \langle u, v \rangle_\rho := \int_{\partial\Omega} \rho uv\, d\sigma. \tag{3.6}$$

Consider the variational principle $(\mathscr{S}_1)$ of maximizing $\mathscr{B}$ on K and define

$$\beta_1 := \sup_{u \in K} \mathscr{B}(u). \tag{3.7}$$

We shall show that the maximizer u_1 of this problem is an eigenfunction of the Steklov problem (3.4) corresponding to the least eigenvalue μ_1 and that $\beta_1 = \mu_1^{-1}$. To do this we first need some technical results.

Theorem 3.1. *Assume that* $\Omega, \partial\Omega, c, \rho$ *satisfy* (B2), (A1) *and* (A2). *Then* $\mathscr{B}$ *and* $\mathscr{D}_c$ *are convex, continuous and G-differentiable on* $H^1(\Omega)$ *with*

$$\langle \mathscr{D}_c{}'(u), v \rangle = 2 \int_\Omega [\nabla u \cdot \nabla v + cuv]\, dx, \tag{3.8}$$

and

$$\langle \mathscr{B}'(u), v \rangle = 2 \int_{\partial\Omega} \rho uv\, d\sigma \quad \textit{for all } u, v \in H^1(\Omega). \tag{3.9}$$

Moreover $\mathscr{B}$ *is also weakly continuous on* $H^1(\Omega)$.

Proof. When u, v are in $H^1(\Omega)$ and $n \geq 3$ then from the Sobolev theorem, u^2, v^2 will be in $L^q(\Omega)$ for $1 \leq q \leq n/(n-2)$. Holder's inequality yields that

$$\left| \int_\Omega c\,(u^2 - v^2)\, dx \right| \leq \|c\|_p \left\|u^2 - v^2\right\|_{p'}$$

where p and p' are conjugate indices. When c satisfies (A1), this implies that $\mathscr{D}_c$ is continuous. This proof also holds when $n = 2$.

Suppose that $\{u_m : m \geq 1\}$ converges weakly to u in $H^1(\Omega)$. The compact trace theorem implies that Γu_m converges strongly to Γu in $L^2(\partial\Omega, d\sigma)$. Apply Holder's inequality then we see that $\mathscr{B}$ is weakly continuous on $H^1(\Omega)$ when ρ satisfies (A2).

The proofs that the G-derivatives $\mathscr{B}', \mathscr{D}_c'$ exist and are given by (3.8)–(3.9) are straightforward. Since these functionals are positive, quadratic and G-differentiable on $H^1(\Omega)$, they are convex. □

The following result is needed to prove that K is bounded in $H^1(\Omega)$.

Theorem 3.2. *Assume that* Ω, $\partial\Omega$, c *satisfy* (B2) *and* (A1). *Then there is an* $\alpha > 0$ *such that*

$$\mathscr{D}_c(u) \geq \alpha \int_\Omega u^2\, dx \quad \text{for all } u \in H^1(\Omega). \tag{3.10}$$

Proof. To prove this inequality consider the variational problem of minimizing $\mathscr{D}_c(u)$ on the subset S of $H^1(\Omega)$ of functions satisfying $\|u\|_2 = 1$.

Let $\{u_m : m \geq 1\}$ be a minimizing sequence for this problem and define

$$\alpha := \inf_{u \in S} \mathscr{D}_c(u)$$

For all sufficiently large m, $\|u_m\|_{1,2}^2 < \alpha + 2$, so this sequence is bounded in $H^1(\Omega)$. Thus it has a weakly convergent subsequence $\{u_{m_j} : j \geq 1\}$ which converges weakly to a limit $\hat{u}$ in $H^1(\Omega)$. From Rellich's theorem this subsequence converges strongly to $\hat{u}$ in $L^2(\Omega)$ so $\hat{u}$ is in S. Thus $\mathscr{D}_c(\hat{u}) = \alpha$ as the functional is weakly l.s.c.

If $\alpha = 0$, then $\nabla\hat{u} \equiv 0$ on Ω so $\hat{u}$ must be constant as Ω is connected. In this case, the assumption (A1) on c provides a contradiction, so $\alpha > 0$ as claimed. The inequality (3.10) now follows for all u in $H^1(\Omega)$ by homogeniety. □

When (B2) and (A1) hold, we will find it convenient to use the weighted inner product

$$[u, v]_c := \int_\Omega [\nabla u \cdot \nabla v + cuv]\, dx. \tag{3.11}$$

and the associated norm $\|u\|_c$. The preceding theorem then yields

Corollary 3.3. *Assume* (A1) *and* (B2) *hold, then* $\|\cdot\|_c$ *is an equivalent norm on* $H^1(\Omega)$ *and* K *is a bounded closed convex subset of* $H^1(\Omega)$.

Proof. There is a C_1 such that $\|u\|_c^2 \leq C_1 \|u\|_{1,2}^2$ since $\mathscr{D}_c$ is continuous and quadratic on $H^1(\Omega)$.

Conversely, from (3.10), we have

$$\|u\|_{1,2}^2 \leq (1 + \alpha^{-1})\, \|u\|_c^2$$

Thus the two inner products are equivalent and K has the claimed properties. □

This result enables us to prove the following existence result for solutions of the variational problem $(\mathscr{S}_1)$.

Theorem 3.4. *Assume that* Ω, $\partial\Omega$, c, ρ *satisfy* (B2), (A1) *and* (A2). *Then* β_1 *is finite and there are maximizers* $\pm u_1$ *of* $\mathscr{B}$ *on* K. *These maximizers satisfy* $\|u_1\|_c = 1$ *and* (3.4). *The corresponding eigenvalue* μ_1 *is the least eigenvalue of* (3.4) *and* $\beta_1 = \mu_1^{-1}$.

Proof. From the results of Corollary 3.3, K is weakly compact in $H^1(\Omega)$. Since $\mathscr{B}$ is weakly continuous, it attains its supremum on K at a point u_1 in K and this supremum is finite. If $\|u_1\|_c < 1$ then there is a $k > 1$ such that ku_1 is in K and then $\mathscr{B}(ku_1) = k^2\mathscr{B}(u_1) > \mathscr{B}(u_1)$. This contradicts the maximality of u_1 so we must have $\|u_1\|_c = 1$.

A Lagrangian functional for the problem $(\mathscr{S}_1)$ is given by $\mathscr{L} : H^1(\Omega) \times [0,\infty) \to \mathbb{R}$ defined by

$$\mathscr{L}(u,\lambda) := \lambda\left[\int_\Omega [|\nabla u|^2 + cu^2]\, dx - 1\right] - \int_{\partial\Omega} \rho u^2\, d\sigma. \tag{3.12}$$

The problem of maximizing $\mathscr{B}$ on K is equivalent to finding an inf-sup point of L on its domain. Any such maximizer will be a critical point of $L(\cdot,\lambda)$ on $H^1(\Omega)$ so it is a solution of

$$\lambda \int_\Omega [\nabla u\cdot\nabla v + cuv]\, dx \; - \int_{\partial\Omega} \rho uv\, d\sigma = 0 \quad \text{for all } v \in H^1(\Omega). \tag{3.13}$$

When $\lambda > 0$ this has the form (3.4) with $\mu = \lambda^{-1}$. If $\lambda = 0$, then (3.13) implies that the maximum value is zero which is not true. Thus (3.4) holds at the maximizer.

If u_1 is a maximizer, then the corresponding eigenvalue μ_1 in (3.4) satisfies

$$\|u_1\|_c^2 = 1 = \mu\mathscr{B}(u_1)$$

upon putting $u = v = u_1$. Hence $\beta_1 = \mu_1^{-1}$.

If μ_1 is not the least positive eigenvalue of (3.4), there will be a nonzero $\tilde{u}$ in $H^1(\Omega)$ satisfying (3.4) with $\tilde{\mu} < \mu_1$. Normalize it to have c-norm 1. Then (3.4) implies that $\tilde{\mu}$ satisfies

$$\tilde{\mu}\mathscr{B}(\tilde{u}) = 1.$$

Hence $\mathscr{B}(\tilde{u}) > \beta_1$ which is impossible so μ_1 is minimal. □

Corollary 3.5. *Assume* Ω, $\partial\Omega$, c, ρ *satisfy* (B2), (A1) *and* (A2). *Then, for all* $u \in H^1(\Omega)$,

$$\int_\Omega [|\nabla u|^2 + c\,u^2]\, dx \geq \mu_1 \int_{\partial\Omega} \rho u^2\, d\sigma, \tag{3.14}$$

where $\mu_1 > 0$ *is the least Steklov eigenvalue of* (3.4). *If equality holds here then u is a multiple of an eigenfunction of* (3.4) *corresponding to* μ_1.

Proof. The inequality holds if $u \equiv 0$. Otherwise let $v := u/\|u\|_c$. Then $v \in K$ and $\mathscr{B}(v) \leq \beta_1$. Homogeniety of these functionals then yields (3.14). □

This inequality (3.14) is the H^1-trace inequality for the operator L. The case $c(x) \equiv 1$, is discussed in Horgan (1979) where some lower bounds for μ_1 on 1- and 2-d regions are found. The choice $u(x) \equiv 1$ here yields an upper bound on the first Steklov eigenvalue:

$$\mu_1 \ \leq \ \int_\Omega c(x)\, dx.$$

Note that the requirements (A2) for ρ permit the choice $\rho(x) := c\chi_\Sigma(x)$ where Σ is any σ-measurable subset of $\partial\Omega$, χ_Σ is the characteristic function of Σ and c is chosen to normalize ρ. Then (3.14) provides an upper bound on

$$\int_\Sigma u^2\, d\sigma \quad \text{in terms of the } c\text{-norm of } u \text{ on } \Omega.$$

4. VARIATIONAL PRINCIPLES FOR SUCCESSIVE STEKLOV EIGENVALUES

Given the first J Steklov eigenvalues and corresponding c-orthonormal eigenfunctions of (L, ρ) we shall now describe how to find the next eigenvalue μ_{J+1} and a corresponding normalized eigenfunction. Assume that the first J eigenvalues are $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_J$ and that $\{u_1, u_2, \ldots, u_J\}$ is a corresponding family of c-orthonormal eigenfunctions of (3.4). This implies that

$$\langle \Gamma u_j, \Gamma u_k \rangle_\rho = \mu_j^{-1}\ \delta_{jk} \tag{4.1}$$

To find μ_{J+1}, let

$$K_J := \{\, u \in K : \langle \Gamma u, \Gamma u_j \rangle_\rho = 0 \text{ for } 1 \leq j \leq J\} \tag{4.2}$$

Consider the variational problem $(\mathscr{S}_{J+1})$ of maximizing $\mathscr{B}$ on K_J and define

$$\beta_{J+1} := \sup_{u \in K_J}\ \mathscr{B}(u). \tag{4.3}$$

Theorem 4.1. *Assume that* Ω, $\partial\Omega$, c, ρ *satisfy* (B2), (A1) *and* (A2). *Then* K_J *is a bounded closed convex set in* $H^1(\Omega)$, β_{J+1} *is finite and there are maximizers* $\pm u_{J+1}$ *of* $\mathscr{B}$ *on K. These maximizers satisfy* $\|u_{J+1}\|_c = \mu_{J+1}\|\Gamma u_{J+1}\|_\rho^2 = 1$, (3.4) *with* $\mu_{J+1} := \beta_{J+1}^{-1}$ *and*

$$[u_{J+1}, u_j]_c = \langle \Gamma u_{J+1}, \Gamma u_j \rangle_\rho = 0 \quad \textit{for } 1 \leq j \leq J. \tag{4.4}$$

Moreover μ_{J+1} *is the smallest eigenvalue of this problem greater than or equal to* μ_J.

Proof. The linear functionals $b_j(u) := \langle \Gamma u, \Gamma u_j \rangle_\rho$ are continuous on $H^1(\Omega)$ since (A2) and the trace theorem hold. Hence K_J is a bounded closed convex subset, as K is. Thus K_J is weakly compact in $H^1(\Omega)$, so $\mathscr{B}$ has a finite maximum on K and attains this maximum on K. By symmetry of the functionals, if u_{J+1} is a maximizer so is $-u_{J+1}$.

The fact that $\|u_{J+1}\|_c = 1$ holds just as in the proof of Theorem 3.4. Hence if (3.4) holds then $\mu_{J+1} \cdot \beta_{J+1} = 1$. The proof that (3.4) holds is described below. When it holds substitute u_j for v and u_{J+1} for u, then the definition of K_J implies (4.4). The proof that μ_{J+1} is the smallest eigenvalue greater than or equal to μ_J is the same as the last part of the proof of Theorem 3.4. □

To complete the above proof, it is necessary to show that the maximizers satisfy (3.4). This may be done using a multiplier type argument similar to that of the proof of Theorem 3.4. A more informative proof, using elementary ideas from convex analysis is as follows.

When C is a closed convex set in a real Hilbert space H, let $I_C : H \to [0, \infty]$ be the indicator functional of C defined by $I_C(u) := 0$ for $u \in C$, and $I_C(u) := \infty$ when $u \notin C$.

When C is the closed unit ball of radius 1 in a closed subspace V of H, then its subdifferential is given, when $u \in C$, by $\partial I_C(u) = V^\perp$ when $\|u\| < 1$ and $\partial I_C(u) = \{\lambda u + w : \lambda \geq 0 \text{ \& } w \in V^\perp\}$ when $\|u\| = 1$. Here $V^\perp$ is the orthogonal complement of V in H. The proof of this a nice exercise using the sharp form of Schwarz' inequality.

The extremality result that will be used is the following.

Theorem 4.2. *Let C be a closed convex subset of a real Hilbert space H and $\mathscr{F} : H \to \mathbb{R}$ be a G-differentiable functional on H. If $\hat{u}$ maximizes $\mathscr{F}$ on C, then $\hat{u}$ satisfies*

$$D\mathscr{F}(u) \in \partial I_C(u) \tag{4.5}$$

When C is a closed ball, centered at the origin, in a closed subspace V of H, and $\hat{u}$ maximizes $\mathscr{F}$ on C, then $\hat{u}$ satisfies

$$[D\mathscr{F}(u), h] = [\lambda u + w, h] \quad \text{for some } \lambda \geq 0,\ w \in V^\perp \text{ and all } h \in H. \tag{4.6}$$

This result is Theorem 2.1. In Auchmuty (2004) and the proof is straightforward. For the problem $(\mathscr{S}_{J+1})$ take K_J for C, $\mathscr{B}$ for $\mathscr{F}$ and $H^1(\Omega)$ for H. Then the extremality condition satisfied at a maximizer of $\mathscr{B}$ on K_J is that u_{J+1} satisfies

$$\langle u, v \rangle_\rho = [\lambda u + w, v]_c \quad \text{for all } v \in H^1(\Omega) \tag{4.7}$$

where $\lambda \geq 0$ and w is in the subspace spanned by $\{u_1, u_2, \ldots, u_J\}$. Substitute u_j for v here then, since u_{J+1} is in K_J, one finds that

$$[w, u_j]_c = 0 \quad \text{for each } 1 \leq j \leq J$$

so $w = 0$. If $\lambda = 0$, then $\mathscr{B}(u_{J+1}) = 0$, so u_{J+1} is not a maximizer. Hence $\lambda > 0$, or (3.4) holds with $\mu = \lambda^{-1}$.

This process may be iterated to produce a countable increasing sequence $\{\mu_j : j \geq 1\}$ of Steklov eigenvalues for (L, ρ). These eigenvalues have the following property.

Theorem 4.3. *Assume that* Ω, $\partial\Omega$, c, ρ *satisfy* (B2), (A1) *and* (A2). *Each eigenvalue* μ_j *of* (L, ρ) *has finite multiplicity and* $\mu_j \to \infty$ *as* $j \to \infty$.

Proof. Suppose the sequence is bounded above by a finite $\hat{\mu}$. The corresponding sequence of eigenfunctions is an c-orthonormal set in $H^1(\Omega)$. Hence it converges weakly to zero. The traces $\{\Gamma u_j : j \geq 1\}$ of these functions will converge strongly to 0 in $L^2(\partial\Omega, d\sigma)$ as Γ is compact. Then (A2) implies that $\mathscr{B}(u_j)$ converges to zero as $j \to \infty$. However (4.1) implies that

$$\mathscr{B}(u_j) \geq \hat{\mu}^{-1} > 0 \quad \text{for all } j \geq 1.$$

This contradiction implies there is no such upper bound $\hat{\mu}$ and the theorem follows. $\square$

5. ORTHOGONAL TRACE SPACES FOR $H^1(\Omega)$

In this section, we shall describe a c-orthogonal decomposition of $H^1(\Omega)$ and show that the Steklov eigenfunctions for (L, ρ) will be a basis of the c-orthogonal complement of $H_0^1(\Omega)$. Throughout this section, Ω will be assumed to satisfy (B2).

Let $C_c^1(\Omega)$ be the set of all real-valued functions on Ω which are C^1 on Ω and have compact support. Let $H_0^1(\Omega)$ be the closure of $C_c^1(\Omega)$ in the H^1-norm.

A function $u \in H^1(\Omega)$ is said to be a H^1-weak solution of

$$Lu(x) := c(x)u(x) - \Delta u(x) = 0 \quad \text{on } \Omega \tag{5.1}$$

whenever

$$[u, \varphi]_c := \int_\Omega [\, cu\varphi + \nabla u \cdot \nabla\varphi \,]\; dx = 0 \tag{5.2}$$

for all $\varphi \in C_c^1(\Omega)$. That is, u is H^1-weak solution of (5.1) if and only if u is c-orthogonal to $C_c^1(\Omega)$. Define W to be the subspace of $H^1(\Omega)$ which is c-orthogonal to $H_0^1(\Omega)$, then the following lemma follows from the definition of $H_0^1(\Omega)$

Lemma 5.1. *Assume* Ω, $\partial\Omega$, c *satisfy* (B2) *and* (A1) *and* W *as above. A function* $u \in H^1(\Omega)$ *is a* H^1*-weak solution of* (5.1) *if and only if* $u \in W$.

The subspace $H_0^1(\Omega)$ may be characterized as the null space of the trace operator Γ defined in Sec. 2. When the following condition holds, this may be expressed in terms of $\mathscr{B}$.

(A3). ρ satisfies (A2) and is strictly positive σ a.e. on $\partial\Omega$.

Proposition 5.2. *Assume* Ω, $\partial\Omega$, c *satisfy* (B2) *and* (A1) *and* ρ *satisfies* (A3). *Then* $u \in H^1(\Omega)$ *and* $\mathscr{B}(u) = 0$ *if and only if* $u \in H_0^1(\Omega)$.

Proof. When $u \in H^1(\Omega)$ and $\mathscr{B}(u) = 0$ then $\Gamma u = 0$ in $L^2(\partial\Omega, \rho d\sigma)$ and thus it is $0\,\sigma\,a.e.$ on Ω as (A3) holds. From Corollary 1.5.1.6 of Grisvard (1985), this implies that $u \in H_0^1(\Omega)$.

Conversely when $u \in H_0^1(\Omega)$, there is a sequence $\{u_m : m \geq 1\} \subset C_c^1(\Omega)$ such that $u_m \to u$ in the c-norm. Since $\mathscr{B}$ is continuous and $\mathscr{B}(u_m) = 0$ for all m, then $\mathscr{B}(u) = 0$. □

These results may be written as

$$H^1(\Omega) = H_0^1(\Omega) \oplus_c W \quad \text{or} \quad H^1(\Omega) = \ker \Gamma \oplus_c \ker L.$$

Here $\oplus_c$ indicates a c-orthogonal direct sum. In many treatments of elliptic boundary value problems the closed subspace W is identified with the fractional Hilbert space $H^{1/2}(\partial\Omega)$. Here we shall characterize it in terms of the coefficients in expansions involving normalized Steklov eigenfunctions.

Theorem 5.3. *Assume* Ω, $\partial\Omega$, c *satisfy* (B2) *and* (A1), ρ *satisfies* (A3). *The sequence* $\{u_j : j \geq 1\}$ *of Steklov eigenfunctions for* (L, ρ) *is a maximal c-orthonormal subset of W.*

Proof. Each u_j is in W as the choice $v \in C_c^1(\Omega)$ in (3.5) yields that (5.2) holds. They are c-orthonormal from Theorem 4.1. If the sequence defined in Sec. 4 is not maximal then there is a $w \in W$ with $\|w\|_c = 1$ and $[w, u_j]_c = 0$ for all $j \geq 1$.

If $\mathscr{B}(w) > 0$, then there will be a J such that $\mathscr{B}(w) > \beta_{J+1}$ from Theorem 4.3. This contradicts the definition of u_{J+1} as w will be in K_J. If $\mathscr{B}(w) = 0$, then Proposition 5.2 implies $w = 0$, which contradicts the definition of w. Hence the theorem follows. □

This result may be interpreted as saying that W is the closed subspace of $H^1(\Omega)$ with the Schroedinger Steklov eigenfunctions $\{u_j : j \geq 1\}$ as a c-orthonormal basis. Then Parseval's theorem for orthogonal expansions in a real Hilbert space yields that each function u in W has a unique representation of the form

$$u = \sum_{j=1}^{\infty} c_j u_j \text{ with } c_j := [u, u_j]_c \quad \text{and} \quad \|u\|_c^2 = \sum_{j=1}^{\infty} |c_j|^2. \tag{5.3}$$

The trace of such a function on $\partial\Omega$ is given by

$$\Gamma u = \sum_{j=1}^{\infty} c_j \, \Gamma u_j \quad \text{with } \|\Gamma u\|_\rho^2 = \sum_{j=1}^{\infty} \mu_j^{-1} \, |c_j|^2. \tag{5.4}$$

This follows from the formulae in Theorem 4.1 for $\|\Gamma u\|_\rho$. In particular, the space W is precisely the space of all functions on Ω with expansions of the form (5.3) and for which the last sum in (5.3) is finite. The trace of such functions on $\partial\Omega$ will be the set of all functions of the form (5.4) for which the last sum in (5.3) is finite. Such traces will be in the weighted space $L^2(\partial\Omega, \rho d\sigma)$ and the trace operator $\Gamma : H^1(\Omega) \to L^2(\partial\Omega, \rho d\sigma)$ will be a compact linear map with operator norm $\|\Gamma\| = \mu_1^{-1/2}$.

Let $w_J := \sum_{j=1}^{J} c_j u_j$ be the Jth partial sum of the Steklov expansion (5.3) and $\Gamma_J : H^1(\Omega) \to L^2(\partial\Omega, \rho d\sigma)$ be the corresponding partial trace defined by

$$\Gamma_J u := \Gamma w_J. \tag{5.5}$$

Then

$$\|(\Gamma - \Gamma_J)u\|_\rho = \mu_{J+1}^{-1/2}\|u\|_c \tag{5.6}$$

so these partial Steklov expansions provide very good approximations for the trace of an H^1-function in $L^2(\partial\Omega, \rho d\sigma)$.

6. THE A-HARMONIC STEKLOV EIGENPROBLEM

The A-harmonic Steklov eigenproblem is that of finding non-trivial solutions of the system

$$\nabla(A(x)\,\nabla s) = 0 \quad \text{on } \Omega \quad \text{subject to} \tag{6.1}$$

$$(A(x)\,\nabla s\,) \cdot \nu = \delta\rho u \quad \text{on } \partial\Omega. \tag{6.2}$$

The $n \times n$ matrix valued function $A(x) := (a_{jk}(x))$ will be assumed to satisfy the following conditions:

(A4). $A(x)$ is a real symmetric matrix whose entries are continuous on $\overline{\Omega}$ and there exist constants $a_1 \geq a_0 > 0$ such that

$$a_0|\xi|^2 \leq (A(x)\xi)\cdot\xi \;\leq\; a_1|\xi|^2 \quad \text{for all } x \in \overline{\Omega},\ \xi \in \mathbb{R}^n. \tag{6.3}$$

The weak form of (6.1)–(6.2) is to find non-trivial (δ, s) in $\mathbb{R} \times H^1(\Omega)$ satisfying

$$\int_\Omega (A(x)\nabla s(x)) \cdot \nabla v(x)\, dx - \delta \int_{\partial\Omega} \rho(x)s(x)v(x)\, d\sigma = 0 \tag{6.4}$$

for all $v \in H^1(\Omega)$. This will be called the A-harmonic Steklov eigenproblem with weight ρ on $\partial\Omega$. When $A(x) \equiv I_n$, Eq. (6.1) is Laplace's equation and then (6.4) will be called the harmonic Steklov eigenproblem.

The harmonic version of this problem has been studied for a long time, especially as it has been arises as a model for the sloshing of a perfect fluid in a tank. See Fox and Kuttler (1983) or McIver (1989) for treatments of this problem.

Whenever Ω obeys (B1), then $\delta_0 = 0$ is a simple eigenvalue of (6.4) with the associated eigenfunction $s_0(x) \equiv 1$. Upon substituting $v = s$ in (6.2), one sees that, provided (A2) and (A4) hold, the A-harmonic Steklov eigenvalues are positive.

In this section, a variational problem for the first strictly positive eigenvalue of (6.4) will be described and the associated trace inequality derived.

Consider the bilinear and quadratic forms on $H^1(\Omega)$ defined respectively by

$$\mathscr{A}(u, v) := \int_\Omega (A(x)\nabla u(x)) \cdot \nabla v(x)\, dx + \int_{\partial\Omega} \rho(x)u(x)v(x)d\sigma, \tag{6.5}$$

$$\mathscr{A}_0(u) := \int_\Omega (A(x)\nabla u(x)) \cdot \nabla u(x)\, dx. \tag{6.6}$$

When ρ, A satisfy (A2) and (A4), $\mathscr{A}$ is an inner product on $H^1(\Omega)$ and the associated norm is denoted $\|u\|_A$. The following provides some basic results about these functionals and the inequality will enable us to show that the A-norm and the standard norm on $H^1(\Omega)$ are equivalent.

Theorem 6.1. *Assume that* Ω, $\partial\Omega$, ρ, A *satisfy* (B2), (A2) *and* (A4). *Then* $\mathscr{A}_0$ *is convex, continuous and G-differentiable on* $H^1(\Omega)$ *with*

$$\mathscr{A}_0'(u)(v) = 2\int_\Omega (A\nabla u) \cdot \nabla v\; dx, \quad \textit{for all } u, v \in H^1(\Omega). \tag{6.7}$$

There is a constant $\alpha_0 > 0$ *such that*

$$\|u\|_A^2 \; := \int_\Omega (A\nabla u) \cdot \nabla u\, dx + \int_{\partial\Omega} \rho u^2\, d\sigma \geq \; \alpha_0 \int_\Omega u^2\, dx \quad \textit{for all } u \in H^1(\Omega). \tag{6.8}$$

Proof. The assumptions (A4) are sufficient to prove the first sentence using straightforward arguments.

To prove the inequality consider the variational problem of minimizing $\|u\|_A^2$ on the subset S of $H^1(\Omega)$ of functions satisfying $\|u\|_2 = 1$. Define

$$\alpha_0 := \inf_{u \in S} \|u\|_A^2.$$

The theorem will hold provided we can show that $\alpha_0 > 0$.

Let $\{u_m : m \geq 1\}$ be a minimizing sequence for this variational problem. Such a sequence is bounded in $H^1(\Omega)$ since (6.3) holds. Thus it has a weakly convergent subsequence with a weak limit $\hat{u}$. $\hat{u}$ is in S from Rellich's theorem and the functional is weakly l.s.c. on S, so $\|\hat{u}\|_A^2 = \alpha_0$. Suppose $\alpha_0 = 0$ then $\hat{u}$ is constant on Ω and $\mathscr{B}(\hat{u}) = 0$. This implies that $\hat{u} \equiv 0$ so it will not be in S. This contradiction implies that $\alpha_0 > 0$ and the inequality (6.8) follows by homogeniety. $\square$

It is worth noting that, the extremal conditions for the variational problem described in the proof, imply that the constant α_0 above is the least eigenvalue of the problem

$$-\nabla(A\nabla u) = \alpha u \quad \text{on } \Omega \text{ and subject to } (A\nabla u)\cdot\nu + \rho u = 0 \quad \text{on } \partial\Omega. \tag{6.9}$$

The weak form of this is to find non-trivial (α, u) in $\mathbb{R} \times H^1(\Omega)$ satisfying

$$\int_\Omega [(A\nabla u)\cdot\nabla v - \alpha uv]\,dx + \int_{\partial\Omega} \rho uv\,d\sigma = 0 \quad \text{for all } v \in H^1(\Omega). \tag{6.10}$$

This shows that the least eigenvalue of (6.9) is the optimal choice of α_0 in (6.8) and that equality holds here for corresponding eigenfunctions of (6.9).

Corollary 6.2. *Assume* (A2), (A4) *and* (B2) *hold, then* $\|\cdot\|_A$ *and the standard norm on* $H^1(\Omega)$ *are equivalent.*

Proof. From (A2) and Holder's inequality,

$$|\mathscr{B}(u)| \le \|\rho\|_\infty \, \|\Gamma u\|^2_{2,\partial\Omega} \le C\|\rho\|_\infty \, \|u\|^2_{1,2}$$

as Γ is continuous. Substitute in the definition of the norm then

$$\|u\|_A^2 \le (a_1 + C\,\|\rho\|_\infty)\,\|u\|^2_{1,2}. \tag{6.11}$$

Conversely, use (A4) and the inequality (6.8) of Theorem 6.1 to obtain

$$\|u\|^2_{1,2} \le ({a_0}^{-1} + {\alpha_0}^{-1})\|u\|_A^2$$

These two inequalities show that the norms are equivalent on $H^1(\Omega)$. □

With this result, a variational principle for the least non-zero eigenvalue of the harmonic Steklov eigenproblem may be described.

Define B_A to be the unit ball of $H^1(\Omega)$ in the A-norm. It is the subset of functions in $H^1(\Omega)$ satisfying

$$\int_\Omega (A\nabla u)\cdot\nabla u\,dx + \int_{\partial\Omega} \rho u^2\,d\sigma \;\le\; 1. \tag{6.12}$$

Let B_{1A} be the subset of B_A of functions which also satisfy

$$[u, s_0]_A = \int_{\partial\Omega} \rho\Gamma u\,d\sigma = 0. \tag{6.13}$$

Here $s_0(x) \equiv 1$ on Ω so B_{1A} will also be a bounded closed convex subset of $H^1(\Omega)$.

Consider the variational principle $(\mathscr{S}\mathscr{H}_1)$ of maximizing $\mathscr{B}$ on B_{1A} and define

$$\gamma_1(\rho, A) := \sup_{u \in B_{1A}} \mathscr{B}(u). \tag{6.14}$$

The next theorem shows that the maximizer s_1 of this problem is an eigenfunction of the harmonic Steklov problem (6.14) corresponding to the least non-zero eigenvalue δ_1 and that γ_1 and δ_1 are related in a simple way.

Theorem 6.3. *Assume that* Ω, $\partial\Omega$, ρ, A *satisfy* (B2), (A2) *and* (A4). *Then* δ_1 *is finite and there are maximizers* $\pm s_1$ *of* $\mathscr{B}$ *on* B_{1A}. *These maximizers satisfy* $\|s_1\|_A = 1$ *and* (6.4). *The corresponding eigenvalue* δ_1 *is the least non-zero eigenvalue of* (6.4) *and* $\gamma_1 = (1 + \delta_1)^{-1}$.

Proof. The existence argument is the same as that of Theorem 3.4 with B_{1A} in place of K and the A-norm in place of the c-norm. The equations satisfied at the maximizers can be found from Theorem 4.2.

Let V_1 be the subspace of $H^1(\Omega)$ of all functions that satisfy (6.13) and use $\mathscr{B}$ in place of $\mathscr{F}$. Then (4.6) says that s_1 satisfies

$$\int_{\partial\Omega} \rho s v \, d\sigma = \mathscr{A}(\lambda s + w, v) \quad \text{for all } v \in H^1(\Omega) \tag{6.15}$$

where $\lambda \geq 0$ and w is a multiple of $s_0(x)$. In terms of integrals, this is

$$(1 - \lambda) \int_{\partial\Omega} \rho s v \, d\sigma - \lambda \int_{\Omega} (A\nabla s) \cdot \nabla v \; dx = \mu \int_{\partial\Omega} \rho v \, d\sigma \tag{6.16}$$

for all $v \in H^1(\Omega)$, some μ in $\mathbb{R}$ and some $\lambda \geq 0$. Put $s = s_1, v \equiv 1$ here, then $\mu = 0$. Put $s = v = s_1$ here, then $\mathscr{B}(s_1) = \lambda$, so $\lambda = \gamma_1 > 0$. Thus the maximizers satisfy (6.4) with $\delta = (1 - \gamma_1)/\gamma_1$. This proves that s_1 is an eigenfunction of the harmonic Steklov problem with δ_1 as stated in the theorem. If δ_1 is not the minimal non-zero eigenvalue, one can show that γ_1 is not the supremum of this problem. $\square$

This result yields a different trace inequality for H^1-functions. Let $H^1_\partial(\Omega)$ be the subspace of functions in $H^1(\Omega)$ which satisfy (6.13). Given $u \in H^1(\Omega)$, define

$$\bar{u}_\partial := \int_{\partial\Omega} \rho \Gamma u \, d\sigma \quad \text{and} \quad Mu(x) := u(x) - \bar{u}_\partial. \tag{6.17}$$

Then $Mu \in H^1_\partial(\Omega)$.

Corollary 6.4. *Assume* Ω, $\partial\Omega$, ρ, A *satisfy* (B2), (A2) *and* (A4) *and* δ_1 *as above. Then, for all* $u \in H^1_\partial(\Omega)$,

$$\int_{\Omega} (A\nabla u) \cdot \nabla u \; dx \geq \delta_1 \int_{\partial\Omega} \rho \, |\Gamma u|^2 \, d\sigma. \tag{6.18}$$

Proof. This follows from (6.14) by homogeniety of the functional and the constraint and uses the expression in Theorem 6.3 for γ_1. $\square$

Note that this inequality holds for all $u \in H^1(\Omega)$ with Γu on the right hand side replaced by $\Gamma M u$.

When $A(x) \equiv I_n$, this inequality has been studied by a number of authors including Kuttler and Sigillito (1968), Payne (1970) and Wheeler and Horgan (1976). Their interest centered on finding lower bounds for δ_1 in terms of geometrical quantities of Ω and $\partial\Omega$.

7. THE SUBSPACE OF *A*-HARMONIC FUNCTIONS

In this section, results analogous to those of Secs. 4 and 5 will be described for the A-harmonic Steklov eigenproblem and an orthonormal basis of the subspace of A-harmonic functions on Ω will be described.

Successive A-harmonic Steklov eigenvalues and eigenfunctions may be found using a variational characterization similar to that for the Schroedinger type operators in Sec. 4. Assume we know the first J non-zero A-harmonic Steklov eigenvalues $0 = \delta_0 < \delta_1 \leq \cdots \leq \delta_J$ and a corresponding family $\{s_0, s_1, \ldots, s_J\}$ of A-orthonormal eigenfunctions of (6.4). From (6.4), they satisfy

$$\langle \Gamma s_j, \Gamma s_k \rangle_\rho = (1+\delta_j)^{-1}\, \delta_{jk} \quad \text{for } 1 \leq j,\ k \leq J. \tag{7.1}$$

To find δ_{J+1}, define

$$B_{JA} := \{\, u \in B_A : \langle \Gamma u, \Gamma s_j \rangle_\rho = 0 \text{ for } 0 \leq j \leq J\}. \tag{7.2}$$

Consider the variational problem $(\mathscr{SH}_{J+1})$ of maximizing $\mathscr{B}$ on B_{JA} and define

$$\gamma_{J+1} := \sup_{u \in B_{JA}} \mathscr{B}(u). \tag{7.3}$$

The following theorem describes the essential properties of this variational problem.

Theorem 7.1. *Assume that* Ω, $\partial\Omega$, ρ, A *satisfy* (B2), (A2) *and* (A4). *Then* B_{JA} *is a bounded closed convex set in* $H^1(\Omega)$, γ_{J+1} *is finite and there are maximizers* $\pm s_{J+1}$ *of* $\mathscr{B}$ *on* B_{JA}. *These maximizers satisfy* $\|s_{J+1}\|_A = 1$, (6.4) *with* $\gamma_{J+1} := (1+\delta_{J+1})^{-1}$ *and*

$$\mathscr{A}(s_{J+1}, s_j) = \langle \Gamma s_{J+1}, \Gamma s_j \rangle_\rho = 0 \quad \text{for } 0 \leq j \leq J. \tag{7.4}$$

Moreover δ_{J+1} *is the smallest eigenvalue of this problem greater than or equal to* δ_J.

Proof. The proof of existence is similar to that of Theorem 4.1. The fact that the maximizers are solutions of (6.4) with $\delta_{J+1} := \gamma_{J+1}^{-1} - 1$ follows in a similar manner to the proof of Theorem 6.3 with a subspace V_J in place of V_1. The minimality of δ_{J+1} is a consequence of the maximality of γ_{J+1}. $\square$

This process may be iterated to produce a countable increasing sequence $\{\delta_j : j \geq 1\}$ of harmonic Steklov eigenvalues. These eigenvalues have the following property – whose proof is similar to that of Theorem 4.3.

Theorem 7.2. *Assume that* Ω, $\partial\Omega$, ρ, A *satisfy* (B2), (A2) *and* (A4). *Each A-harmonic Steklov eigenvalue* δ_j *has finite multiplicity and* $\delta_j \to \infty$ *as* $j \to \infty$.

A function $u \in H^1(\Omega)$ is said to be a A-harmonic on Ω provided

$$\int_\Omega (A\nabla u) \cdot \nabla\varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^1(\Omega). \tag{7.5}$$

This is a distributional version of Eq. (6.1).

Define $\mathscr{H}_A(\Omega)$ to be the subspace of $H^1(\Omega)$ which is A-orthogonal to $H_0^1(\Omega)$ then, just as in Sec. 5, the density of $C_c^1(\Omega)$ in $H_0^1(\Omega)$ implies that there is an decomposition

$$H^1(\Omega) = H_0^1(\Omega) \oplus_A \mathscr{H}_A(\Omega).$$

with the subspaces here being A-orthogonal.

The following result shows that the family of A-orthonormal harmonic Steklov eigenfunctions obtained above is a basis of the space $\mathscr{H}_A(\Omega)$ of all A-harmonic functions on Ω. It is proved using exactly the same argument as in the proof of Theorem 5.3.

Theorem 7.3. *Assume* Ω, $\partial\Omega$, ρ, A *satisfy* (B2), (A3) *and* (A4). *Then the sequence of A-harmonic Steklov eigenfunctions* $\{s_j : j \geq 0\}$ *is a maximal A-orthonormal subset of* $\mathscr{H}_A(\Omega)$.

8. EXAMPLES OF HARMONIC STEKLOV SPECTRA

It is of interest to describe the harmonic eigenvalues and eigenfunctions for some standard regions in $\mathbb{R}^n$. Suppose that the matrix $A(x) \equiv I_n$ and that $\rho(x) \equiv \rho_1$ on $\partial\Omega$ where the constant ρ_1 is normalized so that (3.3) holds.

In the case $n = 2$ and Ω is the unit disc, then $\rho_1 = 1/2\pi$ and the harmonic Steklov eigenfunctions are given by s_0 as before and, in polar coordinates $x = (r, \theta)$,

$$s_{2k-1}(x) := r^k \sin k\theta, \quad s_{2k}(x) := r^k \cos k\theta, \quad \text{for } k \geq 1, \tag{8.1}$$

$$\delta_{2k-1} = \delta_{2k} = k \quad \text{when } k \geq 1. \tag{8.2}$$

Similarly when $n = 3$ and Ω is the unit sphere, then $\rho_1 = 1/4\pi$ and the harmonic Steklov eigenfunctions will be $s_0 \equiv 1$ and, in spherical polar coordinates $x = (r, \theta, \phi)$, with θ being the azimuthal angle,

$$s_{kl}(x) := r^k Y_{kl}(\theta, \phi) \quad \text{for } k \geq 1, \; -k \leq l \leq k. \tag{8.3}$$

Here $Y_{kl}(\theta, \phi)$ is the $(k, l,)$th spherical harmonic given by

$$Y_{k0}(\theta, \phi) := P_k(\cos \phi), \quad \text{when } l = 0, \tag{8.4}$$

$$Y_{kl}(\theta, \phi) := P_k{}^l(\cos \phi) \cos l\theta, \quad \text{when } 1 \leq l \leq k, \tag{8.5}$$

$$Y_{kl}(\theta, \phi) := P_k{}^l(\cos \phi) \sin l\theta, \quad \text{when } -k \leq l \leq -1. \tag{8.6}$$

The Steklov eigenvalues will again be $\{k : k \geq 0\}$ and the eigenvalue k has multiplicity $(2k + 1)$. For a general theory of these issues, see Groemer (1996).

9. STEKLOV SERIES REPRESENTATIONS OF *A*-HARMONIC FUNCTIONS

In this section, the preceding results will be used to describe Steklov spectral representations of the solutions of Eq. (6.1) subject to various boundary conditions.

First consider the Dirichlet problem for this equation and assume the region Ω satisfies (B2). That is, consider the problem of finding a solution $\hat{u}$ of (7.5) which is in $H^1(\Omega)$ and such that $\Gamma u = g \in L^2(\partial\Omega, d\sigma)$. Any such solution will be in $\mathscr{H}_A(\Omega)$. From Theorem 7.3, the fact that $\{s_j : j \geq 0\}$ is an A-orthonormal basis of $\mathscr{H}_A(\Omega)$ implies that

$$\hat{u}(x) = \sum_{j=0}^{\infty} c_j\, s_j(x) \quad \text{with } c_j := \mathscr{A}(\hat{u}, s_j). \tag{9.1}$$

Since $\Gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ is compact, the trace of $\hat{u}$ on $\partial\Omega$ will be

$$\Gamma\hat{u}(x) = \sum_{j=0}^{\infty} c_j\, \Gamma s_j(x) \tag{9.2}$$

Multiply this by $\rho\, \Gamma s_k$ and integrate over $\partial\Omega$, then the Dirichlet boundary data yields

$$c_k = (1 + \delta_k)\ \langle g, \Gamma s_k \rangle_\rho \quad \text{for } k \geq 0. \tag{9.3}$$

That is, the solution of this Dirichlet problem is given by the series in (9.1) with the coefficients defined by (9.3).

Parseval's theorem then yields that

$$\|u\|_A^2 = \sum_{k=0}^{\infty} c_k^2 \ = \sum_{k=0}^{\infty} (1 + \delta_k)^2\ |\langle g, \Gamma s_k \rangle_\rho|^2. \tag{9.4}$$

This shows that this Dirichlet problem has a H^1-solution if and only if g satisfies

$$\sum_{k=0}^{\infty} (1 + \delta_k)^2\ |\langle g, \Gamma s_k \rangle_\rho|^2 \ < \ \infty \tag{9.5}$$

This is a spectral form of the usual criterion that $g \in H^{1/2}(\partial\Omega)$ and the above results may be summarized as follows.

Theorem 9.1. *Assume* Ω, $\partial\Omega$, ρ, A *satisfy* (B2), (A3) *and* (A4), $\{\delta_j : j \geq 0\}$ *is the set of A-harmonic Steklov eigenvalues for* Ω *and* $\{s_j : j \geq 0\}$ *is a corresponding sequence of orthonormal A-harmonic Steklov eigenfunctions. Then there is a solution* $\hat{u}$ *in* $H^1(\Omega)$ *of the Dirichlet problem for* (6.1) *if and only if g satisfies* (9.5). *In this case, the solution can be represented in the form* (9.1)–(9.3) *and the series converges strongly in the* H^1*-norm.*

Let $\hat{u}_M$ be the Mth partial sum of the series in (9.1) then, from (9.3), one has

$$\hat{u}_M(x) = \int_{\partial\Omega} P_M(x, y) g(y) \rho(y) d\sigma(y) \tag{9.6}$$

with

$$P_M(x, y) := \sum_{k=0}^{M} (1 + \delta_k) s_k(x) \Gamma s_k(y). \tag{9.7}$$

This provides a finite rank approximation to the solution of the problem in terms of an integral operator. These partial sums converge strongly to $\hat{u}$ when g satisfies (9.5).

When $A(x) \equiv I_n$ on Ω and $\rho(x)$ is constant on $\partial\Omega$, this result may be interpreted as a representation of the Poisson kernel for the Laplacian on the region Ω. This Poisson kernel may be regarded as the integral kernel associated with the limit as $M \to \infty$ in (9.6)–(9.7).

This methodology may be used to obtain similar representations of H^1-solutions of Eq. (6.1) for general Robin, or Neumann, boundary data. Suppose now that the boundary condition is

$$(1 - \tau)(A\nabla u) \cdot \nu(x) + \tau\rho(x) u(x) = g(x) \quad \text{on } \partial\Omega;\ 0 \leq \tau < 1. \tag{9.8}$$

A function $\hat{u}$ in $H^1(\Omega)$ is defined to be an H^1-solution of Eq. (6.1) subject to (9.8) provided

$$\int_\Omega (A\nabla u) \cdot \nabla v \, dx + (1 - \tau)^{-1} \int_{\partial\Omega} (\tau\rho u - g) v \, d\sigma = 0 \quad \text{for all } v \in H^1(\Omega). \tag{9.9}$$

These weak solutions may be described using a variational principle. Consider the functional $\mathscr{D} : H^1(\Omega) \times [0, 1) \to \mathbb{R}$ defined by

$$\mathscr{D}(u, \tau) := \int_\Omega (A\nabla u) \cdot \nabla u \, dx + (1 - \tau)^{-1} \int_{\partial\Omega} (\tau\rho u - 2g) u \, d\sigma. \tag{9.10}$$

The variational problem is to minimize $\mathscr{D}(\cdot, \tau)$ on $H^1(\Omega)$ and to find

$$\beta(\tau) := \inf_{u \in H^1(\Omega)} \mathscr{D}(u, \tau). \tag{9.11}$$

This is a standard variational problem and the essential results for the Robin problem ($0 < \tau < 1$) may be summarized as follows.

Theorem 9.2. *Assume* $\Omega,\ \partial\Omega,\ \rho,\ A$ *satisfy* (B2), (A2) *and* (A4), g *is in* $L^2(\partial\Omega, d\sigma)$ *and* $0 < \tau < 1$. *Then there is a unique minimizer* $\hat{u}$ *of* $\mathscr{D}(\cdot,\tau)$ *on* $H^1(\Omega)$ *and it is the unique* H^1*-solution of* (9.9). *Moreover there is a positive* $C(\tau,\Omega)$ *such that*

$$\|\hat{u}\|_{1,2} \le C(\tau,\Omega)\|g\|_{2,\partial\Omega}. \tag{9.12}$$

Proof. The functional $\mathscr{D}(\cdot,\tau)$ is convex and continuous on $H^1(\Omega)$, so it is weakly l.s.c. From (A4) and Theorem 6.1, there is a constant $\alpha_1(\tau) > 0$ such that

$$\begin{aligned}\mathscr{D}(u,\tau) &\ge \frac{a_0}{2}\|\nabla u\|_2^2 + \alpha_1(\tau)\|u\|_2^2 - \ 2(1-\tau)^{-1}\|g\|_{2,\partial\Omega}\|u\|_{2,\partial\Omega}\\ &\ge \alpha_2\|u\|_{1,2}^2 - \ C_1(\tau)\|g\|_{2,\partial\Omega}\|u\|_{1,2}\end{aligned}$$

upon using the definition of the $H^1(\Omega)$-norm and the trace theorem for u. This implies that $\mathscr{D}(\cdot,\tau)$ is coercive and strictly convex on $H^1(\Omega)$, so it attains its infimum on $H^1(\Omega)$ and this minimizer is unique. From the definition, $\beta(\tau) \le 0$, so the last inequality implies that (9.12) holds with $C(\tau,\Omega) \le C_1(\tau)/\alpha_2$. $\square$

This solution will have a representation of the form (9.1) as (9.9) implies that $\hat{u}$ is in $\mathscr{H}_A(\Omega)$. Put $v = s_k$ in (9.9) and use the properties of the eigenfunctions to deduce that

$$c_k = \frac{(1+\delta_k)\,\langle g, \Gamma s_k\rangle_\partial}{(1-\tau)\delta_k + \tau} \quad \text{for } k \ge 0. \tag{9.13}$$

Thus the unique solution described in Theorem 9.2, has the representation

$$\hat{u}(x) = \sum_{k=0}^{\infty} \frac{(1+\delta_k)\,\langle g, \Gamma s_k\rangle_\partial}{(1-\tau)\delta_k + \tau}\ s_k(x) \tag{9.14}$$

when $g \in L^2(\partial\Omega, d\sigma)$ and $0 < \tau < 1$. The partial sums of this series converge strongly to $\hat{u}$ in $H^1(\Omega)$ as the $\{s_k : k \ge 0\}$ constitute an orthonormal basis of $\mathscr{H}_A(\Omega)$ from Theorem 7.3. Again these partial sums may be written in terms of a boundary integral operator which is a sum involving the Steklov eigenvalues and eigenfunctions. Namely

$$\hat{u}_M(x) = \mathscr{R}_M(\tau)\,g(x) := \int_{\partial\Omega} R_M(x,y;\tau)g(y)\rho(y)d\sigma(y) \tag{9.15}$$

with

$$R_M(x,y;\tau) := \sum_{k=0}^{M} \frac{(1+\delta_k)}{(1-\tau)\delta_k + \tau}\ s_k(x)\,\Gamma s_k(y). \tag{9.16}$$

The estimate (9.12) shows that the solution operator $\mathscr{R}(\tau)$ will be a bounded linear map of $L^2(\partial\Omega, d\sigma)$ into $H^1(\Omega)$ and the integral operators $\mathscr{R}_M(\tau)$ defined above converge strongly to $\mathscr{R}(\tau)$ as $M \to \infty$.

The Neumann problem corresponds to taking $\tau = 0$ in (9.8)–(9.10). In this case, $\beta(0)$ defined by (9.11) need not be finite and (9.9) need not have a solution. Put $v(x) \equiv 1$ on $\overline{\Omega}$ and substitute, then a necessary condition for (9.9) to have a solution is that

$$\int_{\partial\Omega} g\, d\sigma = 0. \tag{9.17}$$

The following result shows that this condition is also sufficent when $g \in L^2(\partial\Omega, d\sigma)$.

Theorem 9.3. *Assume* Ω, $\partial\Omega$, ρ, A *satisfy* (B2), (A2) *and* (A4) *and* g *is in* $L^2(\partial\Omega, d\sigma)$. *Then* $\beta(0)$ *is finite if and only if* (9.17) *holds. In this case, there is a unique minimizer* $\hat{u}$ *of* $\mathscr{D}(\cdot, 0)$ *in* $H^1_\partial(\Omega)$ *and there is a* 1*-parameter family of* H^1*-solutions of* (9.9) *given by* $u := \hat{u} + ks_0(x)$ *where* k *is any constant.*

Proof. From (9.10),

$$\mathscr{D}(u, 0) = \int_\Omega (A\nabla u) \cdot \nabla u\, dx - 2 \int_{\partial\Omega} gu\, d\sigma. \tag{9.18}$$

If (9.17) does not hold take $u(x) \equiv t$. Let $|t| \to \infty$, then one sees that $\beta(0) = -\infty$. Suppose it does hold, and use the decomposition of (6.17). Then $\mathscr{D}(u, 0) = \mathscr{D}(v, 0)$ where $v := Mu \in H^1_\partial(\Omega)$. The functional $\mathscr{D}(\cdot, 0)$ is strictly convex, continuous and coercive on $H^1_\partial(\Omega)$, upon using Theorem 6.1 and Corollary 6.4. Hence a unique minimizer exists on this subspace. The theorem then follows. $\square$

This solution will have a representation of the form (9.1) with $c_0 = 0$. Substituting this in (9.9) with $\tau = 0, v = s_k$ and using the properties of the functions s_k leads to

$$c_k = (1 + \delta_k)\delta_k{}^{-1} \int_{\partial\Omega} g(\Gamma s_k) d\sigma. \tag{9.19}$$

Thus the unique solution $\hat{u} \in H^1_\partial(\Omega)$ of the Neumann problem for (6.1) is given by

$$\hat{u}(x) = \sum_{k=1}^{\infty} (1 + \delta_k)\delta_k{}^{-1} \langle g, \Gamma s_k \rangle_\partial \; s_k(x) \tag{9.20}$$

when $g \in L^2(\partial\Omega, d\sigma)$. The partial sums of this series converge strongly to $\hat{u}$ in $H^1(\Omega)$; they are given by

$$\hat{u}_M(x) = \int_{\partial\Omega} N_M(x, y) g(y) \rho(y) d\sigma(y) \tag{9.21}$$

with

$$N_M(x,y) := \sum_{k=1}^{M} (1+\delta_k)\delta_k^{-1} s_k(x)\, \Gamma s_k(y). \tag{9.22}$$

Thus the solution operator for this problem can be regarded as the strong limit of the family of integral operators defined by (9.21)–(9.22) and (9.20) provides a representation result.

These results may well be compared to those obtained using the theory of single and double layer potentials described, for example, in DiBenedetto (1995, Chap. 3), or Kress (1989, Sec. 6.4).

This result enables us to show that the traces of the A-harmonic Steklov eigenfunctions when ρ is constant on $\partial\Omega$ will be a basis of the space $L^2(\partial\Omega, d\sigma)$. First let $\tilde{\sigma}$ be the probability measure associated with the surface area measure on $\partial\Omega$. That is,

$$\tilde{\sigma}(E) := \sigma(E)/\sigma(\partial\Omega)$$

for all Borel measurable subsets E of $\partial\Omega$. This corresponds to taking the density function $\rho_1(x) \equiv 1/\sigma(\partial\Omega)$ on $\partial\Omega$.

Let $\{\tilde{\delta}_j : j \geq 0\}$ be the set of A-harmonic Steklov eigenvalues for Ω, ρ_1 and $\{\tilde{s}_j : j \geq 0\}$ is a corresponding sequence of orthonormal A-harmonic Steklov eigenfunctions.

Define $z_0(x) \equiv 1$ and

$$z_j(x) := \tilde{\delta}_j^{1/2}\, \Gamma \tilde{s}_j(x) \quad \text{for } x \in \partial\Omega, \;\; j \geq 1. \tag{9.23}$$

Theorem 9.4. *Assume* Ω, $\partial\Omega$, ρ, A *satisfy* (B2), (A3) *and* (A4). *Then the sequence* $\{z_j : j \geq 0\}$ *defined as above is a maximal orthonormal set in* $L^2(\partial\Omega, d\tilde{\sigma})$

Proof. From Theorem 4.1, this family is orthonormal. Suppose it is not maximal and there is a function $g \in L^2(\partial\Omega, d\tilde{\sigma})$ with $g \not\equiv 0$ and $\langle g, z_j\rangle_\partial = 0$ for all $j \geq 0$. Then (9.17) holds, so there will be a unique solution $\hat{u} \in H^1_\partial(\Omega)$ of the Neumann case of (9.9). This solution is given by (9.20), so it will be identically zero. This contradicts the assumption that g is non-zero so the sequence must be maximal. $\square$

This leads to a different characterization of the space $H^{1/2}(\partial\Omega)$ in terms of this orthonormal basis. Suppose $g \in L^2(\partial\Omega, d\tilde{\sigma})$, then g has the representation

$$g(x) = g_0 + \sum_{j=1}^{\infty} g_j\, z_j(x) \quad \text{with } g_j := \langle g, z_j\rangle_\partial. \tag{9.24}$$

This will be called the *Fourier–Steklov expansion* of g on $\partial\Omega$. From Eq. (9.23),

$$g_j = \tilde{\delta}_j^{1/2}\, \langle g, \Gamma \tilde{s}_j\rangle_\partial \quad \text{for } j \geq 1 \tag{9.25}$$

in terms of the Steklov eigenvalues and eigenfunctions of (6.1)–(6.2) and with ρ_1 in place of ρ. This leads to the following criterion for the H^1-solvability of the Dirichlet problem for (7.5).

Corollary 9.5. *Assume* Ω, $\partial\Omega$, ρ, *A satisfy* (B2), (A3) *and* (A4), $g_j, \tilde{\delta}_j$ *are defined as above. The Dirichlet problem for* (7.5) *subject to* $u = g$ *on* $\partial\Omega$ *has a solution in* $H^1(\Omega)$ *if and only if*

$$\sum_{j=1}^{\infty} \tilde{\delta}_j \, g_j^2 < \infty. \tag{9.26}$$

Proof. Substitute (9.24) and (9.25) in (9.2) and (9.5). Then the Dirichlet problem will have an H^1-solution if and only if

$$\sum_{j=1}^{\infty} (1 + \tilde{\delta}_j)^2 \tilde{\delta}_j^{-1} \, g_j^2 < \infty$$

This condition is equivalent to (9.26) as the $\tilde{\delta}_j$ does not remain small as j increases. □

This result may also be regarded as a characterization of $H^{1/2}(\partial\Omega)$ as a subspace of $L^2(\partial\Omega, d\tilde{\sigma})$. This characteization could be used as a definition of the space $H^{1/2}(\partial\Omega)$. This definition has the advantage that we only require weak regularity conditions (Lipschitzness) for the boundary in this construction.

It should be noted that these Robin and Neumann problems will have H^1-solutions when the boundary data $g \in H^{-1/2}(\partial\Omega)$. This space contains $L^q(\partial\Omega, d\sigma)$ for $q_T \leq q < 2$ where $q_T = 2(n-1)/(n-2)$ when $n \geq 3$ and for $1 < q < 2$ when $n = 2$. This is proved using a stronger version of the trace theorem and requires a more careful analysis of the variational principles for the solution. In these cases the Steklov series representations of the solutions (9.14) and (9.20) remain valid.

10. STEKLOV SERIES REPRESENTATIONS OF SOLUTIONS OF SCHROEDINGER'S EQUATION

Here the problem of representing the solutions of the homogeneous Schroedinger equation (5.2) subject to various boundary conditions will be treated.

First consider the case of prescribed Robin ($0 < \tau < 1$), or Neumann ($\tau = 0$) boundary conditions of the form

$$(1 - \tau)\frac{\partial u}{\partial \nu}(x) + \tau\rho(x)u(x) = g(x) \quad \text{on } \partial\Omega;\ 0 \leq \tau < 1. \tag{10.1}$$

Here g is given and will be assumed to be in $L^2(\partial\Omega, d\sigma)$ – though this can be relaxed as described at the end of the preceding section.

The weak form of this problem is to find $\hat{u} \in H^1(\Omega)$ satisfying

$$\int_\Omega [\nabla u \cdot \nabla v + c\, uv]\, dx + (1 - \tau)^{-1} \int_{\partial\Omega} (\tau\rho u - g)\, v \, d\sigma = 0 \quad \text{for all } v \in H^1(\Omega). \tag{10.2}$$

There is a variational principle for this problem. Consider the problem of minimizing the functional $\mathscr{F} : H^1(\Omega) \times [0,1) \to \mathbb{R}$ defined by

$$\mathscr{F}(u,\tau) := \int_\Omega [|\nabla u|^2 + c\,u^2]\,dx + (1-\tau)^{-1} \int_{\partial\Omega} (\tau\rho u - 2g)u\,d\sigma. \tag{10.3}$$

Theorem 10.1. *Assume* $\Omega,\ \partial\Omega,\ c,\ \rho$ *satisfy* (B2), (A1) *and* (A2), g *is in* $L^2(\partial\Omega, d\sigma)$ *and* $0 \le \tau < 1$. *Then there is a unique minimizer* $\hat{u}$ *of* $\mathscr{F}(\cdot,\tau)$ *on* $H^1(\Omega)$, *it is the unique* H^1*-solution of* (10.2) *and is in the subspace* W *defined in Sec.* 5. *Moreover there is a positive* $C(\tau,\Omega)$ *such that*

$$\|\hat{u}\|_{1,2} \le C(\tau,\Omega)\|g\|_{2,\partial\Omega}. \tag{10.4}$$

Proof. The functional $\mathscr{F}(\cdot,\tau)$ is convex and continuous on $H^1(\Omega)$ from Theorem 3.1. It is coercive and strictly convex from Theorems 3.1 and 3.2 and standard inequalities. Hence this problem has a unique minimizer. The formulae for the G-derivatives in Theorem 3.1 imply that $\mathscr{F}(\cdot,\tau)$ is G-differentiable on $H^1(\Omega)$ and the minimizer satisfies (10.2). Choosing v to have compact support implies that $\hat{u}$ is in W. The last inequality is proved as in Theorem 9.2. $\square$

Since this solution is in the subspace W of $H^1(\Omega)$, Theorem 5.3 implies that it has an expansion in Steklov eigenfunctions of the form

$$\hat{u}(x) = \sum_{j=1}^{\infty} c_j\,u_j(x) \quad \text{with } c_j := [\hat{u}, u_j]_c. \tag{10.5}$$

Substitute u_j for v in (10.2), to see that

$$c_j = \frac{\mu_j\,\langle g, \Gamma u_j\rangle_\partial}{(1-\tau)\mu_j + \tau} \quad \text{for } j \ge 1. \tag{10.6}$$

Hence the unique solution $\hat{u}$ of (10.2) has the Steklov series representation

$$\hat{u}(x) = \sum_{j=1}^{\infty} \frac{\mu_j\,\langle g, \Gamma u_j\rangle_\partial}{(1-\tau)\mu_j + \tau}\; u_j(x), \tag{10.7}$$

for $0 \le \tau < 1$. In particular, the solution of the Neumann problem is given by

$$\hat{u}(x) = \sum_{j=1}^{\infty} \langle g, \Gamma u_j\rangle_\partial\; u_j(x). \tag{10.8}$$

Moreover the partial sums of this series converge strongly to $\hat{u}$ in $H^1(\Omega)$ as the $\{u_j\}$ are an orthonormal basis of W. These partial sums are given by

$$\hat{u}_M(x) = \int_{\partial\Omega} G_M(x,y;\tau)g(y)d\sigma(y) \tag{10.9}$$

with

$$G_M(x,y;\tau) := \sum_{j=1}^{M} \frac{\mu_j\, u_j(x)}{(1-\tau)\mu_j + \tau}\, \Gamma u_j(y). \tag{10.10}$$

These problems were treated extensively in Part B of Bergman and Schiffer (1953) using a variety of classical methods and restricted to $n = 2$. The kernel function defined there by Eq. 2.6, p. 281 is the same operator as in (10.8) – but without requiring the functions in their expansion to be Steklov eigenfunctions.

The Dirichlet problem for (5.2) may be regarded as the limit $\tau \to 1^-$ of the above problem with ρ_1 as in the previous Sec. 9 in place of ρ. It need not have an H^1-solution for each $g \in L^2(\partial\Omega, d\sigma)$. If Eq. (5.2) has an H^1-solution of the form (10.5), then the boundary condition $\Gamma u = g$ implies that

$$c_j = \mu_j\, \langle g, \Gamma u_j\rangle_{\rho_1} \quad \text{for } j \geq 1. \tag{10.11}$$

Thus Parseval's theorem yields that

$$\|u\|_c^2 := \int_\Omega [|\nabla u|^2 + cu^2]\, dx = \sum_{j=1}^{\infty} \mu_j^2\, |\langle g, \Gamma u_j\rangle_{\rho_1}|^2. \tag{10.12}$$

Thus the Dirichlet problem has an H^1-solution if and only if this last sum is finite. This is a spectral form of the usual criterion that $g \in H^{1/2}(\Omega)$ and the above results may be summarized as follows.

Theorem 10.2. *Assume* Ω, $\partial\Omega$, c, ρ *satisfy* (B2), (A1) *and* (A3), $\{\mu_j : j \geq 1\}$ *is the set of Steklov eigenvalues for* (L, ρ) *and* $\{u_j : j \geq 1\}$ *is a corresponding sequence of orthonormal Steklov eigenfunctions. Then there is a solution* $\hat{u}$ *in* $H^1(\Omega)$ *of the Dirichlet problem for* (5.2) *if and only if the sum on the right hand side of* (10.12) *is finite. In this case the solution can be represented in the form* (10.5) *with coefficients given by* (10.11) *and the series converges strongly in the* H^1*-norm.*

This result shows when the solution is in $H^1(\Omega)$ it may be approximated by formulae of the form (10.9)–(10.10) with $\tau = 1$, and $\rho_1 d\sigma$ in place of $d\sigma$. It also allows the proof of the completeness of the traces of the Steklov eigenfunctions in $L^2(\partial\Omega, d\sigma)$.

Let $\{\tilde{\mu}_j : j \geq 1\}$ be the set of Steklov eigenvalues for (L, ρ_1) and $\{\tilde{u}_j : j \geq 1\}$ is a corresponding sequence of orthonormal Steklov eigenfunctions. Define

$$z_j(x) := \sqrt{\tilde{\mu}}\, \Gamma\tilde{u}_j(x) \quad \text{for } x \in \partial\Omega, \;\; j \geq 1. \tag{10.13}$$

Theorem 10.3. *Assume* Ω, $\partial\Omega$, ρ_1, c *satisfy* (B2), (A1) *and* (A3). *Then the sequence* $\{z_j : j \geq 1\}$ *defined as above is a maximal orthonormal set in* $L^2(\partial\Omega, d\tilde{\sigma})$

Proof. From Theorem 5.3, this family is orthonormal. Suppose it is not maximal and there is a function $g \in \$L^2(\partial\Omega, d\tilde{\sigma})$ with $g \not\equiv 0$ and $\langle g, z_j\rangle_\partial = 0$ for all $j \geq 1$.

Then there will be a unique solution $\hat{u} \in H^1(\Omega)$ of the Neumann case of (10.2). This solution is given by (10.7), so it is identically zero. This contradicts the assumption that g is non-zero so the sequence must be maximal. □

11. NEUMANN TO DIRICHLET MAPS AND ROBIN TO DIRICHLET MAPS

The Steklov series representations of the solutions of the boundary value problems described in the last two sections permits us to compare the solutions of an equation subject to different boundary conditions. In particular it allows a spectral representation of the Neumann to Dirichlet (NtD) map and its inverse, the Dirichlet to Neumann (DtN), map. For an introduction to this theory, see Sylvester and Uhlmann (1990). Similar constructions may also be studied with Robin boundary data substituted for either the Dirichlet or Neumann data.

First consider the case of the Schroedinger Steklov problem for (L, ρ_1). The solution of the Neumann problem (10.2) with $\tau = 0$ is given by Eq. (10.8), which may be written

$$\hat{u}(x) := Ng(x) := \sum_{j=1}^{\infty} \tilde{\mu}_j^{-1/2} \langle g, z_j \rangle_\partial \, u_j(x), \tag{11.1}$$

where z_j is defined by (10.13). Thus $Nz_k(x) = \tilde{\mu}_k^{-1/2} u_k(x)$ for $x \in \Omega$ and the trace of this function on $\partial\Omega$ is given by

$$\Gamma N z_k(x) = \tilde{\mu}_k^{-1} z_k(x) \quad \text{for } k \geq 1. \tag{11.2}$$

The operator ΓN is the NtD map and this shows that the restrictions to the boundary of the Steklov eigenfunctions for (L, ρ_1) are the eigenfunctions of this map corresponding to the eigenvalues $\tilde{\mu}_k^{-1}$. In particular, this shows that the operator is a compact linear map of $L^2(\partial\Omega, d\tilde{\sigma})$ to itself.

The Dirichlet to Neumann map is the inverse of this map and will be a closed, unbounded linear map of $L^2(\partial\Omega, d\tilde{\sigma})$ to itself.

This also permits the description of a general Robin to Dirichlet (RtD) map. The H^1-solution of a Schroedinger equation subject to the Robin conditions (10.1) is given by Eq. (10.7)

$$\hat{u}(x) = R(\tau)g(x) := \sum_{j=1}^{\infty} \frac{\mu_j^{1/2} \langle g, z_j \rangle_\partial}{(1-\tau)\mu_j + \tau} \, u_j(x). \tag{11.3}$$

The RtD map will be the operator $\Gamma R(\tau)$ and this is a continuous linear map of $L^2(\partial\Omega, d\tilde{\sigma})$ to itself with

$$\Gamma R(\tau) z_k(x) = [(1-\tau)\tilde{\mu}_k + \tau]^{-1} z_k(x) \quad \text{for } k \geq 1. \tag{11.4}$$

This and Theorem 4.3 imply that $\Gamma R(\tau)$ is actually a compact linear map of $L^2(\partial\Omega, d\tilde{\sigma})$ to itself, and provides a simple spectral representation in terms of the Steklov eigenfunctions.

A similar analysis holds for the A-harmonic equation. The H^1-solution of (6.1) subject to the Robin boundary condition (9.8) is given by (9.14) so its trace on $\partial\Omega$ may be written

$$\Gamma\hat{u}(x) = \Gamma R(\tau)g(x) \ := \bar{g}_\partial/\tau + \sum_{k=1}^{\infty} \frac{(1+\delta_k^{-1})\,\langle g, z_k\rangle_\partial}{(1-\tau)\delta_k + \tau}\ z_k(x) \tag{11.5}$$

The z_k here are defined by (9.23). In particular, this shows that the z_k are eigenfunctions of the RtD operator with

$$\Gamma R(\tau) z_k(x) = \frac{1+\delta_k^{-1}}{(1-\tau)\delta_k + \tau}\, z_k(x) \quad \text{for } k \geq 1. \tag{11.6}$$

$$\Gamma R(\tau) z_0(x) = \tau^{-1}\ z_0(x) \tag{11.7}$$

This and Theorem 7.2 shows that $\Gamma R(\tau)$ is a compact linear map of $L^2(\partial\Omega, d\tilde{\sigma})$ to itself with a simple spectral representation in terms of the Steklov eigenfunctions when $0 < \tau < 1$.

The Neumann to Dirichlet case corresponds to the case $\tau = 0$ and then the compatibility condition (9.17) is required. Let $L^2_m(\partial\Omega, d\tilde{\sigma})$ be the codimension 1 subspace of $L^2(\partial\Omega, d\tilde{\sigma})$ of functions on the surface whose surface integral is 0. The NtD operator ΓN will be a compact linear transformation of $L^2_m(\partial\Omega, d\tilde{\sigma})$ to itself with the $z_k, k \geq 1$ defined by (9.23) as eigenfunctions and

$$\Gamma N z_k(x) = \delta_k^{-2}(1+\delta_k)\, z_k(x) \quad \text{for } k \geq 1. \tag{11.8}$$

It may be observed that the results of this section do not require that the boundary $\partial\Omega$ be a (union of) C^1-manifold(s); our requirements are just that (B1) and (B2) hold. Hence these results apply to polygonal regions in 2 dimensions and to polyhedral regions in $\mathbb{R}^3$.

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	City	
	State/Province	
	Postal code	
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	Institution or Company	
	Suite, floor, room no.	
	Street address	
	City	
	State/Province	
	Postal code	
	Country	
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