OPTIMAL COERCIVITY INEQUALITIES IN $W^{1,p}(\Omega)$.

GILES AUCHMUTY

Abstract. This paper describes the characterization of optimal constants for some coercivity inequalities in $W^{1,p}(\Omega)$, $1 < p < \infty$. A general result involving inequalities of p-homogeneous forms on a reflexive Banach space is first proved. The constants are shown to be the least eigenvalues of certain eigenproblems with equality holding for the corresponding eigenfunctions. This result is applied to 3 different classes of coercivity results on $W^{1,p}(\Omega)$. The inequalities include very general versions of the Friedrichs' and Poincaré inequalities. Scaling laws for the inequalities are also given.

1. Introduction

This paper describes the characterization of optimal constants, and corresponding optimal functions, for some inequalities satisfied by functions in the Sobolev spaces $W^{1,p}(\Omega), 1 < p < \infty$. Here Ω is a bounded, connected, open set in $\mathbb{R}^n$ which satisfy assumptions (A1) and (A2) of section 2.

First some general properties of p-homogeneous inequalities on a reflexive Banach space are derived in sections 3 - 5. It is shown that, for certain classes of problems, the optimal constants may be found using either a constrained, or an associated unconstrained, variational principle. The extremality condition for the minimizer of the unconstrained problem is used to show that the optimal constants are the least eigenvalue of a related eigenproblem. The corresponding eigenfunctions will be optimal functions for the inequality.

These results are then applied to three different classes of inequalities on $W^{1,p}(\Omega)$. These include $W^{1,p}-$ versions of Friedrichs'-type inequalities in sections 6 and 7, some different inequalities involving boundary integrals in sections 8 and 9, and some generalized Poincaré type inequalities in section 10. For each of the inequalities, an associated scaling law is described.

These inequalities generalize some well-known, and often-used, results from the theory of Sobolev spaces. In particular the inequalities are used to prove the equivalence of various norms on $W^{1,p}(\Omega)$. Such results already appear in Necas [14] Chapter 7.4, and more recently in Atkinson and Han [3], section 6.3.5. Most of the published proofs are non-constructive and for numerical, and other, purposes it is of considerable interest to know how the constants depend on the geometry and size of the underlying region.

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The results are illustrated in section 11 by describing the constants in some special cases for rectangular boxes in the plane.

Here three different classes of inequalities will be distinguished. They differ in only one the functionals involved, but the associated extremality conditions lead to different types of eigenproblems at optimality. The inequalities have a variety of names in the literature. Currently *Poincaré inequality* is commonly used for inequalities of the form (2.5) below. Here we will follow the older usage (see Hellwig [12], Section 5.3) of calling results similar to (6.1) *Friedrichs' inequality.* Recently there also has been interest in the discrete analogues of these inequalities; see Brenner [6] and the references therein.

In this paper we shall use various standard results from the calculus of variations and convex analysis. Background material on such methods may be found in Blanchard and Bruning [7] or Zeidler [15], both of which have discussions of the variational principles for the Dirichlet eigenvalues and eigenfunctions of second order elliptic operators. The variational principles used here are variants of the principles described there and are analogous to those for the Laplacian described in section 5 of Auchmuty [5]. Some different unconstrained variational principles for eigenvalue problems are described in [4].

2. Definitions and Notation.

Let Ω be a non-empty, bounded, connected, open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$. Such a set Ω is called a *region.* Let $L^p(\Omega)$ be the usual real Lebesgue space of all functions $u : \Omega \to [-\infty, \infty]$ which are p-th power integrable with respect to Lebesgue measure on Ω, $1 < p < \infty$. Let $\sigma, d\sigma$ represent Hausdorff $(n-1)-$dimensional measure and integration with respect to this measure respectively. The space $L^p(\partial\Omega, d\sigma)$ is the space of all such p-th power integrable functions on $\partial\Omega$. The corresponding norms are $\|u\|_p$ and $\|u\|_{p,\partial\Omega}$ and are defined by

$$\|u\|_p^p \ := \int_\Omega |u|^p \, dx \quad \text{and} \quad \|u\|_{p,\partial\Omega}^p \ := \int_{\partial\Omega} |u|^p \, d\sigma. \tag{2.1}$$

All functions in this paper will take values in $\overline{\mathbb{R}} := [-\infty, \infty]$ and we shall write

$$\overline{u} \ := \ |\Omega|^{-1} \int_\Omega u \, dx \quad \text{and} \quad \overline{u}_\partial \ := \ |\sigma(\partial\Omega)|^{-1} \int_{\partial\Omega} u \, d\sigma.$$

for the mean values of u over the region Ω and the boundary $\partial\Omega$ respectively. Also $p^* := p/(p-1)$ is the dual index to p.

When $u \in L^p(\Omega)$ its weak j-th derivative is denoted $D_j \, u$. The Sobolev space $W^{1,p}(\Omega)$ is defined to be the space of all functions in $L^p(\Omega)$, whose weak first derivatives $D_j \, u, 1 \le j \le n$ are all in $L^p(\Omega)$. The standard norm on $W^{1,p}(\Omega)$ is denoted $\|u\|_{1,p}$ and is defined by

$$\|u\|_{1,p}^p \ := \int_\Omega \left[\sum_{j=1}^n |D_j \, u|^p \, + \, |u|^p \right] \, dx. \tag{2.2}$$

This space is a Banach space. When $p = 2$, this becomes the Hilbert space $H^1(\Omega)$ with the standard H^1- inner product

$$[u, v]_1 := \int_\Omega [u(x)v(x) \ + \ \nabla u(x) \cdot \nabla v(x)] \ dx. \tag{2.3}$$

Here $\nabla u \ := \ (D_1 u, D_2 u, ..., D_n u)$ is the gradient of the function u and we shall write

$$\|\nabla u\|_p^p \ := \ \int_\Omega \sum_{j=1}^n |D_j \, u|^p \ dx \tag{2.4}$$

For our analysis we also require some mild regularity conditions on Ω and $\partial\Omega$. First we shall require that the Sobolev imbedding theorem and the Rellich-Kondrachov theorem hold for $W^{1,p}(\Omega)$. Specifically

(A1): *The imbedding $i : W^{1,p}(\Omega) \to C^0(\overline{\Omega})$ is compact when $p > n$ and $i : W^{1,p}(\Omega) \to L^q(\Omega)$ is compact for $1 \leq q < q_c$ when $p \leq n$ and $q_c = np/(n-p)$.*

Criteria for this assumption are given in Adams and Fournier [1] and in Edmunds and Evans [10] chapter V. In particular when (A1) holds, then the *Poincaré inequality* follows. Namely there is a $C_p = C_p(\Omega) > 0$, which depends on Ω, p only, such that

$$\|\nabla u\|_p \ \geq \ C_p \ \|u - \overline{u}\|_p \quad \text{for all } u \in W^{1,p}(\Omega). \tag{2.5}$$

For a (non-constructive) proof see section 5.8 of [8]. A detailed analysis of criteria that guarantee inequalities such as this is found in [10] chapter V, sections 4 and 5.

We also require a trace condition. Assume that $\partial\Omega$ has finite surface $\sigma-$measure and is a finite union of disjoint Lipschitz surfaces. When this holds there is an outward unit normal ν defined at σ *a.e.* point of $\partial\Omega$. For the definition of this, and related terms, see Evans and Gariepy [9] chapter 4. Let Γ denote the boundary trace operator, then we will require

(A2): *The boundary trace operator $\Gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega, d\sigma)$ is continuous.*

A functional $\mathcal{F} : W^{1,p}(\Omega) \to (-\infty, \infty]$ is said to be *homogeneous of degree p (or p-homogeneous)* provided

$$\mathcal{F}(cu) \ = \ |c|^p \, \mathcal{F}(u) \quad \text{for all } c \in \mathbb{R}, \ u \in W^{1,p}(\Omega).$$

We say the functional is *positive* if $\mathcal{F}(u) \geq 0$ for all u. It is said to be *G-differentiable* at $u \in W^{1,p}(\Omega)$ if there is a continuous linear functional$D\mathcal{F}(u)$ such that

$$\lim_{t \to 0} \ t^{-1} \left[\mathcal{F}(u + th) - \mathcal{F}(u)\right] \ = \ D\mathcal{F}(u)(h) \qquad \text{for all } h \in W^{1,p}(\Omega),$$

In this case, $D\mathcal{F}(u)$ is called the *G-derivative* of $\mathcal{F}$ at u and this expression will also be denoted $\langle D\mathcal{F}(u), h\rangle$.

A real sequence $\{a_m : m \geq 1\}$ is said to be *(strictly) decreasing* if $a_{m+1}(<) \leq a_m$ for all m. A function u is said to be *(strictly) positive* on a set E, if $u(x)(>) \geq 0$ on E.

3. The $p-$homogeneous Inequality

Our interest in this paper is in finding the optimal constant C_0, in inequalities of the form;

$$\mathcal{F}(u) \ := \ \int_\Omega \sum_{j=1}^n |D_j\, u|^p \, dx \ + \ \mathcal{B}(u) \ \geq \ C_0 \, \mathcal{P}(u) \tag{3.1}$$

where $1 < p < \infty$, $\mathcal{B}, \mathcal{P}$ are p-homogeneous functionals on $W^{1,p}(\Omega)$ and $C_0 > 0$. The constant C_0 in 3.1 is said to be *optimal* if it is the largest number such that (3.1) holds. A non-zero function $\hat{u}$ in $W^{1,p}(\Omega)$ *optimizes* (3.1) if equality holds in (3.1) with the optimal choice of C_0. When $\hat{u}$ optimizes (3.1), so does any multiple of $\hat{u}$.

We will be particularly interested in the case where $\mathcal{P} : W^{1,p}(\Omega) \to [0, \infty]$ is defined by

$$\mathcal{P}(u) \ := \ \int_\Omega \rho(x) |u(x)|^p \, dx. \tag{3.2}$$

and $\rho : \Omega \to [0, \infty]$ satisfies
(A3): *The function ρ is in $L^1(\Omega)$ when $p > n$ or else ρ is in $L^q(\Omega)$ for some $q > q_0$ with $q_0 := n/p$ when $1 < p \leq n$ and also $\int_\Omega \rho \, dx > 0$.*

Some properties of this functional may be summarized as follows.

Proposition 3.1. *Assume Ω satisfies (A1), ρ satisfies (A3) and $\mathcal{P}$ is defined by (3.2). Then $\mathcal{P}$ is positive, bounded, convex, weakly continuous and G-differentiable on $W^{1,p}(\Omega)$ with*

$$\langle D\mathcal{P}(u), h \rangle \ = \ p \int_\Omega \rho |u|^{p-2} u h \, dx \qquad \text{for all } u, h \in W^{1,p}(\Omega). \tag{3.3}$$

Proof. First consider the case $p > n$. From (A1), $u \in W^{1,p}(\Omega)$ implies that u is in $C^0(\overline{\Omega})$ and this imbedding is compact. Use Holder's inequality and (A3), then

$$0 \ \leq \ \mathcal{P}(u) \ \leq \ \|\rho\|_1 \, \|u\|_\infty^p$$

so $\mathcal{P}$ is continuous and bounded. $\mathcal{P}$ is convex as $|s|^p$ is convex on $\mathbb{R}$.

Assume $\{u_m : m \geq 1\}$ converges weakly to $\hat{u}$ in $W^{1,p}(\Omega)$. From (A1), it converges to $\hat{u}$ in the uniform norm on $C^0(\overline{\Omega})$. Thus

$$\rho(x) \, |u_m(x)|^p \ \to \ \rho(x) \, |\hat{u}(x)|^p \quad \text{pointwise on } \ \Omega.$$

The Lebesgue dominated convergence theorem now implies that $\mathcal{P}$ is weakly continuous on $W^{1,p}(\Omega)$.

When $1 < p \leq n$ and (A3) holds then Holder's inequality yields

$$0 \ \leq \ \mathcal{P}(u) \ \leq \ \|\rho\|_q \, \||u|^p\|_{q^*} \, .$$

Choose $r = pq/(q-1)$ then, when (A1)holds, the imbedding $i : W^{1,p}(\Omega) \to L^r(\Omega)$ is compact as $p < r < np/(p-1)$. Also

$$\| |u|^p \|_{q^*} \ = \ \|u\|_r^{p/r},$$

so $\mathcal{P}$ is positive and bounded on $W^{1,p}(\Omega)$. It is convex as before.

If $\{u_m : m \geq 1\}$ converges weakly to $\hat{u}$ in $W^{1,p}(\Omega)$ then, from (A1), it converges strongly in $L^r(\Omega)$. Thus there is a subsequence $\{u_{m_j} : j \geq 1\}$ which converges a.e. to $\hat{u}$ on Ω. Apply the Lebesgue dominated convergence theorem then $\mathcal{P}$ is weakly continuous as claimed.

The function $\psi(s) := |s|^p$ with $p > 1$ is continuously differentiable on $\mathbb{R}$ with $\psi'(s) \ = \ p\,|s|^{p-2}s$ for $s \neq 0$ and $\psi'(0) = 0$. Define

$$\Psi(t) \ := \ \mathcal{P}(u + th) \ = \ \int_\Omega \rho |u + th|^p \, dx$$

with $u, h \in W^{1,p}(\Omega)$. Then the G-derivative of $\mathcal{P}$ at u is given by $\Psi'(0) \ = \ \langle D\mathcal{P}(u), h \rangle$. The conditions of Corollary 1.2.2 of [13], page 124 hold in our case, so the t-derivative can be taken under the integral and (3.3) follows. □

Note that this result implies that $\mathcal{P}(u)^{1/p}$ is a weakly continuous semi-norm on $W^{1,p}(\Omega)$.

The essential requirement for the functional $\mathcal{B} : W^{1,p}(\Omega) \to [0, \infty]$ will be
(A4): *The functional $\mathcal{B}$ is weakly lower semi-continuous (l.s.c) on $W^{1,p}(\Omega)$.*

We will describe inequalities based on three examples of this functional $\mathcal{B}$. The first is

$$\mathcal{B}_1(u) \ := \ \int_{\partial\Omega} b \ |\Gamma u|^p \ d\sigma. \tag{3.4}$$

In the following the trace operator Γ will often be omitted. We will require that
(B1): *$b : \partial\Omega \to [0, \infty)$ is in $L^\infty(\partial\Omega, d\sigma)$ and*

$$\int_{\partial\Omega} b \ d\sigma \ := \ b_0 \ > 0. \tag{3.5}$$

A second example is

$$\mathcal{B}_2(u) \ := \ \left| \int_{\partial\Omega} b \ \Gamma u \ d\sigma \right|^p \tag{3.6}$$

where we require that
(B2): *$b : \partial\Omega \to [0, \infty]$ is in $L^{p^*}(\partial\Omega, d\sigma)$ and (3.5) holds.*

A third example will be

$$\mathcal{B}_3(u) \ := \ \left| \ \int_\Omega c(x) u(x) \ dx \ \right|^p. \tag{3.7}$$

We will require
(B3): *$c : \Omega \to [0,\infty]$ is in $L^{p^*}(\Omega)$ and*

$$\int_\Omega c(x)\, dx \ := \ \overline{c}\ |\Omega| \ > 0. \tag{3.8}$$

We shall treat the inequalities associated with each of these choices of $\mathcal{B}$ separately in later sections. In each of these examples the functions ρ, b and/or c may be zero on sets of positive measure; in many important applications they will be characteristic functions of specific subsets.

Both $\mathcal{B}_2, \mathcal{B}_3$ are functionals of the form

$$\mathcal{B}(u) \ := \ |b(u)|^p \tag{3.9}$$

with b being a continuous linear functional on $W^{1,p}(\Omega)$. We will use the following general result for functionals of this form.

Proposition 3.2. *Assume b is a continuous linear functional on $W^{1,p}(\Omega)$, $\mathcal{B}$ is defined by (3.9) and $1 < p < \infty$. Then $\mathcal{B}$ is continuous, convex and satisfies (A4). $\mathcal{B}$ is G-differentiable with $D\mathcal{B}(u) = 0$ when $b(u) = 0$ and*

$$\langle D\mathcal{B}(u), h\rangle \ = \ p\ |b(u)|^{p-2} b(u) b(h) \qquad \textit{for all } u, h \in W^{1,p}(\Omega). \tag{3.10}$$

Proof. When $p > 1$, let $\psi(s) := |s|^p$ as above. ψ is convex and continuously differentiable on $\mathbb{R}$. Using standard results on compositions, $\mathcal{B}$ will be continuous and convex. Thus it is weakly l.s.c. on $W^{1,p}(\Omega)$. Applying the chain rule for G-derivatives and the expression for ψ', the third sentence follows. □

4. Scaling of $p-$homogeneous Inequalities

The inequalities studied here arise in the numerical analysis of elliptic equations, so it is of interest to know how they scale with the size of the domain. Given a reference region Ω, and $L > 0$, define the scaled region $\Omega_L := \{Lx : x \in \Omega\}$.

Define the dilation operator $\mathcal{S}_L : W^{1,p}(\Omega) \to W^{1,p}(\Omega_L)$ by

$$\mathcal{S}_L u(y) \ := \ u_L(y) \ := \ u(y/L) \qquad \text{for} \quad y \in \Omega_L. \tag{4.1}$$

$\mathcal{S}_L$ is a linear isomorphism and the change of variables rule yields that

$$\int_{\Omega_L} \sum_{j=1}^n |D_j u_L(y)|^p dy \ = \ L^{n-p} \int_\Omega \sum_{j=1}^n |D_j\, u(x)|^p\, dx. \tag{4.2}$$

For $L > 0$, define the functional $\mathcal{P}_L : W^{1,p}(\Omega_L) \to [0,\infty]$ by

$$\mathcal{P}_L(u_L) \ := \ \int_{\Omega_L} \rho_L(y) |u_L(y)|^p\, dy \tag{4.3}$$

with $\rho_L := \mathcal{S}_L \rho$. This functional satisfies

$$\mathcal{P}_L(u_L) \ = \ L^n \mathcal{P}(u) \qquad \text{for each} \quad u \in W^{1,p}(\Omega). \tag{4.4}$$

Similarly define the function $b_L : \partial\Omega_L \to [0,\infty]$ by $b_L(y) := b(y/L)$. Then the functionals $\mathcal{B}_{jL}$ defined by (3.4) and (3.6)-(3.7) on $W^{1,p}(\Omega_L)$ scale according to

$$\mathcal{B}_{jL}(u_L) \;=\; L^{n-1}\mathcal{B}_j(u) \quad \text{for } j = 1,2 \text{ and } \quad \mathcal{B}_{3L}(u_L) \;=\; L^n \, \mathcal{B}_3(u).$$

In the following sections we shall describe the scaled versions of the inequalities of the form (3.1). In each case the optimal functions will be dilations of the optimal function with $L = 1$.

5. Variational Principles for Optimal Constants

In this section, we shall show that the problem of finding the optimal constants, and functions, in inequalities of the form (3.1) can be described using an unconstrained variational principle. This reformulation enables a much simpler description of the extremality conditions.

We will now assume that X is a real reflexive Banach space and also

(C1): *$\mathcal{F}, \mathcal{P}$ are continuous functionals on X which are homogeneous of degree $p > 1$.*

(C2): *$\mathcal{F}$ is convex and there exists $c_0 > 0$ such that*

$$\mathcal{F}(u) \;\geq\; c_0 \, \|u\|_X^p \qquad \text{for all} \quad u \in X. \tag{5.1}$$

(C3): *$\mathcal{P}$ is weakly continuous and bounded on X and there exists $v \in X$ such that $\mathcal{P}(v) > 0$.*

Define

$$B \;:=\; \{u \in X : \mathcal{F}(u) \;\leq\; 1\} \quad \text{and} \quad B_1 \;:=\; \{u \in X : \mathcal{F}(u) \;=\; 1\}. \tag{5.2}$$

Consider the variational problem of finding

$$\beta \;:=\; sup_{u \in B_1} \, \mathcal{P}(u). \tag{5.3}$$

When this β is finite, then the homogeniety condition (C1) implies that

$$\beta\mathcal{F}(u) \;\geq\; \mathcal{P}(u) \qquad \text{for all} \quad u \in \; X. \tag{5.4}$$

Now (C2) and (C3) imply that $\beta > 0$, so we also have

$$\mathcal{F}(u) \;\geq\; \beta^{-1} \, \mathcal{P}(u) \qquad \text{for all } u \in \; X. \tag{5.5}$$

Theorem 5.1. *Assume (C1) - (C3) hold, then β defined by (5.3) is finite and there is a $\hat{u} \in B_1$ with $\mathcal{P}(\hat{u}) = \beta$.*

Proof. First we shall show that $\beta \;:=\; sup_{u \in B} \, \mathcal{P}(u)$ is finite and that this supremum is attained. Then we prove this supremum is attained at a point in B_1 which leads to the theorem.

The set B is closed, convex and bounded so it is weakly compact as $W^{1,p}(\Omega)$ is reflexive. $\mathcal{P}$ is weakly continuous from (C3), so there is a finite $\beta_0 > 0$ such that $\beta_0 = \sup_{u \in B} \; \mathcal{P}(u)$ and this infimum is attained so there is a $\hat{u}$ in B with $\mathcal{P}(\hat{u}) = \beta_0$.

If $\mathcal{F}(\hat{u}) < 1$, then $\tau\hat{u} \in B$ for some $\tau > 1$. Then $\mathcal{P}(\tau\hat{u}) = |\tau|^p\beta_0 > \beta_0$. This contradicts the definition of β_0, so $\mathcal{F}(\hat{u}) = 1$. This implies that $\beta = \beta_0$ and the theorem follows. □

Corollary 5.2. *Assume (C1)-(C3) hold, then (3.1) holds with $C_0 = \beta^{-1}$. Moreover this constant is optimal and equality holds in (3.1) when u is any multiple of $\hat{u}$.*

Proof. (3.1) follows from theorem 5.1 and (5.5). Substitution shows that equality holds in (5.5) whenever u is a multiple of any function $\hat{u}$ in B_1 for which $\mathcal{P}(\hat{u}) = \beta$. This implies the same for (3.1). □

The variational principle described above provides the usual constrained variational characterization of the best constants in inequalities such as (3.1). Now consider the functional $\mathcal{J} : X \to \mathbb{R}$ defined by

$$\mathcal{J}(u) \;:=\; \frac{1}{2}\,\mathcal{F}(u)^2 - \;\mathcal{P}(u), \tag{5.6}$$

and the unconstrained problem of finding the infimum of $\mathcal{J}$ on X. The following holds for this problem.

Theorem 5.3. *Assume (C1) - (C3) hold with $\beta, \mathcal{J}$ defined by (5.3) and (5.6). Then $\mathcal{J}$ is weakly l.s.c. and coercive on X and attains its infimum at points $\pm\,\tilde{u}$ in X where $\tilde{u} = \beta^{1/p}\hat{u}$ and $\hat{u}$ is a maximizer of $\mathcal{P}$ on B_1.*

Proof. When $\mathcal{F}$ is convex and continuous on X, so is $\mathcal{F}^2$ as it is positive. Thus $\mathcal{F}^2$ and $\mathcal{J}$ are weakly l.s.c. on X. Define $c_1 := \sup_{u\in B_1} |\mathcal{P}(u)|$. This is finite from assumption (C3) and homogeniety implies that $\mathcal{P}(u) \le c_1\|u\|_X^p$ for all $u \in X$. Thus

$$\mathcal{J}(u) \;\ge {c_0}^2\|u\|_X^{2p} - \;C_1\|u\|_X^p$$

for all $u \in X$. This shows that $\mathcal{J}$ is coercive on X. Hence $\mathcal{J}$ is bounded below on X and attains its infimum. If $\tilde{u}$ is a minimizer, so is $-\tilde{u}$ as $\mathcal{J}$ is even.

Given $u \in X, u \neq 0, p > 1$, consider the ray $\Gamma_u \;:=\; \{su : s > 0\}$. Let z be the unique point on this ray satisfying $\mathcal{F}(z) = 1$. Then $u = tz$ where $t^p = \mathcal{F}(u)$. Now

$$\mathcal{J}(sz) = s^{2p}/2 - s^p\mathcal{P}(z).$$

If $\mathcal{P}(z) \le 0$, this expression is minimized at $s = 0$ and the infimum of $\mathcal{J}$ along the ray Γ_u is zero. If $\mathcal{P}(z) > 0$, this expression is minimized at $\tilde{s}$ where

$$\tilde{s}^p = \mathcal{P}(z) \qquad \text{and} \qquad \inf_{s>0}\,\mathcal{J}(sz) = -\mathcal{P}(z)^2/2. \tag{5.7}$$

For $z \in B_1$, define $\mathcal{P}_+(z) \;:=\; \max(0, \mathcal{P}(z))$. Then

$$\inf_{u\in X}\mathcal{J}(u) \;=\; \inf_{z\in B_1}\inf_{s>0}\;\mathcal{J}(sz) \;=\; inf_{z\in B_1}[-\mathcal{P}_+(z)^2/2] \;=\; -\beta^2/2. \tag{5.8}$$

That is, the minimizers of $\mathcal{J}$ on X are $\tilde{u} = \pm\beta^{1/p}\hat{u}$ where $\hat{u}$ is a maximizer of $\mathcal{P}$ on B_1. Moreover at these minimizers

$$\mathcal{F}(\tilde{u}) = \beta, \quad \mathcal{P}(\tilde{u}) = \beta^2 \quad \text{and} \quad \mathcal{J}(\tilde{u}) = -\beta^2/2. \tag{5.9}$$

□

To describe the conditions satisfied by the solutions of these problems we shall require
(C4): *$\mathcal{F}, \mathcal{P}$ are G-differentiable on X.*

When a functional $\mathcal{P}$ is p-homogeneous and G-differentable at $u \in X$, then differentiation of $\mathcal{P}(tu)$ at $t = 1$ yields *Euler's rule* that

$$\langle D\mathcal{P}(u), u\rangle \ = \ p\, \mathcal{P}(u). \tag{5.10}$$

Corollary 5.4. *Assume (C1)-(C4) hold and $\tilde{u}$ minimizes $\mathcal{J}$ on X. Then $\tilde{u}$ is a solution of*

$$D\mathcal{F}(u) \ = \ \beta^{-1} D\mathcal{P}(u). \tag{5.11}$$

Proof. Apply the chain rule to (5.6), then

$$D\mathcal{J}(u) \ = \ \mathcal{F}(u)\, D\mathcal{F}(u) - D\mathcal{P}(u).$$

From (5.9) the minimizers of $\mathcal{J}$ on X have $\mathcal{F}(\tilde{u}) = \beta$, so (5.11) follows. □

Consider now the general eigenvalue problem of solving

$$D\mathcal{F}(u) \ = \ \mu\, D\mathcal{P}(u) \tag{5.12}$$

That is, we wish to find those $(\mu, u) \in \mathbb{R} \times X$, with $u \neq 0$, which solve (5.12). This will be interpreted in the weak form that

$$\langle D\mathcal{F}(u), h\rangle \ = \ \mu\, \langle D\mathcal{P}(u), h\rangle \quad \text{for all } h \in X. \tag{5.13}$$

Theorem 5.5. *Assume (C1) - (C4) hold, and β is defined by (5.3). Then β^{-1} is the least value of μ such that (5.13) has a non-zero solution in X.*

Proof. Let $\hat{v}$ be a non-zero solution of (5.13). Put $u = h = \hat{v}$ in (5.13), then Euler's rule (5.10) and (5.4) yield that

$$0 \ = \ p\, [\mathcal{F}(\hat{v}) - \mu\, \mathcal{P}(\hat{v})] \ \geq \ p(1 - \mu\beta)\mathcal{F}(\hat{v}).$$

From (C2), $\mathcal{F}(\hat{v}) > 0$, so $\mu \geq \beta^{-1}$. Moreover $\hat{u}$ is a non-zero solution of (5.13) with $\mu \ = \ \beta^{-1}$ so the result follows. □

6. $p-$versions of Friedrichs' Inequality

K.O Friedrichs' is credited with H^1-coercivity inequalities of the form

$$\int_\Omega \sum_{j=1}^n |D_j\, u|^2 \, dx \, + \int_{\partial\Omega} |u|^2 d\sigma \ \geq \ C_0 \int_\Omega |u|^2 dx. \tag{6.1}$$

See Necas [14] theorem 1.9 and Hellwig [12], Section 5.3 for proofs of this result and references to the earlier literature.

Here we shall describe the p-analogue of this with $1 < p < \infty$ and allow weights in the last two terms. That is, we prove an inequality of the form (3.1) with $\mathcal{B}_1$ given by (3.4) in place of $\mathcal{B}$ and $\mathcal{P}$ defined by (3.2). The intent is to identify the optimal constant C_F and corresponding optimal functions for the inequality

$$\mathcal{F}_1(u) \ := \ \int_\Omega \sum_{j=1}^n |D_j\, u|^p \, dx \ + \int_{\partial\Omega} b|u|^p \, d\sigma \ \geq \ C_F \int_\Omega \rho|u|^p \, dx. \tag{6.2}$$

for all $u \in W^{1,p}(\Omega)$. Here ρ, b obey (A3) and (B1) respectively. The value of C_F depends on the region Ω, the value of p and the functions b, ρ. It will be called *Friedrichs' constant* and some variational characterizations of it will be developed.

We need some basic properties of the functional $\mathcal{B}_1$.

Proposition 6.1. *Assume (A2) and (B1) hold and $\mathcal{B}_1$ is defined by (3.4). Then $\mathcal{B}_1$ is convex, positive and continuous on $W^{1,p}(\Omega)$. Also (A4) holds and $\mathcal{B}_1$ is G-differentiable with*

$$\langle D\mathcal{B}_1(u), h\rangle \ = \ p \int_{\partial\Omega} b|u|^{p-2}uh \, d\sigma \qquad \textit{for all } u, h \in W^{1,p}(\Omega). \tag{6.3}$$

Proof. The integrand $b(x,s) := b(x)|s|^p$ is positive on $\Omega \times \mathbb{R}$ and $b(x,.)$ is convex on $\mathbb{R}$, so $\mathcal{B}_1$ is positive and convex.

If $\{u_m : m \geq 1\}$ converges to $\hat{u}$ in $W^{1,p}(\Omega)$, then (A2) implies that $\{\Gamma u_m\}$ converges strongly to $\Gamma\hat{u}$ in $L^p(\partial\Omega, d\sigma)$. Thus there is a subsequence $\{\Gamma u_{m_j} : j \geq 1\}$ which converges σ a.e. to $\Gamma\hat{u}$ on $\partial\Omega$, so

$$b(x, u_{m_j}(x)) \ \to \ b(x, \hat{u}(x)) \quad \sigma \ a.e. \text{ on } \partial\Omega \text{ as } \ j \to \infty.$$

When (B1) holds the Lebesgue dominated convergence theorem shows that $\mathcal{B}_1$ is continuous as claimed. Since $\mathcal{B}_1$ is continuous and convex, (A4) holds.

The proof that $\mathcal{B}_1$ is G-differentiable and (6.3) holds parallels that of the corresponding part of the proof of Proposition 3.1. □

A consequence of this result is the observation that $\mathcal{B}_1(u)^{1/p}$ is a continuous seminorm on $W^{1,p}(\Omega)$.

Consider the variational problem of minimizing $\mathcal{F}_1$ on the set

$$S_1 \ := \ \{ \ u \in W^{1,p}(\Omega) : \ \mathcal{P}(u) \ = \ 1\}. \tag{6.4}$$

When ρ satisfies (A3), proposition 3.1 shows that S_1 will be a weakly closed unbounded subset of $W^{1,p}(\Omega)$. Define

$$\alpha \ := \ \inf_{u \in S_1} \ \mathcal{F}_1(u). \tag{6.5}$$

Let $W^{1,p}_m(\Omega)$ be the subspace of $W^{1,p}(\Omega)$ of all functions with mean value zero. Then each $u \in W^{1,p}(\Omega)$ has a unique decomposition of the form

$$u \ = \ \overline{u} \ + \ v \qquad \text{with } v \in W^{1,p}_m(\Omega). \tag{6.6}$$

The triangle inequality for norms yields

$$\|u\|_{1,p} \;\le\; |\overline{u}||\Omega|^{1/p} + \|v\|_{1,p}\,. \tag{6.7}$$

Theorem 6.2. *Assume (A1) - (A3) and (B1) hold and $1 < p < \infty$. Then there is an optimal constant $C_F > 0$ and corresponding optimal functions for (6.2).*

Proof. To prove this we shall show that the infimum of $\mathcal{F}_1$ on S_1 is attained and then (6.2) holds with the optimal $C_F = \alpha > 0$. Since $\mathcal{F}_1(u) \ge 0$ we have $\alpha \ge 0$. Let $\{u_m : m \ge 1\}$ be a decreasing, minimizing sequence for $\mathcal{F}_1$ on S_1, then

$$\|\nabla u_m\|_p^p \;\le\; \mathcal{F}_1(u_1) \qquad \text{for all} \quad m \ge 1.$$

For each m, write $u_m = \overline{u}_m + v_m$ as in (6.6) . Then $\|\nabla u_m\|_p = \|\nabla v_m\|_p$, so $\|v_m\|_{1,p}$ is bounded from the Poincaré inequality (2.5).

When $\mathcal{B}_1$ is defined by (3.4) and (B1) holds then, since ${\mathcal{B}_1}^{1/p}$ is a semi-norm,

$$|\overline{u}_m|{b_0}^{1/p} \;=\; \mathcal{B}_1(\overline{u}_m)^{1/p} \;\le\; \mathcal{B}_1(u_m)^{1/p} + \mathcal{B}_1(v_m)^{1/p} \;\le\; \mathcal{F}_1(u_1)^{1/p} + C_2\,\|v_m\|_{1,p}\,.$$

In the last inequality the fact that Γ is continuous from (A2) was used. This yields that $\{\overline{u}_m : m \ge 1\}$ is bounded. Then (6.7) implies that $\{u_m : m \ge 1\}$ is bounded in $W^{1,p}(\Omega)$. This sequence has a weak limit $\hat{u}$ as $W^{1,p}(\Omega)$ is reflexive when $1 < p < \infty$ and $\hat{u}$ is in S_1 as S_1 is weakly closed. The functional $\mathcal{F}_1$ is weakly l.s.c. on $W^{1,p}(\Omega)$ from proposition 6.1. Thus $\mathcal{F}_1(\hat{u}) = \alpha$ as the sequence $\{u_m : m \ge 1\}$ is a minimizing sequence. Hence the infimum in (6.5) is attained.

If $\alpha = 0$ then $\|\nabla \hat{u}\|_p = 0$. Thus $\hat{u}$ is constant on Ω since Ω is connected. From (3.6) this constant must be 0 which is impossible if $\hat{u} \in S_1$. Hence $\alpha > 0$. By homogeniety (6.2) follows with $C_F := \alpha$ and any multiple of $\hat{u}$ is an optimal function for (6.2). □

This theorem implies that the expression

$$\|u\|_{b,p} \;:=\; \mathcal{F}_1(u)^{1/p} \tag{6.8}$$

is a norm on $W^{1,p}(\Omega)$. Moreover we have the following

Corollary 6.3. *Assume (A1), (A2) and (B1) hold, $1 < p < \infty$, then the (b,p) norm defined by (6.8) is an equivalent norm to the standard norm on $W^{1,p}(\Omega)$.*

Proof. When (B1) holds and $u \in W^{1,p}(\Omega)$, then

$$\mathcal{F}_1(u) \;\le\; \|\nabla u\|_p^p + \|b\|_{\infty,\partial\Omega}\,\|\Gamma u\|_{p,\partial\Omega}^p.$$

Since Γ is continuous from (A2) this yields

$$\|u\|_{b,p} \;\le\; (1+C)\|u\|_{1,p} \qquad \text{for some positive C.}$$

Take $\rho \equiv 1$ in theorem 6.2 then, since $b \ge 0$ and (6.2) holds,

$$2\mathcal{F}_1(u) \;\ge \|\nabla u\|_p^p + C_F\|u\|_p^p \;\ge\; \min(1, C_F)\;\|u\|_{1,p}^p$$

These two inequalities imply the equivalence of these norms on $W^{1,p}(\Omega)$. □

To describe the scaling of these inequalities, let $\Omega,\ \Omega_L$ be as in section 4. Define ρ_L, b_L as before. When (6.2) holds on Ω, multiply through by L^{n-p} and use the formulae of section 4 to find that

$$\int_{\Omega_L} \sum_{j=1}^{n} |D_j\, u|^p\, dy \,+\, L^{1-p} \int_{\partial\Omega_L} b_L |u|^p\, d\sigma \;\geq\; C_F\, L^{-p} \int_{\Omega_L} \rho_L |u|^p\, dy. \tag{6.9}$$

for all $L > 0,\ u \in W^{1,p}(\Omega_L)$. This is the general scale dependent version of (6.2) and equality will hold here for some functions in $W^{1,p}(\Omega_L)$.

7. Friedrichs' constant as an Eigenvalue

In the section we shall show that the optimal constant C_F in (6.2), and also the optimal functions, can be described as the least eigenvalue, and associated eigenfunctions, of a p-Laplacian eigenproblem on Ω.

This will be done by using the unconstrained variational formulation introduced in section 5. Take $X = W^{1,p}(\Omega), \mathcal{F}_1$ in place of $\mathcal{F}$ and assume $\mathcal{P}$ is defined by (3.2). When (A1) and (A3) hold for ρ, then proposition 3.1 shows that $\mathcal{P}$ satisfies (C1), (C3) and (C4). Similarly when (A1)-(A2) hold then $\mathcal{F}_1$ satisfies (C1) and (C3). The characterization of C_F in (6.2) may be compared with (5.5) and theorem 5.5 to show that C_F is the least eigenvalue of an eigenproblem of the form (5.13).

The G-differentiability of $\mathcal{F}_1$ is directly verified and then equation (5.13) becomes, in this case, the problem of finding (μ, u) in $\mathbb{R} \times W^{1,p}(\Omega)$ with $u \neq 0$ which solve

$$\int_{\Omega} \left[\sum_{j=1}^{n} |D_j\, u|^{p-2} D_j\, u D_j h \;-\; \mu\rho |u|^{p-2} u h \right] dx \,+\, \int_{\partial\Omega} b|u|^{p-2} u h\, d\sigma \;=\; 0. \tag{7.1}$$

for all $h \in W^{1,p}(\Omega)$.

When $p = 2$, this is the weak form of a linear eigenvalue problem for the Laplacian on Ω. Namely to find non-zero solutions of

$$\int_{\Omega} \left[\sum_{j=1}^{n} D_j\, u D_j h \;-\; \mu\rho u h \right] dx \,+\, \int_{\partial\Omega} b u h\, d\sigma \;=\; 0 \quad \text{for all } h \in H^1(\Omega). \tag{7.2}$$

This is the weak form of the eigenproblem

$$-\Delta u \;=\; \mu\, \rho\, u \qquad \text{in } \Omega \tag{7.3}$$

$$(\nabla u)\cdot \nu \,+\, b u \;=\; 0 \qquad \text{on } \partial\Omega. \tag{7.4}$$

This boundary condition is of Robin type, when b is strictly positive and of Neumann type on any subset where $b = 0$. In section 12, we shall determine the value of C_F for rectangles in the plane by direct solution of this problem.

For general $p > 1$, (7.1) is the weak form of an eigenvalue problem for the p-Laplacian on Ω. Namely one seeks non-zero solutions of

$$-\Delta_p u \;=\; -\sum_{j=1}^{n} D_j(|D_j\, u|^{p-2} D_j\, u) \quad = \quad \mu\, \rho\, |u|^{p-2}\, u \qquad \text{in } \Omega \tag{7.5}$$

$$\sum_{j=1}^{n} (|D_j\, u|^{p-2}(D_j\, u)\nu_j \;+\; b|u|^{p-2}u \quad = \quad 0 \qquad \text{on } \partial\Omega. \tag{7.6}$$

Friedrichs' constant may now be characterized as follows.

Theorem 7.1. *Assume (A1) - (A3) and (B1) hold with $1 < p < \infty$. Then the optimal constant $C_F > 0$ in (6.2) is the least eigenvalue μ_1 of (7.1). Equality holds in (6.2) if and only if u is an eigenfunction of (7.1) corresponding to the least eigenvalue μ_1.*

Proof. The fact that C_F is the least eigenvalue of (7.1) follows from theorem 5.5. If u_1 is a corresponding eigenfunction of (7.1) then put $u = h = u_1$ in (7.1) to see that equality holds in (6.2).

Conversely if $\tilde{u}$ is a non-zero function for which equality holds in (6.2) then it is a multiple of a function which maximizes $\mathcal{P}$ on the unit sphere in $W^{1,p}(\Omega)$ with the (b,p)-norm. Hence it is a multiple of a minimizer of the associated $\mathcal{J}$ on $W^{1,p}(\Omega)$. Corollary 5.4 now yields the result when one observes that any multiple of a solution of (7.1) is again a solution of the equation. □

8. Coercivity Inequalities with Boundary Integrals

The analysis of finite element methods for linear elliptic operators uses coercivity inequalities of the form

$$\int_{\Omega} \sum_{j=1}^{n} |D_j\, u|^2\, dx \;+\; \left(\int_{\partial\Omega} bu\; d\sigma\right)^2 \;\geq\; C_1 \int_{\Omega} |u|^2 dx. \tag{8.1}$$

for all $u \in H^1(\Omega)$. See Brenner [6] or Arnold et al [2] for discussions of this. This inequality differs from (6.2) and I'm not aware of a published proof of this result which includes an estimate of the constant C_1. A slightly different inequality is given as example 6.3.16 in [3].

A more general form of this is that for $1 < p < \infty$, there is a constant $C_B > 0$ such that

$$\mathcal{F}_2(u) \;:=\; \int_{\Omega} \sum_{j=1}^{n} |D_j\, u|^p\, dx \;+\; \left|\int_{\partial\Omega} bu\; d\sigma\right|^p \;\geq\; C_B \int_{\Omega} \rho|u|^p\, dx. \tag{8.2}$$

for all $u \in W^{1,p}(\Omega)$. Here b, ρ satisfy (B2) and (A3). The value of C_B will depend on Ω, p, b, ρ. Here we shall characterize the optimal constant C_B, and the optimizing functions in this inequality via some variational principles. In particular we will show that C_B is the least eigenvalue of a eigenproblem with integro-differential boundary conditions.

The analysis of this inequality parallels that for Friedrichs' inequality. We first consider the variational problem of minimizing $\mathcal{F}_2$ on the set S_1 defined by (6.4). Write

$$\alpha_2 \;:=\; \inf_{u\in S_1}\; \mathcal{F}_2(u). \tag{8.3}$$

The proof of the following theorem shows that the minimizers of this problem exist and are functions for which equality holds in (8.2) with the value $\alpha_2 = C_B$.

Theorem 8.1. *Assume (A1) - (A3) and (B2) hold and $1 < p < \infty$. Then there is an optimal constant $C_B > 0$ and corresponding optimal functions for (8.2).*

Proof. First note that $\alpha_2 > 0$. Let $\{u_m : m \geq 1\}$ be a decreasing, minimizing sequence for $\mathcal{F}_2$ on S_1, then

$$\|\nabla u_m\|_p^p \;\leq\; \mathcal{F}_2(u_1) \qquad \text{for all} \quad m \geq 1.$$

For each m, write $u_m = \overline{u}_m + v_m$ as in (6.6) . Then $\|\nabla u_m\|_p = \|\nabla v_m\|_p$, so $\|v_m\|_{1,p}$ is bounded from the Poincaré inequality (2.5).

Define the linear functional b on $W^{1,p}(\Omega)$ by

$$b(u) \;:=\; \int_{\partial\Omega} b\,\Gamma u\; d\sigma. \tag{8.4}$$

Then $b(u_m) = b_0\overline{u}_m + b(v_m)$. When (A2) and (B2) hold, b will be continuous, so

$$b_0|\overline{u}_m| \;\leq\; |b(u_m)| + C\,\|v_m\|_{1,p}\,.$$

Now $|b(u_m)|$ is bounded as we have a descent sequence for $\mathcal{F}_2$ thus $|\overline{u}_m|$ is uniformly bounded. Hence the sequence $\{u_m : m \geq 1\}$ is bounded in $W^{1,p}(\Omega)$ from (6.7). The concluding arguments in the proof of theorem 6.2 now apply and complete the proof of this theorem. □

This result implies that the expression

$$\|u\|_{\partial,p} \;:=\; \mathcal{F}_2(u)^{1/p} \tag{8.5}$$

defines a norm on $W^{1,p}(\Omega)$. This may be strengthened to the following

Corollary 8.2. *Assume (A1), (A2) and (B2) hold, $1 < p < \infty$, then the (∂, p) norm defined by (8.5) is an equivalent norm to the standard norm on $W^{1,p}(\Omega)$.*

Proof. When (B2) holds and $u \in W^{1,p}(\Omega)$, then

$$\mathcal{F}_2(u) \;\leq\; \|\nabla u\|_p^p \,+\, \|b\|_{p^*,\partial\Omega}^p\,\|\Gamma u\|_{p,\partial\Omega}^p$$

Since Γ is continuous from (A2) this yields

$$\|u\|_{\partial,p} \;\leq\; (1+C)\|u\|_{1,p} \qquad \text{for some positive C.}$$

Take $\rho \equiv 1$ in theorem 8.1 then, since $b \geq 0$ and (8.2) holds,

$$2\mathcal{F}_2(u) \;\geq \|\nabla u\|_p^p \,+\, C_B\|u\|_p^p \;\geq\quad \min(1, C_B)\;\|u\|_{1,p}^p$$

These two inequalities imply the equivalence of these norms on $W^{1,p}(\Omega)$. □

When Ω_L is defined as in section 4, the scaled version of (8.2) is obtained by multiplying by L^{n-p} and using the formulae in section 4. This yields

$$\int_{\Omega_L} \sum_{j=1}^{n} |D_j\, u|^p \; dy \,+\, L^{n(1-p)} \left| \int_{\partial\Omega_L} b_L u \; d\sigma \right|^p \;\geq\; C_B L^{-p} \int_{\Omega_L} \rho_L |u|^p \; dy. \tag{8.6}$$

for all $L > 0,\; u \in W^{1,p}(\Omega_L)$. Moreover there are functions in $W^{1,p}(\Omega_L)$ for which equality holds here.

9. The Optimal constant C_B

In the last section the constant C_B was identified as the value of a variational problem. This problem has the form of the problem described in section 5 with X replaced by $W^{1,p}(\Omega)$, $\mathcal{F}$ by $\mathcal{F}_2$ and β by α_2.

Just as in section 5, an unconstrained variational principle for this problem may be introduced and we find that the minimizers of our problem are given by the eigenfunctions of the analogue of equation (5.13) corresponding to the least eigenvalue. It is straightforward to complete the verification that $\mathcal{F}_2$ is G-differentiable on $W^{1,p}(\Omega)$. In this case equation (5.13) becomes the problem of finding non-zero solutions (μ, u) in $\mathbb{R} \times W^{1,p}(\Omega)$ of

$$\int_{\Omega} \left[\sum_{j=1}^{n} |D_j\, u|^{p-2} D_j\, u D_j h \;-\; \mu\rho |u|^{p-2} u h \right] \, dx \,+\, |b(u)|^{p-2} b(u) b(h) \;=\; 0 \tag{9.1}$$

for all $h \in W^{1,p}(\Omega)$. When $p = 2$, this reduces to

$$\int_{\Omega} \left[\sum_{j=1}^{n} D_j\, u D_j h \;-\; \mu\rho u h \right] \, dx \,+\, b(u) b(h) \;=\; 0 \quad \text{for all } h \in H^1(\Omega). \tag{9.2}$$

This is the weak form of the eigenproblem

$$-\Delta u \;=\; \mu\, \rho u \qquad \text{in } \Omega \tag{9.3}$$

$$(\nabla u)\cdot \nu \,+\, b(u) b \;=\; 0 \qquad \text{on } \partial\Omega. \tag{9.4}$$

This boundary condition is an integro-differential equation. Nevertheless standard elliptic spectral theory applies to this eigenproblem with only minimal changes.

For general $p > 1$, (9.1) is the weak form of an eigenvalue problem for the p-Laplacian on Ω. Namely one seeks non-zero solutions of (7.5) subject to the integro-differential boundary condition

$$\sum_{j=1}^{n} (|D_j\, u|^{p-2}(D_j\, u)\nu_j \; + \; b|b(u)|^{p-2}b(u) \;=\; 0 \qquad \text{on } \partial\Omega. \tag{9.5}$$

The optimal constant C_B may now be characterized in a similar way to that of the Friedrichs' constant in section 7. The proof of the following is essentially the same as that of thoerem 7.1.

Theorem 9.1. *Assume (A1) - (A3) and (B2) hold with $1 < p < \infty$. Then the optimal constant $C_B > 0$ in (8.2) is the least eigenvalue μ_1 of (9.1). Equality holds in (8.2) if and only if u is an eigenfunction of (9.1) corresponding to the least eigenvalue μ_1.*

10. Generalized Poincaré Inequalities

The name *Poincaré Inequality* is attached to a number of different results. In Gilbarg and Trudinger, [11] Section 7.9 or Edmunds and Evans [10], Chapter V, section 3, inequalities of the form

$$\int_\Omega \sum_{j=1}^{n} |D_j\, u|^p \; dx \;\geq\; C_p \int_\Omega |u - \overline{u}|^p dx. \tag{10.1}$$

are described with specific simple formulae for (in fact general lower bounds on) C_p.

Here we shall consider the question of finding the optimal constant C_P in the inequality

$$\mathcal{F}_3(u) \;:=\; \int_\Omega \sum_{j=1}^{n} |D_j\, u|^p \; dx \;+\; \left| \int_\Omega cu \; dx \right|^p \;\geq\; C_P \int_\Omega \rho |u|^p \; dx. \tag{10.2}$$

for all $u \in W^{1,p}(\Omega)$. Here c, ρ satisfy (B3) and (A3). When $p = 2$ and the functions c, ρ are constants, this is one of the forms given in Necas [14], Chapter 1. The value of C_P will depend on Ω, p, c, ρ.

Here we shall characterize the optimal constant C_P, and the optimizing functions in this inequality via some variational principles. In particular we will show that C_P is the least eigenvalue of a Neumann eigenproblem for an integrodifferential operator on Ω.

Just as before, consider the problem of minimizing the functional $\mathcal{F}_3$ on the set S_1 defined by (6.4). Write

$$\alpha_3 \;:=\; \inf_{u \in S_1} \; \mathcal{F}_3(u). \tag{10.3}$$

The proof of the following theorem shows that the minimizers of this problem exist, there are functions for which equality holds in (10.2) and the value $\alpha_3 = C_P$.

Theorem 10.1. *Assume (A1), (A3) and (B3) hold and $1 < p < \infty$. Then there is an optimal constant $C_P > 0$ and corresponding optimal functions for (10.2).*

Proof. First note that $\alpha_3 > 0$. Let $\{u_m : m \geq 1\}$ be a decreasing, minimizing sequence for $\mathcal{F}_3$ on S_1, then

$$\|\nabla u_m\|_p^p \;\leq\; \mathcal{F}_3(u_1) \qquad \text{for all} \quad m \geq 1.$$

For each m, write $u_m = \overline{u}_m + v_m$ as in (6.6) . Then $\|\nabla u_m\|_p = \|\nabla v_m\|_p$, so $\|v_m\|_{1,p}$ is bounded from the Poincaré inequality (2.5).

Define the linear functional c on $W^{1,p}(\Omega)$ by

$$c(u) \;:=\; \int_\Omega c\, u \; dx. \tag{10.4}$$

Then $c(u_m) = \overline{c}\;|\Omega|\overline{u}_m + c(v_m)$. When (B3) holds, c will be continuous, so the fact that $|c(u_m)|$ is uniformly bounded implies that $|\overline{u}_m|$ is uniformly bounded. Hence the sequence $\{u_m : m \geq 1\}$ is bounded in $W^{1,p}(\Omega)$ from (6.7). The concluding arguments in the proof of theorem 6.2 apply again here to yield the proof of this theorem. □

This result implies that the expression

$$\|u\|_{c,p} \;:=\; \mathcal{F}_3(u)^{1/p} \tag{10.5}$$

defines a norm on $W^{1,p}(\Omega)$. This may be strengthened to the following

Corollary 10.2. *Assume (A1), (A3) and (B3) hold, $1 < p < \infty$, then the (c,p) norm defined by (10.5) is an equivalent norm to the standard norm on $W^{1,p}(\Omega)$.*

Proof. When (B3) holds and $u \in W^{1,p}(\Omega)$, then

$$\mathcal{F}_3(u) \;\leq\; \|\nabla u\|_p^p + \|c\|_{p^*}^p\,\|u\|_p^p \;\leq\; C\,\|u\|_{1,p}^p$$

with $C > 0$. Take $\rho \equiv 1$ in theorem 10.1 then, since $c \geq 0$ and (10.2) holds,

$$2\mathcal{F}_2(u) \;\geq \|\nabla u\|_p^p + C_P\|u\|_p^p \;\geq\; \min(1, C_P)\;\|u\|_{1,p}^p$$

These two inequalities imply the equivalence of these norms on $W^{1,p}(\Omega)$. □

When Ω_L is defined as in section 4, the scaled version of (10.2) is obtained by using the formulae in section 4 and multiplying by L^{n-p}. This yields

$$\int_{\Omega_L} \sum_{j=1}^n |D_j\, u|^p \; dy \;+\; L^{n-p(n+1)} \left| \int_{\Omega_L} c_L u \; dy \right|^p \;\geq\; C_P L^{-p} \int_{\Omega_L} \rho_L |u|^p \; dy. \tag{10.6}$$

for all $L > 0,\; u \in W^{1,p}(\Omega_L)$. Equality holds here for some functions in $W^{1,p}(\Omega_L)$.

For this case the analog of (5.13) is to find non-zero solutions (μ, u) in $\mathbb{R} \times W^{1,p}(\Omega)$ of

$$\int_\Omega \left[\sum_{j=1}^{n} |D_j u|^{p-2} D_j u D_j h \;-\; \mu\rho |u|^{p-2} u h \right] dx \;+\; |c(u)|^{p-2} c(u) c(h) \;=\; 0 \tag{10.7}$$

for all $h \in W^{1,p}(\Omega)$. When $p = 2$, this reduces to

$$\int_\Omega \left[\sum_{j=1}^{n} D_j u D_j h \;-\; \mu\rho u h \right] dx \;+\; c(u) c(h) \;=\; 0 \quad \text{for all } h \in H^1(\Omega). \tag{10.8}$$

This is the weak form of the eigenproblem

$$-\Delta u \;+\; c(u) c \;=\; \mu\, \rho u \qquad \text{in } \Omega \tag{10.9}$$

$$(\nabla u) \cdot \nu \;=\; 0 \qquad \text{on } \partial\Omega. \tag{10.10}$$

This is an eigenvalue problem for an integro-differential equation on Ω. It is worth noting that when the function c here is itself an eigenfunction of the Neumann problem for equation (7.4), then the eigenfunctions of this problem are precisely the eigenfunctions of the Neumann problem for (7.4) and only one eigenvalue is different.

For general $p > 1$, (10.7) is the weak form of an eigenvalue problem for the p-Laplacian on Ω. Namely one seeks non-zero solutions of the system

$$-\sum_{j=1}^{n} D_j(|D_j u|^{p-2} D_j u) \;+\; |c(u)|^{p-2} c(u) c \;=\; \mu\, \rho\, |u|^{p-2}\, u \qquad \text{in } \Omega \tag{10.11}$$

$$\sum_{j=1}^{n} (|D_j u|^{p-2} (D_j u) \nu_j \;=\; 0 \qquad \text{on } \partial\Omega. \tag{10.12}$$

11. Optimal Inequalities for Boxes

In her paper [6], Brenner describes discrete analogues of the boundary inequality (11.2) and the Poincaré inequality (11.3) below for 2d polygons and 3d polyhedra. To illustrate the preceding analysis and for comparison with the results in [6], we will find explicit formulae for the optimal constants when the region Ω is taken to be a rectangle. Take $p = 2$ and $\Omega \;:=\; (0, \pi) \times (0, h)$ with $h > 0$ being the height of the rectangle and let all the coefficient functions be identically 1.

The three different inequalities may be written

$$\int_\Omega \sum_{j=1}^n |D_j u|^2 \, dx \;+\; \int_{\partial\Omega} |u|^2 \, ds \;\ge\; C_F(h) \int_\Omega |u|^2 \, dx \tag{11.1}$$

$$\int_\Omega \sum_{j=1}^n |D_j u|^2 \, dx \;+\; \left| \int_{\partial\Omega} u \, ds \right|^2 \;\ge\; C_B(h) \int_\Omega |u|^2 \, dx, \quad \text{and} \tag{11.2}$$

$$\int_\Omega \sum_{j=1}^n |D_j u|^2 \, dx \;+\; \left| \int_\Omega u \, dx \right|^2 \;\ge\; C_P(h) \int_\Omega |u|^2 \, dx. \tag{11.3}$$

Here ds replaces $d\sigma$ as it represents arc-length on $\partial\Omega$,

From (7.4)-(7.4), the value of $C_F(h)$ in (11.1) is the least eigenvalue of the Robin-Laplacian problem

$$-\Delta u \;=\; \mu\, u \qquad \text{in } \Omega \text{ and} \qquad (\nabla u)\cdot\nu \,+\, u \;=\; 0 \qquad \text{on } \partial\Omega. \tag{11.4}$$

Similarly the value of $C_B(h)$ in (11.2) is the least eigenvalue of the Laplacian eigenproblem

$$-\Delta u \;=\; \mu\, u \qquad \text{in } \Omega \text{ and} \qquad (\nabla u)\cdot\nu \,+\, \int_{\partial\Omega} u \, ds \;=\; 0 \qquad \text{on } \partial\Omega. \tag{11.5}$$

Finally the value of $C_P(h)$ in (11.3) is the least eigenvalue of the modified Laplacian eigenproblem

$$-\Delta u \,+\, \left(\int_\Omega u \; dx \right) \;=\; \mu\, u \qquad \text{in } \Omega \text{ and} \qquad (\nabla u)\cdot\nu \;=\; 0 \qquad \text{on } \partial\Omega. \tag{11.6}$$

The eigenfunctions of (11.6) are precisely the eigenfunctions of the Neumann-Laplacian on Ω. The first non-zero eigenvalue of the Neumann-laplacian is $\lambda_1^{(N)} = \min(1, \pi/h^2)$. Thus the optimal constant in (11.3) for this rectangle is

$$C_P(h) \;=\; \min(h\pi, 1, \pi/h^2). \tag{11.7}$$

A careful analysis of a family of constrained variational principles for $C_B(h)$ leads to the result that

$$C_B(h) \;=\; \min(1, \pi/h^2). \tag{11.8}$$

The first eigenvalue, and the corresponding eigenfunction of (11.4) may also be found explicitly. It is

$$C_F(h) \;=\; k_0 + (k_1(h))^2 \tag{11.9}$$

where $k_0 = 0.40742$ and $0.5(\pi/h) < k_1(h) < (\pi/h)$. In fact k_0, k_1 are the smallest positive solutions respectively of

$$\tan k\pi \;=\; (-2k)/(1-k^2) \quad \text{and} \quad \tan kh \;=\; (-2k)/(1-k^2).$$

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Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA

E-mail address: auchmuty@uh.edu