SPECTRAL CHARACTERIZATION OF THE TRACE SPACES $H^s(\partial\Omega)$.

GILES AUCHMUTY

Abstract. The spaces $H^s(\partial\Omega)$ are described via Steklov eigenfunction expansions. This characterization provides explicit formulae for the inner products, norms and allows the description of explicit orthonormal bases for these spaces. It uses a special inner product on $H^1(\Omega)$, and properties of the harmonic Steklov eigenfunctions on the region. This approach only requires the boundary of the region be Lipschitz and generalizes the classical definitions that are used when the boundary is a smooth manifold.

1. Introduction

This paper develops a spectral characterization of Hilbert trace spaces on Lipschitz regions in $\mathbb{R}^n$. The description uses Steklov eigenfunction expansions and provides explicit formulae for the inner products, norms and orthonormal bases.

The approach used here is intrinsic and is based on the use of special inner products and related decompositions for $H^1(\Omega)$. The Steklov eigenfunctions then simultaneously provide orthogonal sets in $H^1(\Omega)$ and of $L^2(\partial\Omega, d\sigma)$. This enables the description of $H^s(\partial\Omega)$ for all real s. The resulting spaces form an interpolatory family of spaces and explicit formulae for the inner products in $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ are obtained. These results depend on completeness and related results developed in Auchmuty [4].

The usual theory of trace spaces as described in Adams and Fournier [2], Dautray and Lions [8], Lions and Magenes [12] or McLean [13] requires the use of Fourier transforms and local diffeomorphisms of domains onto a half-space. Such an approach is difficult to implement computationally and does not lead to a satisfactory approximation theory. The approach developed here has good constructive properties and is much more amenable to numerical simulation, see Kloucek et al [11].

2. Definitions and Notation.

This paper will first provide a spectral characterization of the trace spaces $H^s(\partial\Omega)$ when Ω is a regions in $\mathbb{R}^n$. A region is a non-empty, connected, open subset of $\mathbb{R}^n$. Its closure is denoted $\overline{\Omega}$ and its boundary is $\partial\Omega := \overline{\Omega} \setminus \Omega$. The basic assumption on this region is the following.

(B1): *Ω is a bounded region in $\mathbb{R}^n$ and its boundary $\partial\Omega$ is the union of a finite number of disjoint closed Lipschitz surfaces; each surface having finite surface area.*

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When this holds there is an outward unit normal ν defined at σ $a.e.$ point of $\partial\Omega$. The definitions and terminology of Evans and Gariepy [9], will generally be used except that $\sigma, d\sigma$ will represent Hausdorff $(n-1)-$dimensional measure and integration with respect to this measure respectively. All functions in this paper will take values in $\overline{\mathbb{R}} := [-\infty, \infty]$ and derivatives should be taken in a weak sense.

A real sequence $\{x_m : m \geq 1\}$ is said to be (strictly) increasing if $x_{m+1}(>) \geq x_m$ for all m. Similarly a function u is said to be (strictly) positive on a set E, if $u(x) \geq (>)0$ on E. The gradient of a function u will be denoted ∇u.

The real Lebesgue spaces $L^p(\Omega)$ and $L^p(\partial\Omega, d\sigma),\ 1 \leq p \leq \infty$ will be defined in the standard manner and have the usual p-norm denoted by $\|u\|_p$ and $\|u\|_{p,\partial\Omega}$ respectively. Their inner products are defined by

$$\langle u, v\rangle := \int_\Omega u(x)\ v(x)\ dx \qquad \text{and} \quad \langle u, v\rangle_{\partial\Omega} := |\partial\Omega|^{-1} \int_{\partial\Omega} u\ v\ d\sigma.$$

Let $H^1(\Omega)$ be the usual real Sobolev space of functions on Ω. It is a real Hilbert space under the standard H^1- inner product

$$[u, v]_1 := \int_\Omega [u(x).v(x)\ +\ \nabla u(x) \cdot \nabla v(x)]\ dx. \tag{2.1}$$

The corresponding norm will be denoted by $\|u\|_{1,2}$

The region Ω is said to satisfy *Rellich's theorem* provided the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ is compact for $1 \leq p < p_S$ where $p_S(n) \ := \ 2n/(n-2)$ when $n \geq 3$, or $p_S(2) = \infty$ when $n = 2$.

There are a number of different criteria on Ω and $\partial\Omega$ that imply this result. When (B1) holds it is theorem 1 in section 4.6 of [9]. See also Amick [1]. DiBenedetto [7], in theorem 14.1 of chapter 9 shows that the result holds when Ω is bounded and satisfies a "cone property". Adams and Fournier give a thorough treatment of conditions for this result in chapter 6 of [2] and show that it also holds for some classes of unbounded regions.

When (B1) holds and $u \in W^{1,1}(\Omega)$ then the trace of u on $\partial\Omega$ is well -defined and is a Lebesgue integrable function with respect to σ, see [9], Section 4.2 for details. The region Ω is said to satisfy a *compact trace theorem* provided the trace mapping $\Gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ is compact. The trace map is the linear extension of the map restricting Lipschitz continuous functions on $\overline{\Omega}$ to $\partial\Omega$. Occasionally u will be used in place of Γu for the trace of a function on $\partial\Omega$.

Evans and Gariepy [9], section 4.3 show that Γ is continuous when $\partial\Omega$ satisfies (B1). Theorem 1.5.1.10 of Grisvard [10] proves an inequality that implies the compact trace theorem when $\partial\Omega$ satisfies (B1). This inequality is also proved in [7], chapter 9, section 18 under stronger regularity conditions on the boundary.

Here we shall always require that the region satisfy

(B2): Ω *and* $\partial\Omega$ *satisfy (B1), the Rellich theorem and the compact trace theorem.*

Instead of (2.1), we will primarily use the $\partial-$inner product

$$[u,v]_\partial := \int_\Omega \nabla u \cdot \nabla v \, dx \, + \, |\partial\Omega|^{-1} \int_{\partial\Omega} u \, v \, d\sigma. \tag{2.2}$$

The corresponding norm will be denoted by $\|u\|_\partial$. When (B2) holds, this norm is equivalent to the usual $(1,2)-$norm on $H^1(\Omega)$. This is proved in Corollary 6.2 of [4] and is part of theorem 21A of [14].

A function $u \in H^1(\Omega)$ is said to be *harmonic on* Ω provided it is a solution of Laplace's equation in the usual weak sense. Namely

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx \; = \; 0 \qquad \text{for all } \varphi \in C_c^1(\Omega). \tag{2.3}$$

Here $C_c^1(\Omega)$ is the set of all C^1-functions on Ω with compact support in Ω.

Define $\mathcal{H}(\Omega)$ to be the space of all such harmonic functions on Ω. When (B1) holds, the closure of $C_c^1(\Omega)$ in the H^1-norm is the usual Sobolev space $H_0^1(\Omega)$. Then (2.3) is equivalent to saying that $\mathcal{H}(\Omega)$ is $\partial-$orthogonal to $H_0^1(\Omega)$. This may be expressed as

$$H^1(\Omega) \; = \; H_0^1(\Omega) \oplus_\partial \mathcal{H}(\Omega), \tag{2.4}$$

where $\oplus_\partial$ indicates that this is a $\partial-$orthogonal decomposition. This result is also discussed in section 22.4 of [14].

In this paper we shall use various standard results from the calculus of variations and convex analysis. Background material on such methods may be found in Blanchard and Bruning [6] or Zeidler [15], both of which have discussions of the variational principles for the Dirichlet eigenvalues and eigenfunctions of second order elliptic operators. The variational principles used here are variants of the principles described there and are analogous to those for the Laplacian described in section 5 of Auchmuty [3].

In this paper all the variational principles, and functionals will be defined on (closed convex subsets of) $H^1(\Omega)$. When $\mathcal{F} : H^1(\Omega) \to (-\infty, \infty]$ is a functional, then $\mathcal{F}$ is said to be *G-differentiable* at a point $u \in H^1(\Omega)$ if there is a $\mathcal{F}'(u)$ such that

$$\lim_{t \to 0} t^{-1} \left[\mathcal{F}(u + tv) - \mathcal{F}(u)\right] \; = \; \mathcal{F}'(u)(v) \qquad \text{for all } v \in H^1(\Omega),$$

with $\mathcal{F}'(u)$ a continuous linear functional on $H^1(\Omega)$. In this case, $\mathcal{F}'(u)$ is called the G-derivative of $\mathcal{F}$ at u.

3. The Harmonic Steklov Eigenproblem

Assume Ω is a region in $\mathbb{R}^n$ which satisfies (B2). A non-zero function $s \in H^1(\Omega)$ is said to be a *harmonic Steklov eigenfunction* on Ω corresponding to the Steklov eigenvalue δ provided s satisfies

$$\int_\Omega \nabla s \cdot \nabla v \, dx \; = \; \delta \, |\partial\Omega|^{-1} \int_{\partial\Omega} s \, v \, d\sigma \qquad \text{for all} \quad v \in H^1(\Omega). \tag{3.1}$$

This is the weak form of the boundary value problem

$$\Delta s \;=\; 0 \qquad \text{on } \Omega \text{ with } \quad D_\nu\, s \;=\; \delta\, |\partial\Omega|^{-1}\, s \quad \text{on } \partial\Omega. \tag{3.2}$$

Here Δ is the Laplacian and $D_\nu\, s(x) := \nabla s(x) \cdot \nu(x)$ is the unit outward normal derivative of s at a point x on the boundary.

$\delta_0 = 0$ is the least eigenvalue of this problem corresponding to the eigenfunction $s_0(x) \equiv 1$ on Ω. This eigenvalue is simple as Ω is connected. All other eigenvalues of (3.1) are strictly positive.

These eigenvalues and a corresponding family of $\partial-$orthonormal eigenfunctions may be found using variational principles as described in sections 6 and 7 of [4]. A different variational description is developed in Bandle [5], Chapter 3. Let the first k Steklov eigenvalues be $0 = \delta_0 < \delta_1 \leq \delta_2 \leq \ldots \leq \delta_{k-1}$ and $s_0, s_1, \ldots, s_{k-1}$ be a corresponding set of $\partial-$orthonormal eigenfunctions. The k-th eigenfunction s_k will be a maximizer of the functional

$$\mathcal{B}(u) \;:=\; |\partial\Omega|^{-1} \int_{\partial\Omega} u^2\, d\sigma \tag{3.3}$$

over the subset B_k of functions in $H^1(\Omega)$ which satisfy

$$\|s\|_\partial \;\leq\; 1 \qquad \text{and} \qquad [s, s_l]_\partial \;=\; 0 \quad \text{for} \quad 0 \leq l \leq k-1. \tag{3.4}$$

The existence and some properties of these eigenfunctions are described in sections 6 and 7 of [4]. In particular, that analysis shows that each δ_j is of finite multiplicity and $\delta_j \to \infty$ as $j \to \infty$; see Theorem 7.2 of [4]. Let $\mathcal{S} := \{s_j : j \geq 0\}$ be the maximal family of $\partial-$orthonormal eigenfunctions constructed inductively as above. For each $u \in H^1(\Omega)$, consider the series

$$P_H\, u(x) \;:=\; \sum_{j=0}^{\infty} [u, s_j]_\partial\; s_j(x) \tag{3.5}$$

Theorem 3.1. *Assume $\Omega, \partial\Omega$ satisfy (B2) and P_H is defined by (3.5), then P_H is the $\partial-$orthogonal projection of $H^1(\Omega)$ onto $\mathcal{H}(\Omega)$.*

Proof. This follows from standard results about orthogonal expansions and theorem 7.3 of [4] which says that $\mathcal{S}$ is a maximal orthonormal subset of $\mathcal{H}(\Omega)$. □

An expression of the form

$$v(x) \;:=\; \sum_{j=0}^{\infty} c_j\, s_j(x) \qquad \text{with } c_j := [v, s_j]_\partial \tag{3.6}$$

will be called a harmonic Steklov expansion and, since $\mathcal{S}$ is a basis of $\mathcal{H}(\Omega)$, the Riesz-Fischer theorem implies that it represents an H^1- harmonic function on Ω if and only if

$$\sum_{j=0}^{\infty} |c_j|^2 \;<\; \infty. \tag{3.7}$$

As described in section 8 of [4], the Steklov eigenfunctions on the unit disc in $\mathbb{R}^2$ are the functions $r^k \cos k\theta$ and $r^k \sin k\theta$ so the above series are familiar from classical treatments of harmonic functions on a disc. Similarly when Ω is the unit ball in $\mathbb{R}^3$ the Steklov eigenfunctions are spherical harmonics and these series generalize some common expansions in classical mathematical physics.

4. A Spectral Representation of the Trace Operator

The Steklov eigenfunctions s_j described in the preceding section have L^2 traces on the boundary $\partial\Omega$ whenever (B1) holds. Define

$$\hat{s_j}(x) \ := \ \sqrt{1+\delta_j}\ \Gamma\, s_j(x) \qquad \text{for } x \in \partial\Omega, \ \text{ and } \ j \geq 0. \tag{4.1}$$

Then (3.1) and (3.4) imply that the set $\hat{\mathcal{S}} := \{\hat{s_j} : j \geq 0\}$ will be an orthonormal set in $L^2(\partial\Omega, d\sigma)$ with respect to the inner product defined in section 2. The following result provides an explicit expression for the trace operator in terms of the harmonic Steklov expansion of a function $u \in H^1(\Omega)$.

Theorem 4.1. *Assume $\Omega, \partial\Omega$ satisfy (B2), with $\Gamma, \hat{\mathcal{S}}$ as above. Then $\hat{\mathcal{S}}$ is a maximal orthonormal set in $L^2(\partial\Omega, d\sigma)$ and*

$$\Gamma u \ = \ \sum_{j=0}^{\infty} \ (1+\delta_j)^{-1/2}\ [u, s_j]_\partial\ \hat{s_j} \quad \textit{for each } u \in H^1(\Omega). \tag{4.2}$$

Proof. The first claim is a special case of theorem 9.4 in [4]. The null space of the operator Γ is $H_0^1(\Omega)$ from theorem 3.40 of [13]. Hence, from (2.4) and theorem 3.1, $\Gamma u = \Gamma P_H\, u$, where P_H is the projection onto the space $\mathcal{H}(\Omega)$. (4.2) then follows from (4.1). □

Apply Parseval's identity to (4.2) then

$$\|\Gamma\, u\|_{\partial\Omega}^2 \ := \ |\partial\Omega|^{-1} \int_{\partial\Omega} \ |\Gamma\, u|^2\, d\sigma \ = \ \sum_{j=0}^{\infty} \ (1+\delta_j)^{-1}\ [u, s_j]_\partial{}^2. \tag{4.3}$$

for any $u \in H^1(\Omega)$, since $\hat{\mathcal{S}}$ is a basis of $L^2(\partial\Omega, d\sigma)$.

Suppose now that $g = \Gamma\, u$ for some $u \in H^1(\Omega)$, then $g \in L^2(\partial\Omega, d\sigma)$ and

$$g(x) \ = \ \sum_{j=0}^{\infty} \ g_j\ \hat{s_j}(x) \qquad \text{with } g_j \ = \ \langle g, \hat{s_j}\rangle_{\partial\Omega}. \tag{4.4}$$

(4.2) and the orthonormality of $\hat{\mathcal{S}}$ implies that

$$[u, s_j]_\partial \ = \ (1+\delta_j)^{1/2}\ g_j \qquad \text{for all } j \geq 0.$$

Let g_M be the M-th partial sum of the series (4.4) and consider the map $E_M : L^2(\partial\Omega, d\sigma) \to \mathcal{H}(\Omega)$ defined by

$$E_M\, g(x) \ := \ \sum_{j=0}^{M} \ (1+\delta_j)^{1/2} \ \ g_j\, s_j(x). \tag{4.5}$$

This is a harmonic function on Ω with boundary trace g_M. Define

$$E\, g(x) \ := \ \lim_{M\to\infty} \ E_M\, g(x) \tag{4.6}$$

then (3.7) shows that Eg is in $\mathcal{H}(\Omega)$ if and only if

$$\sum_{j=0}^{\infty} (1+\delta_j)\, |g_j|^2 \ < \ \infty. \tag{4.7}$$

Formally E defines a harmonic extension of the boundary data g to Ω.

5. A Spectral Definition of $H^s(\partial\Omega)$

The classical description of trace theorems for H^1- functions on a region requires the description of the boundary using local coordinates and mappings from canonical regions such as a half-space. See chapter 3 of McLean [13] for a detailed description under weak regularity conditions. A comparison of a number of methods for defining these spaces is given in the appendix to chapter 4, volume 2 of [8]. Here a very different definition will be described which should be much more useful for computational purposes. It uses an intrinsic characterization so that no mappings, or special representations, of the region are required.

Specifically $H^s(\partial\Omega)$ is defined as that subspace of $L^2(\partial\Omega, d\sigma)$ of functions whose harmonic Steklov coefficients satisfy certain summability conditions. For $s \geq 0$, we define $H^s(\partial\Omega)$ to be the subspace of all functions $g \in L^2(\partial\Omega, d\sigma)$ with Steklov expansion (4.4), satisfying

$$\sum_{j=0}^{\infty} \ (1+\delta_j)^{2s}\, |g_j|^2 \ < \ \infty. \tag{5.1}$$

Define the *s-inner product* and *s-norm* on $H^s(\partial\Omega)$ by

$$[g,h]_{s,\partial\Omega} \ := \ \sum_{j=0}^{\infty} \ (1+\delta_j)^{2s}\, g_j\, h_j \qquad \text{and} \qquad \|g\|_{s,\partial\Omega}^2 \ := \ \sum_{j=0}^{\infty} \ (1+\delta_j)^{2s}\, {g_j}^2. \tag{5.2}$$

When $s = 0$, one sees that $H^0(\partial\Omega) = L^2(\partial\Omega, d\sigma)$.

When $s = 1/2$, (4.7) shows that the space $H^{1/2}(\partial\Omega)$ will be precisely the class of all boundary values of H^1- functions on Ω - so this definition agrees with the classical definition based on Fourier methods when $\partial\Omega$ is a smooth manifold.

Note that this definition of the spaces $H^s(\partial\Omega)$ only requires that $\partial\Omega$ be smooth enough for the Steklov eigenanalysis to hold. The following results show that this definition satisfies the same intermediate space properties as the original definitions of Lions and Magenes [12] chapter 1, section 7.3 - which required that the boundary be a $C^\infty-$ manifold.

Theorem 5.1. *Assume that $\Omega, \partial\Omega$ satisfy (B2) and $H^s(\partial\Omega)$ is defined as above. If $0 \leq s_1 < s_2$, then $H^{s_2}(\partial\Omega)$ is a dense subspace of $H^{s_1}(\partial\Omega)$ and the imbedding of $H^{s_2}(\partial\Omega)$ into $H^{s_1}(\partial\Omega)$ is compact.*

Proof. For $M \geq 1$, let $P_M : L^2(\partial\Omega, d\sigma) \to L^2(\partial\Omega, d\sigma)$ be the finite rank operator corresponding to the M-th partial sum of the Steklov expansion (4.4). That is

$$P_M\, g(x) \;:=\; \sum_{j=0}^{M} \langle g, \hat{s}_j\rangle_{\partial\Omega}\; \hat{s}_j(x) \qquad \text{for } g \in L^2(\partial\Omega, d\sigma). \tag{5.3}$$

Obviously $P_M\, g \in H^s(\partial\Omega)$ for all $s \geq 0$ and the definition (5.2) yields that

$$\|g\|_{s_1,\partial\Omega} \;\leq\; \|g\|_{s_2,\partial\Omega} \qquad \text{whenever} \quad 0 \leq s_1 < s_2. \tag{5.4}$$

Given $g \in H^{s_1}(\partial\Omega)$, then the sequence $\{P_M\, g : M \geq 1\}$ is a subset of $H^{s_2}(\partial\Omega)$ which converges to g in $H^{s_1}(\partial\Omega)$. Thus $H^{s_2}(\partial\Omega)$ is a dense subspace of $H^{s_1}(\partial\Omega)$.

Consider the linear map $L_\theta : L^2(\partial\Omega, d\sigma) \to L^2(\partial\Omega, d\sigma)$ defined by

$$L_\theta\, g(x) \;:=\; \sum_{j=0}^{\infty} (1+\delta_j)^{-\theta}\; \langle g, \hat{s}_j\rangle_{\partial\Omega}\; \hat{s}_j(x). \tag{5.5}$$

For $\theta > 0$, using the fact that $\delta_j \to \infty$, L_θ is a compact linear operator as it may be uniformly approximated by a finite rank operator. Moreover

$$\|L_\theta\, g\|^2_{s,\partial\Omega} \;=\; \sum_{j=0}^{\infty} (1+\delta_j)^{2(s-\theta)}\; \langle g, \hat{s}_j\rangle^2_{\partial\Omega}. \tag{5.6}$$

Thus L_θ is a linear isometry of $L^2(\partial\Omega, d\sigma)$ onto $H^\theta(\partial\Omega)$ so the imbedding of $H^s(\partial\Omega)$ into $L^2(\partial\Omega, d\sigma)$ is compact whenever $s > 0$. A translation in s, then yields that the imbedding of $H^{s_2}(\partial\Omega)$ into $H^{s_1}(\partial\Omega)$ is compact whenever $s_1 < s_2$. □

The family of spaces $H^s(\partial\Omega)$ with $s \geq 0$ form an interpolatory family (or scale) of real Hilbert spaces as these s-norms satisfy the following log-convexity inequality.

Theorem 5.2. *Assume that $\Omega, \partial\Omega$ satisfy (B2) and $H^s(\partial\Omega)$ is defined as above. If $0 \leq s_1 < s_2$ and $s = (1-\theta)s_1 + \theta s_2$ with $0 \leq \theta \leq 1$, then*

$$\|g\|_{s,\partial\Omega} \;\leq\; \|g\|^{1-\theta}_{s_1,\partial\Omega}\; \|g\|^{\theta}_{s_2,\partial\Omega} \qquad \textit{for all} \quad g \in H^{s_2}(\partial\Omega). \tag{5.7}$$

Proof. This is obviously true when $\theta = 0$ or 1. Assume $0 < \theta < 1$, then from (5.2),

$$\|g\|^2_{s,\partial\Omega} \;:=\; \sum_{j=0}^{\infty} (1+\delta_j)^{2s}\, {g_j}^2. \tag{5.8}$$

Factor each term in the sum, so that $(1+\delta_j)^{2s}\, {g_j}^2 \;=\; c_j\, d_j$ with

$$c_j := (1+\delta_j)^{s_1(1-\theta)}\, {g_j}^{2(1-\theta)}, \qquad d_j := (1+\delta_j)^{s_2\theta}\, {g_j}^{2\theta}.$$

Apply Holder's inequality to (5.8) with $p := 1/(1-\theta), p^* := \theta^{-1}$, then (5.7) follows. □

Suppose F is a continuous linear functional on $H^s(\partial\Omega)$ with $s \;\geq\; 0$. F will be *represented by* a function $f \in L^2(\partial\Omega, d\sigma)$ provided

$$F(g) \;=\; |\partial\Omega|^{-1} \int_{\partial\Omega} f\, g\; d\sigma \;=\; \langle f, g\rangle_{\partial\Omega} \qquad \text{for all} \quad g \in H^s(\partial\Omega). \tag{5.9}$$

When f, g have Steklov expansions of the form (4.4) with Steklov coefficients f_j, g_j, then this becomes

$$F(g) \;=\; \sum_{j=0}^{\infty} f_j\, g_j \tag{5.10}$$

Use of Schwarz' inequality here shows that each $f \in L^2(\partial\Omega, d\sigma)$ represents a continuous linear functional on such $H^s(\partial\Omega)$.

For $s < 0$, define the space $H^s(\partial\Omega)$ to be the completion of the space $L^2(\partial\Omega, d\sigma)$ with respect to the inner product and norm of (5.2). Elements of this space will be called *generalized functions* on $\partial\Omega$. Below we shall show that $H^{-s}(\partial\Omega)$ is precisely the dual space of $H^s(\partial\Omega)$ with respect to the pairing induced by the L^2-inner product on $\partial\Omega$. When $\partial\Omega$ is a C^∞ manifold, these will be spaces of distributions on $\partial\Omega$.

It is straightforward to verify that $H^s(\partial\Omega)$ with $s < 0$ is a real Hilbert space under the inner product of (5.2). The following theorem specifies the duality relationship.

Theorem 5.3. *Assume that $\Omega, \partial\Omega$ satisfy (B2), $H^s(\partial\Omega)$ is defined as above with $s > 0$ and F is a continuous linear functional on $H^s(\partial\Omega)$. Then there is a unique generalized function $f \in H^{-s}(\partial\Omega)$ such that*

$$F(g) \;=\; \langle f, g\rangle_{\partial\Omega} \qquad \textit{for all} \quad g \in H^s(\partial\Omega). \tag{5.11}$$

Moreover the dual norm of F is $\|f\|_{-s,\partial\Omega}$.

Proof. Rewrite each term in the sum (5.10) as the product of

$$c_j := \mu_j^{-1}\, f_j \quad \text{and} \quad d_j := \mu_j\, g_j \qquad \text{with} \quad \mu_j := (1+\delta_j)^s.$$

Apply Schwarz' inequality to (5.10), then the definitions of the norms yield

$$|F(g)| \;\leq\; \|f\|_{-s,\partial\Omega}\, \|g\|_{s,\partial\Omega} \tag{5.12}$$

Moreover equality holds here whenever $f_j = (1+\delta_j)^2\, g_j$ for all $j \geq 0$. Since F is continuous if and only if it is bounded, we see that each continuous linear functional on $H^s(\partial\Omega)$ will be represented by a generalized function in $H^{-s}(\partial\Omega)$. The dual norm is defined by

$$\|F\|_{*s} \;:=\; \sup_{\|g\|_{s,\partial\Omega} \;\leq\; 1} \quad |F(g)|.$$

so (5.12) shows that it is given by the norm on $H^{-s}(\partial\Omega)$. □

6. Spectral Representation of the Normal Derivative

When $\partial\Omega$ is locally a C^1-manifold, then the exterior unit normal ν is a continuous vector field. The exterior normal derivative of a C^1-function u on $\overline{\Omega}$ is then given by

$$D_\nu \, u(x) \ := \ \nabla u(x) \cdot \nu(x) \qquad \text{for } x \in \partial\Omega. \tag{6.1}$$

When the Steklov eigenfunctions are sufficiently smooth, then (3.2) and (4.1) imply that

$$D_\nu \, s_j(x) \ = \ \frac{\delta_j}{|\partial\Omega| \sqrt{(1+\delta_j)}} \ \hat{s}_j(x) \qquad \text{for } x \in \partial\Omega \text{ and } j \geq 0. \tag{6.2}$$

Take this to hold for each Steklov eigenfunction. When $v \in \mathcal{H}(\Omega)$ has a Steklov expansion of the form (3.6), define the linear extension of D_ν to be

$$D_\nu \, v \ = \ |\partial\Omega|^{-1} \sum_{j=1}^{\infty} \frac{\delta_j}{\sqrt{(1+\delta_j)}} \ [v, s_j]_\partial \ \hat{s}_j. \tag{6.3}$$

The H^s- norm of this generalized function on $\partial\Omega$ is

$$\|D_\nu \, v\|^2_{s,\partial\Omega} \ = \ |\partial\Omega|^{-2} \sum_{j=1}^{\infty} \ (1+\delta_j)^{2s-1} {\delta_j}^2 \ [v, s_j]^2_\partial. \tag{6.4}$$

In view of this calculation, this operator satisfies the following.

Theorem 6.1. *Assume that $\Omega, \partial\Omega$ satisfy (B2), $H^s(\partial\Omega)$ is defined in section 5. Then the operator D_ν defined by (6.3) is a continuous map from $\mathcal{H}(\Omega)$ to $H^s(\partial\Omega)$ for $s \leq -1/2$.*

Proof. From (6.4), the operator will be continuous if and only if there is a constant $C > 0$ such that

$$(1+\delta_j)^{2s-1} {\delta_j}^2 \ \leq \ C \qquad \text{for all } j \geq 0.$$

Since $\delta_j \to \infty$, this holds if and only if $s \leq -1/2$. □

7. Explicit Inner Product on $H^{1/2}(\partial\Omega)$.

When $f, g \in H^{1/2}(\partial\Omega)$, the inner product on $H^{1/2}(\partial\Omega)$ was defined in section 5 in terms of a Steklov series expansion. Here it will be shown to have an expression in terms of the boundary trace and a normal derivative.

Given $g \in H^{1/2}(\partial\Omega)$, let Eg be its harmonic extension in $\mathcal{H}(\Omega)$ defined by (4.6). Then the outward normal derivative $D_\nu \, Eg$ will be in $H^{-1/2}(\partial\Omega)$ from theorem 6.1.

Theorem 7.1. *Assume that $\Omega, \partial\Omega$ satisfy (B2) and $H^{1/2}(\partial\Omega)$ is defined as above. Then E is an linear isometry from $H^{1/2}(\partial\Omega)$ to $\mathcal{H}(\Omega)$ and*

$$[f, g]_{1/2,\partial\Omega} \ = \ \langle f, g \ + \ |\partial\Omega| \, D_\nu \, Eg\rangle_{\partial\Omega} \qquad \textit{for all } f, g \in H^{1/2}(\partial\Omega). \tag{7.1}$$

Proof. From (4.6),

$$E\,g(x) \;=\; \sum_{j=0}^{\infty} \,(1+\delta_j)^{1/2}\;\, g_j\, s_j(x), \qquad \text{so}$$

$$\|Eg\|_{\partial}^2 \;=\; \sum_{j=0}^{\infty} (1+\delta_j)\,|g_j|^2 \;=\; \|g\|_{1/2,\partial\Omega}^2 \qquad \text{for all } g \in H^{1/2}(\partial\Omega). \tag{7.2}$$

Hence E is an isometry as claimed. Substitute for Eg in (6.3), then

$$D_\nu\, E\, g(x) \;=\; |\partial\Omega|^{-1} \,\sum_{j=0}^{\infty}\, \delta_j\;\, g_j\, \hat{s_j}(x).$$

This and the orthonormality of $\hat{\mathcal{S}}$, yields that

$$\langle f, g \;+\; |\partial\Omega|\, D_\nu\, Eg\rangle_{\partial\Omega} \;=\; \sum_{j=0}^{\infty}\; f_j\,(1+\delta_j)\,g_j \tag{7.3}$$

which is (7.1). □

This result (7.1) may be written formally as

$$[f,g]_{1/2,\partial\Omega} \;=\; \int_{\partial\Omega} f\,(|\partial\Omega|^{-1}\,g \;+\; D_\nu g)\;d\sigma$$

so the $(1/2)-$norm is defined by the quadratic form

$$\|g\|_{1/2,\partial\Omega}^2 \;=\; \int_{\partial\Omega} [\,|\partial\Omega|^{-1}\,g^2 \;+\; g\,D_\nu g\,]\;d\sigma. \tag{7.4}$$

That is, $H^{1/2}(\partial\Omega)$ is the space of all functions in $L^2(\partial\Omega, d\sigma)$ for which this quadratic form is finite. Here $D_\nu g$ is actually the outward normal derivative of the harmonic extension of g to Ω.

8. The Inner Product on $H^{-1/2}(\partial\Omega)$.

The space $H^{-1/2}(\partial\Omega)$ was defined as the completion of $L^2(\partial\Omega, d\sigma)$ with respect to the inner product defined by (5.2) with $s = -1/2$. In this section, this inner product will be characterized in terms of the solution of a Robin boundary value problem for Laplace's equation. More specifically, it will be described using a variational principle for such solutions.

Given $g \in H^{-1/2}(\partial\Omega)$, define the functional $\mathcal{D} : H^1(\Omega) \to \mathbb{R}$ by

$$\mathcal{D}(u) \;:=\; \int_\Omega |\nabla u|^2\;dx \;+\; |\partial\Omega|^{-1} \int_{\partial\Omega}\; |\Gamma u|^2\,d\sigma \;-\; 2\;\langle g, \Gamma u\rangle_{\partial\Omega}. \tag{8.1}$$

Consider the variational principle of minimizing $\mathcal{D}$ on $H^1(\Omega)$ and determining

$$q(g) \;:=\; \inf_{u\in H^1(\Omega)} \; \mathcal{D}(u). \tag{8.2}$$

The essential results about this problem can be stated as follows.

Theorem 8.1. *Assume that $\Omega, \partial\Omega$ satisfy (B2), $g \in H^{-1/2}(\partial\Omega)$ and $\mathcal{D}$ is defined by (8.1). Then there is a unique minimizer $\hat{u}$ of $\mathcal{D}$ on $H^1(\Omega)$ and it satisfies*

$$\int_\Omega \nabla u \cdot \nabla v \; dx \;+\; \langle (\Gamma\, u \,-\, g), \Gamma\, v\rangle_{\partial\Omega} \;=\; 0 \qquad \textit{for all} \quad v \in H^1(\Omega). \tag{8.3}$$

Proof. The existence of a unique minimizer of $\mathcal{D}$ on $H^1(\Omega)$ is theorem 9.2 of [4] with $\tau = 1/2$ and $g \in L^2(\partial\Omega, d\sigma)$. The extension to $g \in H^{-1/2}(\partial\Omega)$ is straightforward.

This functional $\mathcal{D}$ is convex and G-differentiable on $H^1(\Omega)$ and its derivative can be obtained from theorems 3.1 and 6.1 of [4]. The left hand side of (8.3) is the directional derivative of $\mathcal{D}$, so the extremality conditions imply that $\hat{u}$ will be a minimizer of $\mathcal{D}$ on $H^1(\Omega)$ if and only if it is a solution of (8.3). □

Note that equation (8.3) is the weak form of the Robin boundary value problem

$$\Delta\, u \;=\; 0 \qquad \text{on } \Omega \text{ with} \qquad |\partial\Omega|\, D_\nu\, u \,+\, u \;=\; g \quad \text{on } \partial\Omega. \tag{8.4}$$

When $v \in H^1_0(\Omega)$, then $\Gamma v \equiv 0$ so (8.3) implies that (2.3) holds or the solution $\hat{u}$ is harmonic on Ω. Let $\hat{u}$ have a Steklov expansion of the form

$$\hat{u}(x) \;:=\; \sum_{j=0}^{\infty} \; u_j\, s_j(x) \qquad \text{on } \Omega. \tag{8.5}$$

Then, from (4.2), the boundary trace $\Gamma\,\hat{u}$ is given by

$$\Gamma\,\hat{u} \;:=\; \sum_{j=0}^{\infty} \; \frac{u_j}{\sqrt{(1+\delta_j)}}\,\hat{s}_j. \tag{8.6}$$

Assume that $g \in H^{-1/2}(\partial\Omega)$ has the Steklov representation

$$g \;:=\; \sum_{j=0}^{\infty} \; g_j\,\hat{s}_j \qquad \text{with } g_j \;:=\; \langle g, \hat{s}_j\rangle_{\partial\Omega}. \tag{8.7}$$

Substitute $v \;=\; s_0$ in (8.3) to find that $u_0 \;=\; g_0$.

For $k \geq 1$, put $v \;=\; s_k$ in (3.1), then

$$\int_\Omega \nabla u \cdot \nabla s_k \; dx \;=\; \delta_k\; \langle s_k, \Gamma\, u\rangle_{\partial\Omega} \qquad \text{for all} \quad u \in H^1(\Omega). \tag{8.8}$$

Substitute this in (8.3) with $v \;=\; s_k$, to obtain

$$(1 \,+\, \delta_k)\; \langle \Gamma\hat{u}, \hat{s_k}\rangle_{\partial\Omega} \;=\; \langle g, \hat{s_k}\rangle_{\partial\Omega} \qquad \text{for all } k \geq 1. \tag{8.9}$$

The expression (8.6) for $\Gamma\hat{u}$, yields that the Steklov coefficients u_k of the solution of this variational problem are given by

$$u_k \;=\; (1 \,+\, \delta_k)^{-1/2}\; g_k \qquad \text{for each } k \geq 0. \tag{8.10}$$

Define $G_R \,:\, H^{-1/2}(\partial\Omega) \;\to\; \mathcal{H}(\Omega)$ to be the solution operator of this variational problem. Equations (8.5) and (8.10) show that G_R has the Steklov spectral representation

$$\hat{u}(x) \;:=\; G_R\, g(x) \;=\; \sum_{k=0}^{\infty} (1 \,+\, \delta_k)^{-1/2}\; g_k\; s_k(x). \tag{8.11}$$

for any $g \in H^{-1/2}(\partial\Omega)$ as in (8.7). The boundary trace of this function will be

$$\Gamma G_R\, g \;=\; \sum_{k=0}^{\infty} (1 \,+\, \delta_k)^{-1}\; g_k\; \hat{s_k}. \tag{8.12}$$

Moreover a straightforward computation shows that

$$\|\Gamma G_R\, g\|_{1/2,\partial\Omega} \;=\; \|g\|_{-1/2,\partial\Omega}. \tag{8.13}$$

so this operator ΓG_R is an isometric linear mapping of $H^{-1/2}(\partial\Omega)$ onto $H^{1/2}(\partial\Omega)$ More generally ΓG_R will be an isometry from any space $H^s(\partial\Omega)$ onto $H^{s+1}(\partial\Omega)$.

This, together with the orthonormality of $\hat{\mathcal{S}}$, proves the following theorem

Theorem 8.2. *Assume that $\Omega, \partial\Omega$ satisfy (B2) and G_R is the operator defined by (8.12). Then the inner product on $H^{-1/2}(\partial\Omega)$ obeys*

$$[f, g]_{-1/2,\partial\Omega} \;=\; \langle f, \Gamma\, G_R\, g\rangle_{\partial\Omega} \qquad \textit{for all } f, g \in H^{-1/2}(\partial\Omega). \tag{8.14}$$

For other negative values of s, the inner products on $H^s(\partial\Omega)$ may be defined using fractional powers of the operator ΓG_R.

References

[1] C. J. Amick, "Some remarks on Rellich's theorem and the Poincaré inequality," J. London Math. Soc. (2) **18** (1973), 81-93.

[2] R.A. Adams and J.J.F. Fournier, *Sobolev spaces, 2nd ed.*, Academic Press, 2003.

[3] G. Auchmuty, "The Main inequality of Vector Analysis" Math. Models and Methods in Applied Sciences, **14**, (2004) 1-25.

[4] G. Auchmuty, "Steklov Eigenproblems and the Representation of Solutions of Elliptic Boundary Value Problems", Numerical Functional Analysis and Optimization, **25** (2004) 321-348.

[5] C. Bandle *Isoperimetric Inequalities and Applications* Pitman, London 1980.

[6] P. Blanchard and E. Brüning, *Variational Methods in Mathematical Physics*, Springer Verlag, Berlin (1992).

[7] E. DiBenedetto, *Real Analysis*, Birkhauser, Boston, (2001).

[8] R. Dautray and J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol 2, Springer Verlag, Berlin (1988).

[9] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton (1992).

[10] P. Grisvard, *Elliptic problems in non-smooth domains* Pitman, Boston, (1985).
[11] Kloucek, P.
[12] J.L. Lions and E. Magenes, *Problemes aux limites non homogenes et applications, volume 1*, Dunod, Paris (1968).
[13] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge U. Press, (2000).
[14] E. Zeidler, *Nonlinear Functional Analysis and its Applications, IIA:* , Springer Verlag, New York (1985).
[15] E. Zeidler, *Nonlinear Functional Analysis and its Applications, III: Variational Methods and Optimization*, Springer Verlag, New York (1985).

Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA

E-mail address: auchmuty@uh.edu