

1 VARIATIONAL PRINCIPLES FOR SELF-ADJOINT ELLIPTIC EIGENPROBLEMS

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1.1 INTRODUCTION

Variational principles which characterize the eigenvalues and eigenfunctions of self-adjoint linear differential operators have been studied for over a century. Most of them are related to Rayleigh's principle which is usually regarded as a constrained variational problem of extremizing a quadratic form subject to the constraint that another quadratic form is held constant. These constrained extrema are the eigenfunctions and the eigenvalue are given both by the value of the problem and the Lagrange multipliers associated with the constraint. Some texts which describe these principles include Bandle, 1980, Chatelin, 1983, Courant and Hilbert, 1958, Gould, 1966, Mikhlin, 1964 and Weinberger, 1974.

Here we shall describe, and analyze, some unconstrained variational problems which identify the eigenvalues and eigenvectors of self-adjoint linear operators. They involve functionals which are bounded below, non-convex and not quadratic. The critical points of these functions are eigenfunctions of the operator, while the critical values and the norms of the critical points are related to the corresponding eigenvalue. The minimum value of the functional is related to the least eigenvalue of the operator and the nonzero minimizers are eigenfunctions corresponding to this eigenvalue.

The functional has well defined second derivatives. Thus we shall describe a (Morse-) index for the critical points and a Morse theory which enables the identification of the eigenvalue associated with a particular critical point. A non-degenerate critical point of index j of the functional is a simple eigenfunction of the operator corresponding to the $(j + 1)$ -st smallest eigenvalue. This index theory plays the same role as, but is more straightforward than, the Courant-Fischer-Weyl minimax theory for the Rayleigh quotient.

The theory developed here uses works for closed, self-adjoint linear operators on a Hilbert space. The conditions imposed, however, are those appropriate for studying the spectral theory of elliptic boundary value problems. This is done, for linear second order equations, with possibly indefinite weight functions, in section 1.8.

The author has described, in Auchmuty, 1989a and Auchmuty, 1989b, related principles for characterizing the eigenvalues and eigenvectors of matrices and of compact linear operators. Some other functionals for these problems were

analyzed in Auchmuty, 1986, where they were obtained using nonconvex duality theory.

Our first interest is in studying regular, and weighted, eigenproblems for a closed, self-adjoint linear operator which is semibounded and has compact resolvent. In section 1.2, some basic results on the quadratic forms and functionals associated with these operators are described. The results are needed for the application of methods from convex analysis to these problems. Then section 1.3 describes the evaluation of the Morse index of quadratic forms between dual Banach spaces and obeying certain inequalities of Gårding type.

Section 1.4 describes the functional and the variational principle appropriate to the standard eigenproblem. The critical points, critical values and minima are found. The functional involves a parameter μ and the dependence of critical points, or bifurcation diagram, with respect to μ is described. Then, in section 1.5, the second variation is analyzed and a Morse theory is developed. The use of further, linear orthogonality constraints to find higher eigenvalues and eigenvectors is described in section 1.6.

In section 1.7, certain classes of weighted eigenproblems are analyzed. Our formulation allows us to treat these problems under weaker assumptions on the weights, than is possible with Rayleigh's principle. Results are obtained for indefinite weight operators; including some necessary and sufficient conditions for the existence of eigenvalues and results on the number of linear independent eigenvectors.

Finally in section 1.8, the results are applied to linear second-order elliptic boundary value problems, with both definite and indefinite weight functions. Variational principle for finding successive eigenvalues and eigenfunctions are described. When the weight function is nonnegative, Morse indices of non-degenerate critical points are determined.

1.2 QUADRATIC FORMS AND CLOSED LINEAR OPERATORS

Throughout this paper H will be a real, separable, Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. All topologies and convergence shall be understood to be in this norm topology unless otherwise stated.

Let $L : D_L(\subset H) \rightarrow H$ be closed, densely-defined, linear operator on H and L^* be its adjoint. Then L^* is again a closed densely-defined linear operator and its domain is denoted D_{L^*} .

The operator L is said to be *symmetric* if $D_L \subset D_{L^*}$ and $Lu = L^*u$ for all u in D_L . L is said to be *self-adjoint* if it is symmetric and $D_L = D_{L^*}$. The operator L is said to be *positive semidefinite* if

$$\langle Lu, u \rangle \geq 0 \quad \text{for all } u \text{ in } D_L. \quad (1.1)$$

It is *positive definite* if there is a constant $c > 0$ such that

$$\langle Lu, u \rangle \geq c \|u\|^2 \quad \text{for all } u \text{ in } D_L. \quad (1.2)$$

In this section, various properties of this operator L will be related to those of some associated quadratic forms. Let $\overline{\mathbb{R}} = [-\infty, \infty]$ and define $f : H \rightarrow \overline{\mathbb{R}}$ to

be the quadratic form

$$f(u) = \begin{cases} \frac{1}{2}\langle Lu, u \rangle & u \text{ in } D_L \\ \infty & \text{otherwise.} \end{cases} \quad (1.3)$$

Lemma 1.2.1 *Let L be a closed, densely defined, linear operator on H and f be defined by (1.3). Then the following statements are equivalent:*

- (i) f is convex on H ,
- (ii) L is positive semidefinite operator on H , and
- (iii) f has a finite infimum on H .

PROOF: (i) \Leftrightarrow (ii). Let $u(t) = (1-t)u + tv$ with u, v in D_L . Then $f(u(t)) = at^2 + bt + c$ with $2a = \langle L(u-v), v-v \rangle$. Now f is convex on H if and only if it is convex on D_L and this occurs if and only if $a \geq 0$ for all u, v in D_L . Since L is linear, this holds if and only if L is a positive semidefinite operator on H .

(ii) \Leftrightarrow (iii) Let $\gamma = \inf_{u \in H} f(u)$. When (ii) holds, then $\gamma = 0$ so (iii) holds.

Also $f(tu) = t^2 f(u)$ for any real scalar t , so γ can only be 0 or $-\infty$. Thus

(iii) \Rightarrow (ii). □

When $g : H \rightarrow \overline{\mathbb{R}}$ is a given functional, its *essential domain* is $\text{dom } g = \{u \in H : |g(u)| < \infty\}$. A convex functional $g : H \rightarrow \overline{\mathbb{R}}$ is said to be *closed* if its epigraph

$$\text{epi } g = \{(x, z) \in H \times \mathbb{R} : z \geq f(x)\} \quad \text{is closed.}$$

When g is not closed, its closure \bar{g} is defined to be the maximal, closed, convex minorant of g .

Despite this naming, when a closed operator L obeys the conditions of lemma 1.2.1, the quadratic form f associated with L is not a closed convex functional unless, $D_L = H$. We have the following information on the closure \bar{f} .

Lemma 1.2.2 *Let L be a closed, densely defined, positive semidefinite operator on H , f be defined by (1.3) and \bar{f} is the closure of f . Then*

(i) \bar{f} is a nonnegative convex, l.s.c. function on H .

(ii) $\bar{f}(v) = f(v)$ for all v in $D_L \cap D_{L^*}$, and

(iii) $\frac{1}{2}(Lv + L^*v) \in \partial \bar{f}(v)$ for all v in $D_L \cap D_{L^*}$.

PROOF: From these assumptions and lemma 1.2.1, f and $\text{epi } f$ are convex. From Zeidler, 1985, proposition 51.6, $\text{epi } \bar{f}$ is the closure of $\text{epi } f$. Hence \bar{f} will be convex and lower semicontinuous (l.s.c.). It is nonnegative as 0 is a convex minorant of f . Since L is a positive semidefinite, (1.1) yields

$$\langle L(u - v), v - v \rangle \geq 0 \quad \text{for all } u, v \text{ in } D_L.$$

$$\begin{aligned} \text{Thus } f(u) &\geq f(v) + \frac{1}{2}\langle Lv, u \rangle + \frac{1}{2}\langle Lu, v \rangle - \langle Lv, v \rangle \\ &= f(v) + \frac{1}{2}\langle Lv + L^*v, u - v \rangle \end{aligned} \quad (1.4)$$

when v is also in D_{L^*} .

Suppose $f(v) = \alpha$ and $\bar{f}(v) = \bar{\alpha}$. Then $\bar{\alpha} \leq \alpha$ by definition and there is a sequence of points $\{(v_n, z_n) : n \geq 1\}$ in $\text{epi } f$ with v_n converging to v and z_n converging to $\bar{\alpha}$. Substitute v_n for u in (1.4), then

$$z_n \geq f(v_n) \geq f(v) - c \|v_n - v\|,$$

for all n where $c = \frac{1}{2} \|Lv + L^*v\|$. Take limits as n increases, then (ii) holds as

$$\bar{\alpha} \geq \bar{f}(v) \geq f(v) = \alpha.$$

When u is any point in $\text{dom } \bar{f}$, there is a sequence of points $\{(u_n, \zeta_n) : n \geq 1\} \subset \text{epi } f$ with $u_n \rightarrow u$, and $\zeta_n \rightarrow \bar{f}(u)$. (1.4) and (ii) yield that

$$\zeta_n \geq f(u_n) \geq \bar{f}(v) + \frac{1}{2} \langle Lv + L^*v, u_n - v \rangle.$$

Take the limit as $n \rightarrow \infty$ of both sides here, then

$$\bar{f}(u) \geq \bar{f}(v) + \frac{1}{2} \langle Lv + L^*v, u - v \rangle,$$

or $\frac{1}{2}(Lv + L^*v) \in \partial \bar{f}(v)$ as claimed. \square

Corollary 1.2.3 *If $D_L \cap D_{L^*}$ is dense in H , then $\partial \bar{f}(v) = \{\frac{1}{2}(Lv + L^*v)\}$ for each v in $D_L \cap D_{L^*}$.*

PROOF: Suppose v is in $D_L \cap D_{L^*}$ and w is in $\partial \bar{f}(v)$. Then, for any h in $D_L \cap D_{L^*}$, $f(v + th) - f(v) \geq t \langle w, h \rangle$. Thus

$$\lim_{t \rightarrow 0^+} t^{-1}[f(v + th) - f(v)] = \frac{1}{2} \langle Lv + L^*v, h \rangle \geq \langle w, h \rangle, \quad \text{or}$$

$$\langle \frac{1}{2}(Lv + L^*v) - w, h \rangle \geq 0 \quad \text{for all } h \text{ in } D_L \cap D_{L^*}.$$

Since $D_L \cap D_{L^*}$ is dense in H , this implies $w = \frac{1}{2}(Lv + L^*v)$. \square

When L is a symmetric, positive definite, closed linear operator, we define the Hilbert space H_L to be the completion of D_L with respect to the inner product

$$[u, v] = \langle Lu, v \rangle. \tag{1.5}$$

The norm on H_L is defined by

$$\|u\|_L^2 = \langle Lu, u \rangle. \quad (1.6)$$

Any Cauchy sequence in D_L with respect to the norm (1.6), will also be a Cauchy sequence in H . Hence H_L may be regarded as a subspace of H and (1.2) implies that the embedding of H_L in H is continuous.

Let H_L^* be the dual space of H_L with respect to the pairing $\langle \cdot, \cdot \rangle$. H_L^* is a Banach space under the usual dual norm,

$$\|g\|_* = \sup_{\|u\|_L=1} \langle u, g \rangle. \quad (1.7)$$

The next result shows that the essential domain of \bar{f} is H_L - as expected.

Theorem 1.2.4 *Suppose L is a symmetric, positive definite, closed, densely defined linear operator on H and f is defined by (1.3). Let \bar{f} be the closure of f and H_L be the completion of D_L with respect to the inner product (1.5). Then*

$$\bar{f}(u) = \begin{cases} \frac{1}{2} \|u\|_L^2 & \text{for } u \in H_L \\ \infty & \text{otherwise.} \end{cases} \quad (1.8)$$

PROOF: Let g be the functional on H defined by the right hand side of (1.8).

Then g is a convex functional on H . Let $\{(x_n, z_n) : n \geq 1\}$ be a sequence of points in $\text{epi } f$ which converge to (\hat{u}, \hat{z}) in $H \times \mathbb{R}$. Given $\epsilon > 0$, for all sufficiently large n ,

$$2f(u_n) = \|u_n\|_L^2 \leq 2\hat{z} + \epsilon.$$

Hence $\{u_n : n \geq 1\}$ is a bounded sequence in H_L . Since the embedding of H_L in H is continuous, there is a v in H_L and a subsequence $\{u_{n_j} : j \geq 1\}$, such that u_{n_j} converges weakly to v in H . But u_{n_j} converges strongly to \hat{u} by assumption, so $\hat{u} = v$ is in H_L .

The norm on H_L is weakly *l.s.c.* so

$$\frac{1}{2} \|\hat{u}\|_L^2 \leq \liminf_{j \rightarrow \infty} \frac{1}{2} \|u_{n_j}\|^2 \leq \liminf_{j \rightarrow \infty} z_{n_j} = \hat{z}.$$

Thus $g(\hat{u}) \leq \hat{z}$, or $\text{epi } g \supset \text{epi } f = \text{epi } \bar{f}$.

Let (\tilde{u}, \tilde{z}) be a point in $\text{epi } g$. Then there is a sequence of points $\{u_n : n \geq 1\}$ in D_L such that $\|u_n - \tilde{u}\|_L^2 \rightarrow 0$ as $n \rightarrow \infty$. Let $z_n = \tilde{z} + f(u_n) - g(\tilde{u})$. Then (u_n, z_n) is in $\text{epi } f$, u_n converges to \tilde{u} in H and z_n converges to \tilde{z} as

$$g(\tilde{u}) = \lim_{n \rightarrow \infty} \frac{1}{2} \|u_n\|_L^2 = \lim_{n \rightarrow \infty} f(u_n).$$

Hence (\tilde{u}, \tilde{z}) is in the closure of $\text{epi } f$, or $\text{epi } g = \text{epi } \bar{f}$. □

Corollary 1.2.5 *Under the assumptions of the theorem, $\partial \bar{f}(v) = \{Lv\}$ for all v in D_L .*

PROOF: Since L is symmetric, $Lv = L^*v$ for all $v \in D_L$ and $D_{L^*} \supset D_L$. Thus $D_L \cap D_{L^*} = D_L$ is dense in H and the result follows from the corollary to lemma 1.2.2. □

In view of this result, we shall use the restriction of \bar{f} to H_L as our basic functional throughout this paper. Define $f_L : H_L \rightarrow \mathbb{R}$ by

$$f_L(u) = \frac{1}{2} \|u\|_L^2. \tag{1.9}$$

When L obeys the conditions of theorem 1.2.4, then from a well-known result of Friedrichs, it has an extension which obeys the following condition:

($\mathcal{L}1$) : The operator L is closed, densely defined, linear, self-adjoint and positive definite.

Theorem 1.2.6 *Suppose ($\mathcal{L}1$) holds and f_L is defined by (1.9). Then*

(i) f_L is convex, continuous and weakly l.s.c. on H_L ,

(ii) there exists a continuous, linear, self-adjoint operator $\mathcal{L} : H_L \rightarrow H_L^*$, with

$$f_L(u) = \frac{1}{2} \langle \mathcal{L}u, u \rangle \text{ for all } u \text{ in } H_L, \text{ and} \quad (1.10)$$

(iii) f_L is twice continuously Gateaux differentiable on H_L with

$$Df_L(u) = \mathcal{L}u \quad \text{and} \quad D^2 f_L(u) = \mathcal{L}. \quad (1.11)$$

PROOF: (i) follows as (1.9) defines a norm.

(ii) Define $\mathcal{L} : H_L \rightarrow H_L^*$ by the equation $\langle \mathcal{L}u, v \rangle = [u, v]$ for all u, v in H_L .

This operator is linear and bounded so it is continuous. Self-adjointness follows from the symmetry of the inner product.

(iii) follows directly from the representation (1.10) and the definition of Gateaux derivative. \square

We shall also use the condition:

($\mathcal{L}2$) $M : D_M(\subset H) \rightarrow H$ is a closed, densely defined, self-adjoint linear operator with $D_L \subset D_M$ obeying

- (i) M is positive semi-definite on D_M
- (ii) there exists a constant $\gamma > 0$ such that

$$\langle Mu, u \rangle \leq \gamma \langle Lu, u \rangle \quad \text{for all } u \text{ in } D_L. \quad (1.12)$$

Consider the quadratic form $g : H \rightarrow \overline{\mathbb{R}}$ defined by

$$g(u) = \begin{cases} \frac{1}{2} \langle Mu, u \rangle & \text{if } u \text{ in } D_M \\ -\infty & \text{otherwise.} \end{cases}$$

From lemma (1.2.1), g is a convex functional when M obeys $(\mathcal{L}2)$. Let \bar{g} be its closure, then the following holds.

Lemma 1.2.7 *Assume $(\mathcal{L}1)$ and $(\mathcal{L}2)$ hold, then $\text{dom } \bar{g} \supseteq H_L$.*

PROOF: Suppose $\{u_n : n \geq 1\}$ is a Cauchy sequence in D_L with respect to the norm defined by (1.6) and \hat{u} is its limit in H_L . Then $\{\langle Mu_n, u_n \rangle : n \geq 1\}$ is a Cauchy sequence since (1.12) holds. Hence $\lim_{n \rightarrow \infty} \langle Mu_n, u_n \rangle = \bar{g}(\hat{u})$ is well-defined and finite, so $H_L \subset \text{dom } \bar{g}$ as claimed. \square

Corollary 1.2.8 *Suppose $(\mathcal{L}1)$ - $(\mathcal{L}2)$ hold, then there is a continuous self-adjoint linear operator $\mathcal{M} : H_L \rightarrow H_L^*$ obeying $\bar{g}(u) = \frac{1}{2} \langle \mathcal{M}u, u \rangle$ for all u in H_L .*

PROOF: Define $G : H_L \times H_L \rightarrow \mathbb{R}$ by

$$G(u, v) = \bar{g}(u + v) - \bar{g}(u) - \bar{g}(v). \quad (1.13)$$

When u, v are in D_M , then

$$\begin{aligned} G(u, v) &= \frac{1}{2} \langle M(u+v), u+v \rangle - \frac{1}{2} \langle Mu, u \rangle - \frac{1}{2} \langle Mu, v \rangle \\ &= \langle Mu, v \rangle \quad \text{using the self-adjointness of } M. \end{aligned}$$

Since $0 \leq \bar{g}(u) \leq \frac{\gamma}{2} \|u\|_L^2$ for all u in D_L , substitution yields

$$|G(u, v)| \leq \frac{\gamma}{2} \|u+v\|_L^2.$$

Thus G is bounded on bounded sets in $H_L \times H_L$ and G is bilinear and symmetric on a dense subspace, so it is bilinear and symmetric on $H_L \times H_L$. Hence G is continuous and there is a continuous, self-adjoint, operator \mathcal{M} mapping H_L into H_L^* and obeying

$$G(u, v) = \langle \mathcal{M}u, v \rangle \quad \text{for all } u, v \text{ in } H_L.$$

Put $v = u$ in (1.13) then $\langle \mathcal{M}u, u \rangle = 2\bar{g}(u)$ so the result holds. \square

In analogy to (1.10) we shall often write $f_M : H_L \rightarrow \mathbb{R}$ where f_M is the restriction of \bar{g} to H_L . Thus

$$f_M(u) = \frac{1}{2} \langle \mathcal{M}u, u \rangle \quad \text{for all } u \text{ in } H_L \quad (1.14)$$

and f_M will be a continuous, convex, nonnegative functional on H_L .

1.3 MORSE INDEX OF A QUADRATIC FORM

The type of a critical point of a variational problem is determined by the quadratic form associated with the second derivative of the functional at the point. A theory of the Morse index for such problems in a Hilbert space was

developed by Hestenes, 1951. In this paper, we will use an extension of this theory along the lines outlined in Zeidler, 1985, Section 37.27b.

We will use the following conditions:

(A1): H is a Hilbert space, X is another infinite dimensional, Banach space which is a subspace of H and the imbedding $I : X \rightarrow H$ is 1 – 1 and continuous.

(A2): Let X^* be the dual space of X with respect to the inner product on H , and $A : X \rightarrow X^*$ be a linear, continuous, self-adjoint map,

(A3): there exist constants $c_1 > 0$ and $c_2 \geq 0$, and a compact, linear, self-adjoint map $B : X \rightarrow X^*$ such that, for all u in X ,

$$\langle Au, u \rangle \geq c_1 \|u\|_X^2 - c_2 \langle Bu, u \rangle. \quad (1.15)$$

Consider the quadratic form $Q : X \rightarrow \mathbb{R}$ defined by

$$Q(u) = \langle Au, u \rangle. \quad (1.16)$$

In Morse theory we seek subspaces X_+, X_- of X and a corresponding splitting of the operator A so that

$$Q(u) = \langle A_+ u, u \rangle - \langle A_- u, u \rangle, \quad (1.17)$$

with A_+ and A_- positive semidefinite operators from X to X^* .

When A obeys an inequality of the form (1.15), then such splittings are related to the eigenvalue problem of finding non-trivial solutions in H_L of

$$Au = \lambda Bu. \quad (1.18)$$

The quadratic form Q is said to be *nondegenerate* if A is bijective. The *nullity* (or *null index*) of Q is the dimension of the null space $N(A)$ of A .

Let $E_- = \{x \in X : Q(x) < 0\}$. The *Morse index* $i(Q)$ of Q is the maximum dimension of those subspaces of X which are subspaces of $E_- \cup \{0\}$.

Theorem 1.3.1 *Assume (A1) - (A3) hold and Q is defined by (1.16). Then Q is nondegenerate if and only if 0 is not an eigenvalue of A .*

PROOF: From (1.15) one sees that $A + c_2 B$ is a bijective linear map of X to X^* . In particular, it is a Fredholm map of index 0. Thus A is a Fredholm map of index 0 as B is compact; see Zeidler, 1986, Section 8.4.

Hence the range of A is closed and A will be bijective if and only if $N(A) = \{0\}$. That is, 0 is not an eigenvalue of A . \square

As was indicated in this proof, for any value of λ , $A - \lambda B$ is a Fredholm map of index zero. When $N(A - \lambda B)$ is nonzero, it must be finite dimensional. We define $m(\lambda) = \dim N(A - \lambda B)$ to be the *multiplicity* of λ as an eigenvalue of (1.18). The results will often require the condition:

(A4): $\langle Bu, u \rangle > 0$ for all u in $H_L \setminus \{0\}$.

When (A1) - (A4) hold, a spectral theory for (1.18) can be described. Namely there is a countable family of eigenvalues $\{\lambda_j : j \geq 1\}$ obeying $-\infty < \lambda_1 \leq \lambda_2 \leq \dots$ with no finite accumulation point. There is a corresponding family of eigenvectors $\{e_j : j \geq 1\}$ of (1.18) with $\langle Be_j, e_k \rangle = \delta_{jk}$. These eigenvectors are said to be B-orthonormal and, for each u in H_L ,

$$u = \sum_{j=1}^{\infty} \langle Bu, e_j \rangle e_j. \quad (1.19)$$

This is proven as proposition 22.31 of Zeidler, Volume IIA.

Theorem 1.3.2 *Assume (A1) - (A4) holds and Q is defined by (1.16). Then the Morse index of Q is finite and equal to the number of negative eigenvalues of (1.18) counting multiplicity. Moreover, if $\{\lambda_j : j \geq 1\}$ is the set of all eigenvalues of (A, B) and $\{e_j : j \geq 1\}$ is a corresponding family of B -orthonormal eigenvectors, then*

$$Q(u) = \sum_{j=1}^{\infty} \lambda_j \langle Bu, e_j \rangle^2. \quad (1.20)$$

PROOF: For each u in H_L , (1.19) holds. Thus

$$Au = \sum_{j=1}^{\infty} \lambda_j \langle Bu, e_j \rangle Be_j \quad (1.21)$$

as each $Ae_j = \lambda_j Be_j$, so (1.20) follows.

Let J be the number of negative eigenvalues of (1.18) counting multiplicity, and $\{e_j : 1 \leq j \leq J\}$ be a corresponding set of B -orthonormal eigenfunctions. Let X_- be the subspace of X spanned by this set. If a non-zero v is in X_- then from (1.20), $Q(v) < 0$. Moreover (1.20) shows that X_- is a maximal subspace for which this holds so $i(Q) = \dim X_- = J$. \square

This result enables us to define the operators A_- , A_+ in (1.17) and corresponding subspaces X_- , X_+ . Let X_- be the subspace defined in this last proof and let $X_+ = \{v \in H_L : \langle Bv, e_j \rangle = 0 \text{ for } 1 \leq j \leq J\}$. Then $X = X_+ \oplus X_-$ with X_- being finite dimensional. Define $A_- : X \rightarrow X^*$ by

$$A_- u = - \sum_{j=1}^J \lambda_j \langle Bu, e_j \rangle Be_j.$$

Then A_- is a positive semi-definite operator with finite dimensional range. Let $A_+ = A + A_-$; then from (1.21) and (1.20) A_+ is positive semidefinite and (1.17) holds.

These results hold when B is any compact self-adjoint linear map obeying (A4) and for which the estimate (1.15) holds. Different choices of B will provide different representations of the form (1.17) but the index J remains invariant. When $I : X \rightarrow H$ is compact, take $B = I$ so that (1.15) is an inequality of Gårding type.

1.4 UNCONSTRAINED VARIATIONAL PRINCIPLES FOR SELF-ADJOINT EIGENPROBLEMS

Suppose L, M are closed, self-adjoint, linear operators on Hilbert space H obeying $(\mathcal{L}1)$ and $(\mathcal{L}2)$. Consider the eigenvalue problem of finding those values λ for which there are nonzero solutions u in D_L of

$$Au = Lu + Mu = \lambda u. \quad (1.22)$$

In this section we shall describe some (unconstrained) functionals whose critical points yield the eigenvalues and eigenfunctions of A . First note that if $M : H \rightarrow H$ is continuous, and L obeys $(\mathcal{L}1)$ then (1.12) always holds. Moreover one can always add a finite multiple of the identity to both sides of (1.22) so there is no loss of generality in requiring that M be positive semidefinite also.

The number λ is an *eigenvalue* of A if there is a nonzero vector v in D_L satisfying (1.22). Any such v is called an *eigenfunction* of A corresponding to

the eigenvalue λ . When v is an eigenfunction of A of norm 1 we say v is *normalized*. Let E_λ be the set of all normalized eigenfunctions of A corresponding to the eigenvalue λ . The number of linearly independent eigenfunctions of A corresponding to the eigenvalue λ is called the *multiplicity* of λ as an eigenvalue of A and is denoted $m(\lambda)$.

When $m(\lambda) = 1$, λ is a *simple* eigenvalue of A and E_λ consists of exactly two points. When $m(\lambda) \geq 2$, then E_λ is diffeomorphic to a sphere of dimension $m(\lambda) - 1$.

Let H_L be the completion of D_L in the norm (1.6) and \mathcal{M} be the continuous linear operator associated with M as defined in the corollary to lemma 1.2.7.

Consider the parameterized functional $\mathcal{F}_p : H_L \times (0, \infty) \rightarrow R$ by

$$\mathcal{F}_p(u; \mu) = \frac{1}{2} \|u\|_L^2 + \frac{1}{2} \langle \mathcal{M}u, u \rangle + \frac{1}{p} \|u\|^p - \frac{\mu}{2} \|u\|^2 \quad (1.23)$$

where $\|\cdot\|_L$, $\|\cdot\|$ are the norms on H_L and H respectively and $2 < p < \infty$.

The variational principle (\mathcal{P}_μ) is the unconstrained problem of minimizing $\mathcal{F}(\cdot; \mu)$ on H_L and finding

$$\alpha_p(\mu) = \inf_{u \in H_L} \mathcal{F}_p(u; \mu). \quad (1.24)$$

A point v in H_L is said to be a *critical point* of $\mathcal{F}_p(\cdot; \mu)$ if $\mathcal{F}_p(\cdot; \mu)$ is G -differentiable at v and

$$\mathcal{D}\mathcal{F}_p(v; \mu) = 0. \quad (1.25)$$

Here $\mathcal{DF}_p(v; \mu)$ is the Gateaux derivative of $\mathcal{F}_p(\cdot; \mu)$ at v in H_L . A number ν is said to be a *critical value* of $\mathcal{F}_p(\cdot; \mu)$ if there is a critical point v with $\mathcal{F}_p(v; \mu) = \nu$. Henceforth we shall also require

($\mathcal{L3}$): The imbedding $I : H_L \rightarrow H$ is compact.

The following theorem describes some properties of the functional \mathcal{F}_p . The subscript p will be omitted henceforth.

Theorem 1.4.1 *Assume ($\mathcal{L1}$) - ($\mathcal{L3}$) hold and \mathcal{F} is defined by (1.23) with $p > 2$, then*

(i) $\mathcal{F}(\cdot; \mu)$ is continuous, coercive and weakly l.s.c. on H_L ,

(ii) $\mathcal{F}(\cdot; \mu)$ is Gateaux differentiable on H_L with

$$\mathcal{DF}(u; \mu) = \mathcal{L}u + \mathcal{M}u + (\|u\|^{p-2} - \mu)u. \quad (1.26)$$

(iii) If v is a critical point of $\mathcal{F}(\cdot; \mu)$ then $v = 0$ or $v = (\mu - \lambda)^q e$ where λ is an eigenvalue of A lying in $(0, \mu)$, e is in E_λ and $q = (p - 2)^{-1}$.

(iv) The critical values of $\mathcal{F}(\cdot; \mu)$ are 0 and $-C_p(\mu - \lambda)^{pq}$ where λ is an eigenvalue of A in $(0, \mu)$ and $C_p = (2pq)^{-1}$.

(v) If λ_1 is the least eigenvalue of A , then

$$\alpha_p(\mu) = \begin{cases} 0 & \text{if } 0 \leq \mu \leq \lambda_1 \\ -C_p(\mu - \lambda_1)^{pq} & \text{when } \mu > \lambda_1 \end{cases} \quad (1.27)$$

(vi) When $\mu \leq \lambda_1$, $\mathcal{F}(\cdot; \mu)$ is minimized at 0 while if $\mu > \lambda_1$, then it is minimized at $(\mu - \lambda_1)^q e$ with e in E_{λ_1} .

PROOF: (i). $\mathcal{F}(\cdot; \mu)$ is continuous as each term in its definition is. When $p > 2$, $\|u\| \geq 0$, it is straightforward to verify that

$$\frac{1}{p} \|u\|^p - \frac{\mu}{2} \|u\|^2 \geq -C_p \mu^{p'}$$

where $C_p = (\frac{1}{2} - \frac{1}{p}) = (p-2)/p$ and $p' = p/(p-1)$.

Thus (1.23) implies that $\mathcal{F}(u; \mu) \geq \frac{1}{2} \|u\|_L^2 - C_p \mu^{p'}$, so $\mathcal{F}(\cdot; \mu)$ is coercive on H_L . If $\{u_n : n \geq 1\}$ converges weakly to u in H_L , then it converges strongly to u in H when (L3) holds. Thus the last two terms in (1.23) are weakly continuous. The first term is weakly *l.s.c.* from theorem 1.2.6 and the second term is because it is continuous and convex on H_L .

(ii). Each of the terms in (1.23) is Gateaux differentiable. From theorem 1.2.6, the chain rule and a direct computation, (1.26) follows.

(iii). If v is a critical point of $\mathcal{F}(\cdot; \mu)$ then (1.26) implies that it obeys

$$\mathcal{A}u = (\mu - \|u\|^{p-2})u \tag{1.28}$$

where $\mathcal{A} = \mathcal{L} + \mathcal{M}$. Obviously 0 is always a solution of this. When v is nonzero, this right hand side is in H_L , and thus in H . Thus $\mathcal{A}v$ is in H , so v must actually be in D_L , as $L + M$ maps D_L onto H . Hence v is a solution of (1.22) and is an eigenfunction of A corresponding to the eigenvalue $\lambda = \mu - \|v\|^{p-2}$. That is, $\|v\|^{p-2} = \mu - \lambda$ or $\|v\| = (\mu - \lambda)^q$ with $q = (p-2)^{-1}$.

(iv). $\mathcal{F}(0; \mu) = 0$, so 0 is a critical value. If $v \neq 0$ is a critical point then, from (1.28),

$$\langle \mathcal{A}v, v \rangle = (\mu - \|v\|^{p-2}) \|v\|^2$$

and $\mathcal{F}_p(v; \mu) = (\frac{1}{p} - \frac{1}{2}) \|v\|^p = -C_p (\mu - \lambda)^{pq}$.

(v)-(vi). From (i), \mathcal{F}_p attains its infimum on H_L , and this must occur at a critical point; so the infimum is a critical value.

When $0 < \mu \leq \lambda_1$, 0 is the only critical point of $\mathcal{F}_p(\cdot; \mu)$, so $\alpha_p(\mu) = 0$ and it is attained at 0 in H_L .

When $\mu > \lambda_1$, then the critical values of $\mathcal{F}_p(\cdot; \mu)$ are $-C_p(\mu - \lambda_j)^{pq}$ from (iv) and this will be smallest when $\lambda = \lambda_1$ is the least eigenvalue of A . This value will be attained at any vector of the form $(\mu - \lambda_1)^q e$ with e in E_{λ_1} . \square

This theorem may be restated to provide the following description of the critical points of $\mathcal{F}(\cdot; \mu)$.

Corollary 1.4.2 *Assume $(\mathcal{L}1)$ - $(\mathcal{L}3)$ hold and \mathcal{F} is defined by (1.22), with $2 < p < \infty$. Then the set of critical points of $\mathcal{F}(\cdot; \mu)$ is a closed, bounded set in H_L with a finite number of connected components. It is a finite set if and only if each eigenvalue of \mathcal{A} in $(0, \mu)$ is simple.*

PROOF: The critical points of (1.23) are the solutions of (1.28). This is an eigenvalue problem of the type (1.18) with $X = H_L$, $A = \mathcal{A}$ and $B = I$ being the imbedding operator of H_L into H_L^* . I obeys (A4) and (A3) holds with $c_1 = 1$, $c_2 = 0$ so \mathcal{A} will only have a finite number of eigenvalues less than μ .

Thus the set of critical points in $\mathcal{F}(\cdot; \mu)$ is the union of $(\mu - \lambda_j)^p E_j$ where λ_j is an eigenvalue of \mathcal{A} in $(0, \mu)$ and $E_j = E_{\lambda_j}$. Each E_j either consists of 2 points or is a infinite set which is connected, closed and bounded. \square

The variational principle (\mathcal{P}_μ) can be used to find both upper and lower bounds on λ . If, for given positive μ , $\alpha_p(\mu) = 0$, then λ_1 must be greater than or equal to μ . This provides a lower bound on λ_1 .

If μ is given and there is a u in H_L with $\mathcal{F}(u; \mu) < 0$, then $C_p(\mu - \lambda_1)^{pq} \geq -\mathcal{F}(u, \mu)$. Rearrangement leads to an upper bound on λ_1 , namely

$$\lambda_1 \leq \mu - \left(\frac{-\mathcal{F}(u; \mu)}{C_p} \right)^{2C_p}. \quad \text{In fact,} \quad \lambda_1 = \inf_{\mathcal{F}(u; \mu) < 0} \left[\mu - \left(\frac{-\mathcal{F}(u; \mu)}{C_p} \right) \right].$$

There are 2 parameters p, μ in this functional. p may be chosen at our convenience provided $p > 2$. The choice of p does not affect the number, or type, of critical points.

In contrast, when the parameter μ is increased, the number of critical points may increase. For all values of μ , 0 is a critical point and it is the unique critical point of $\mathcal{F}_p(\cdot; \mu)$ when $\mu < \lambda_1$. As μ increases through an eigenvalue λ_j of A , then a new branch of critical points of \mathcal{F}_p bifurcates from the origin.

Figure 1.1 is a schematic bifurcation (or solution) diagram for the critical points of \mathcal{F}_p when $p = 3$. In that figure, $C_j = \{(\lambda_j + s, s e_j) : s \geq 0, e_j \in E_j\}$ is the set of critical points corresponding to the j -th distinct eigenvalue of A . Each of these branches is a straight line which extends to infinity in both μ and H_L (or H). There is no secondary bifurcation.

For general p , the branches have the form

$$C_u^{(p)} = \{(\lambda_j + s, s^q e_j) : s \geq 0, e_j \in E_j\}$$

with $q(p-2) = 1$ as before. When $p = 4$, the branches are parabolae and the bifurcation diagram is similar to figure 1.2.

1.5 TYPES OF CRITICAL POINTS AND MORSE INDICES

In this section we shall show that the functional $\mathcal{F}(\cdot; \mu)$ defined by (1.23) has a well-defined second derivative. The quadratic form associated with this sec-

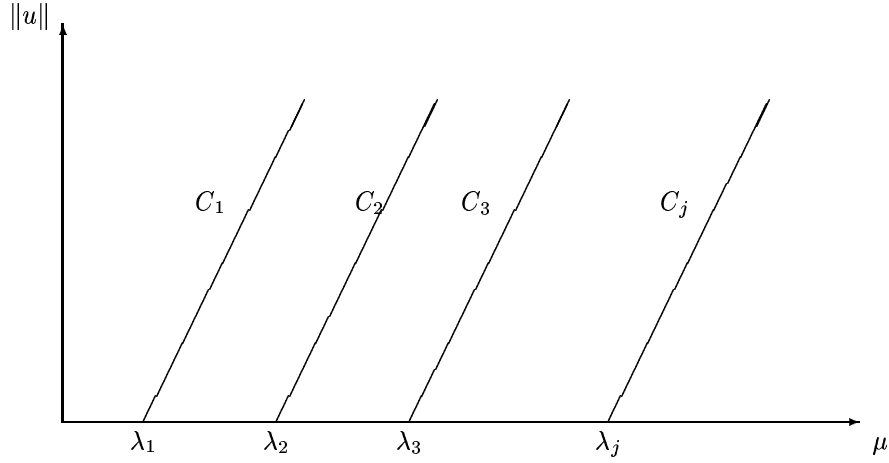


Figure 1.1 Solution diagram of critical points of \mathcal{F}_3 .

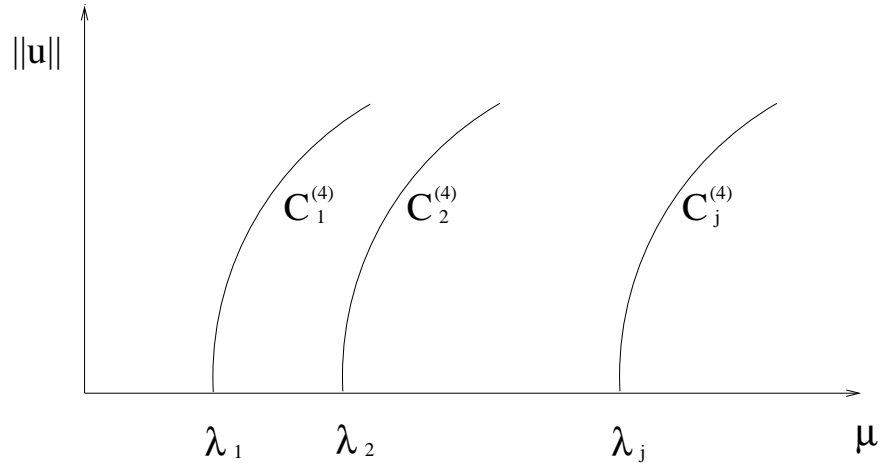


Figure 1.2 Critical point diagram of \mathcal{F}_4 .

ond derivative has a Morse-type index and it provides a Morse theory for this functional. The results described below may be compared with the Courant-Fisher-Weyl minimax theory for Rayleigh's principle.

When $u (\neq 0)$ is in H_L , define the linear operator $P_u : H_L \rightarrow H_L$ by

$$P_u v = (\langle u, v \rangle / \|u\|^2) u. \quad (1.29)$$

P_u is the projection in the direction of u and is a continuous linear operator.

Theorem 1.5.1 *Assume $(\mathcal{L}1)$ -($\mathcal{L}2$) hold and \mathcal{F} is defined by (1.23) with $2 < p < \infty$. Then $\mathcal{DF}(\cdot; u)$ is Gateaux differentiable on H_L and*

$$\mathcal{D}^2\mathcal{F}(u; \mu) = \begin{cases} \mathcal{A} - \mu I & \text{when } u = 0 \\ \mathcal{A} - \mu I + \|u\|^{p-2} [I + (p-2)P_u] & \text{when } u \neq 0 \end{cases} \quad (1.30)$$

Proof: From (1.26), $\mathcal{DF}(u; \mu) = \mathcal{A}u + (\|u\|^{p-2} - \mu I) u$ where $I : H_L \rightarrow H_L^*$ is the natural imbedding.

Consider the mapping $\mathcal{N} : H_L \rightarrow H_L^*$ defined by $\mathcal{N}(u) = \|u\|^{p-2} u$. When $p > 2$, then $\lim_{t \rightarrow 0^+} t^{-1}[\mathcal{N}(th) - \mathcal{N}(0)] = 0$ for all $h \in H_L$. Thus, \mathcal{N} is G -differentiable at 0 with $\mathcal{DN}(0) = 0$ and $\mathcal{D}^2\mathcal{F}(0; \mu) = \mathcal{A} - \mu I$.

When $u \neq 0$, a calculation shows that

$$\lim_{t \rightarrow 0^+} t^{-1}[\mathcal{N}(u + th) - \mathcal{N}(u)] = \|u\|^{p-2} [h + (p-2)P_u h]. \quad (1.31)$$

Thus the second part of (1.30) holds, since the other terms in $\mathcal{DF}(u; \mu)$ are linear and continuous. \square

Now consider the (Hessian) quadratic form $Q_u : H_L \rightarrow R$ defined by

$$Q_u(h) = \langle \mathcal{D}^2\mathcal{F}(u; \mu)h, h \rangle \quad (1.32)$$

$$= \langle \mathcal{A}h, h \rangle + (\|u\|^{p-2} - \mu) \|h\|^2 + (p-2) \|u\|^{p-4} \langle u, h \rangle^2. \quad (1.33)$$

This quadratic form satisfies the conditions described in section 1.3 with $X = H_L$, $A = \mathcal{D}^2\mathcal{F}(u; \mu)$ and $B = I$ being the imbedding of H_L into H_L^* . When $(\mathcal{L}1)$ -($\mathcal{L}3$) hold, then so do (A1)-(A4).

A critical point v of $\mathcal{F}(\cdot; \mu)$ is said to be *nondegenerate* if the quadratic form Q_v defined by (1.32) is nondegenerate. The *Morse index* $i(v)$ of the critical point v will be the Morse index of Q_v . The following provides some results on the degeneracy and indices of the critical points of \mathcal{F} .

Theorem 1.5.2 *Assume (L1)–(L3) hold and \mathcal{F} is defined by (1.23) with $2 < p < \infty$, then*

(i:) *0 is a nondegenerate critical point of \mathcal{F} if and only if μ is not an eigenvalue of (1.22). If $\mu \leq \lambda_1$ then the Morse index of 0 is 0 and when $\mu > \lambda_1$, it is*

$$\sum_{\lambda_j < \mu} m(\lambda_j). \quad (1.34)$$

(ii:) *when $v = (\mu - \lambda_j)^q e_j$ is a nonzero critical point of \mathcal{F} , then v is nondegenerate if and only if λ_j is a simple eigenvalue of (1.22). When $j = 1$ its Morse index is 0, while if $j \geq 2$ it is*

$$\sum_{k=1}^{j-1} m(\lambda_k) \quad (1.35)$$

PROOF: (i) A computation gives $Q_0(h) = \langle \mathcal{A}h, h \rangle - \mu \|h\|^2$. From theorem 1.3.1, 0 is a non-degenerate critical point of $\mathcal{F}(\cdot; \mu)$ if and only if μ is not an eigenvalue of \mathcal{A} . Theorem 1.3.2 says that the Morse index of 0 is the number of negative eigenvalues, counting multiplicity, of $\mathcal{A} - \mu I$, so (1.34) holds.

(ii) When $v = (\mu - \lambda_j)^q e_j$, then $\|v\|^{p-2} = \mu - \lambda_j$ so (1.32) implies

$$Q_v(h) = \langle \mathcal{A}h, h \rangle - \lambda_j \|h\|^2 + (p-2)(\mu - \lambda_j) \langle e_j, h \rangle^2.$$

Using theorem 1.3.1 again, Q_v will be nondegenerate if and only if λ_j is a simple eigenvalue of \mathcal{A} . Now

$$Q_v(h) = \langle [\mathcal{A} - \lambda_j I + (p-2)(\mu - \lambda_j)P_j]h, h \rangle$$

where $P_j h = \langle e_j, h \rangle e_j$. Thus from theorem 1.3.2 the Morse index of v is the number of negative eigenvalues λ of

$$[\mathcal{A} - \lambda_j I + (p-2)(\mu - \lambda_j)P_j] e_j = \lambda e_j.$$

The negative eigenvalues here correspond precisely to the eigenvalues of \mathcal{A} which are less than λ_j so (1.35) follows. \square

These results extend the analysis of section 1.4. There we saw that when $\mu > \lambda_1$, the critical points on the branch $C_1^{(p)}$ were the minimizers of $\mathcal{F}(\cdot; \mu)$ on H_L . From this analysis one sees that, when $\mu > \lambda_2$, the critical points on $C_2^{(p)}$ are saddle points of $\mathcal{F}(\cdot; \mu)$ with index $m(\lambda_1)$. In general, the Morse index of a critical point on any of the branches $C_j^{(p)}$ may be found explicitly and this index is invariant along the branch.

Along the trivial branch, when μ passes through an eigenvalue λ_j of \mathcal{A} the Morse index of 0 changes with $i^+ - i^- = m(\lambda_j)$ where $i^+ = \lim_{\mu \rightarrow \lambda_j^+} i(0)$ and $i^- = \lim_{\mu \rightarrow \lambda_j^-} i(0)$. Moreover, from theorem 1.4.1, when μ crosses λ_j a branch of critical points diffeomorphic to an $(m(\lambda_j) - 1)$ -dimensional sphere bifurcates from the trivial solution. This differs from the more common situation where a finite number of different branches bifurcate from the trivial branch when $m(\lambda_j) > 1$.

1.6 CONSTRAINED VARIATIONAL PRINCIPLES FOR HIGHER EIGENVALUES

In the last two sections, it has been shown that the variational principle (\mathcal{P}_μ) provides information on λ_1 and, when $\mu > \lambda_1$, $\mathcal{F}(\cdot; \mu)$ is minimized at an eigenfunction of \mathcal{A} corresponding to λ_1 . The other eigenfunctions can only be saddle points of $\mathcal{F}(\cdot; \mu)$; never local minimizers.

Just as for Rayleigh's principle, however, we may develop minimization principles for these other eigenfunctions by looking for constrained minima of $\mathcal{F}(\cdot; \mu)$; subject to certain orthogonality constraints.

Let $\{e_j : 1 \leq j \leq J\}$ be a set of eigenvectors of (1.22) which obey

$$\langle e_j, e_k \rangle = \delta_{jk} \quad \text{for } 1 \leq j, k \leq J.$$

Let V_J be the subspace spanned by this set and define

$$W_J = \{u \in H_L : \langle u, e_j \rangle = 0 \text{ for } 1 \leq j \leq J\}. \quad (1.36)$$

W_J will be the orthogonal complement of V_J with respect to the inner product on H .

Consider the problem $(\mathcal{P}_{\mu,J})$ of minimizing $\mathcal{F}(\cdot; \mu)$ on W_J and finding

$$\alpha_J(\mu) = \inf_{u \in W_J} \mathcal{F}_p(u; \mu). \quad (1.37)$$

Let λ_{J+1} be the least eigenvalue of (1.22) corresponding to an eigenvector of \mathcal{A} which lies in W_J . The next theorem describes the minimizers and the value of this variational principle.

Theorem 1.6.1 *Assume $(\mathcal{L}1)$ -($\mathcal{L}3$) hold, \mathcal{F} is defined by (1.23) with $2 < p < \infty$ and W_J , λ_{J+1} as above. Then*

$$\alpha_J(\mu) = \begin{cases} 0 & \text{if } \mu \leq \lambda_{J+1} \\ -C_p(\mu - \lambda_{J+1})^{pq} & \text{when } \mu > \lambda_{J+1}. \end{cases} \quad (1.38)$$

When $\mu > \lambda_{J+1}$, this infimum is attained at $v = (\mu - \lambda_{J+1})^q e$ with e in $E_{J+1} \cap W_J$.

PROOF: W_J is a closed subspace of H_L and $\mathcal{F}(\cdot; \mu)$ is coercive on H_L , so it is coercive on W_J . Thus $\mathcal{F}(\cdot, \mu)$ has a finite infimum on W_J and it is attained.

The Lagrange multiplier rule and (1.26) implies that the local extrema of $\mathcal{F}(\cdot; \mu)$ on W_J obey

$$\mathcal{A}u + (\|u\|^{(p-2)} - \mu) u = \sum_{j=1}^J \beta_j e_j$$

where β_1, \dots, β_J are real numbers. Take inner products of this with e_k , then $\beta_k = 0$ for each k , as \mathcal{A} is self-adjoint and each u is in W_J .

Thus the local extrema obey (1.28). Repeating the analysis in the proof of part (v) of theorem 1.4.1, leads to (1.38). \square

This theorem shows that the successive eigenvalues, and eigenvectors, of \mathcal{A} may be found by a deflation process. Namely, given the first J eigenvalues $\lambda_1, \dots, \lambda_J$ of \mathcal{A} and a corresponding orthonormal set of eigenfunctions, the next smallest eigenvalue λ_{J+1} and a corresponding eigenfunction is obtained by solving this constrained variational problem on W_J and with $\mu > \lambda_{J+1}$.

Just as in section 1.4, this principle may be modified to find upper and lower bounds on λ_{J+1} .

1.7 INDEFINITE WEIGHTED EIGENPROBLEMS

The preceding methodology may be extended to weighted eigenproblems of the form

$$\mathcal{A}u = \mathcal{L}u + \mathcal{M}u = \lambda \mathcal{B}u \quad (1.39)$$

where \mathcal{L}, \mathcal{M} are as before, and \mathcal{B} obeys

($\mathcal{L}4$) : $\mathcal{B} : H_L \rightarrow H_L^*$ is a compact linear operator.

When \mathcal{B} is positive semi-definite, replace the last two terms in (1.23) by

$$\frac{1}{p} \langle \mathcal{B}u, u \rangle^{p/2} - \frac{\mu}{2} \langle \mathcal{B}u, u \rangle.$$

Then the preceding analysis applies to this functional and analogous results to those obtained above may be obtained in a parallel manner.

It is of much greater interest to show how these methods can be used to describe variational principles for indefinite weighted eigenproblems for which \mathcal{B} is not required to be positive semi-definite. For such problems, the usual Rayleigh-type principles do not apply.

Consider the functional $\mathcal{K} : H_L \times (0, \infty) \rightarrow \mathbb{R}$ defined by

$$\mathcal{K}(u; \mu) = \frac{1}{2} \|u\|_L^2 + \frac{1}{2} \langle \mathcal{M}u, u \rangle + \frac{1}{4} \langle \mathcal{B}u, u \rangle^2 - \frac{\mu}{2} \langle \mathcal{B}u, u \rangle. \quad (1.40)$$

This functional reduces to $\mathcal{F}_4(u; \mu)$ when $\mathcal{B} = I$. The variational principle will be to find

$$\gamma(\mu) = \inf_{u \in H_L} \mathcal{K}(u; \mu). \quad (1.41)$$

The basic properties of this functional, and this variational principle, may be summarized as follows.

Theorem 1.7.1 *Assume $(\mathcal{L}1)$ - $(\mathcal{L}4)$ hold and \mathcal{K} is defined by (1.40). Then*

(i) $\mathcal{K}(\cdot; \mu)$ is continuous, coercive and weakly l.s.c. on H_L ,

(ii) $\mathcal{K}(\cdot; \mu)$ is G -differentiable on H_L with

$$\mathcal{DK}(u; \mu) = \mathcal{L}u + \mathcal{M}u + (\langle \mathcal{B}u, u \rangle - \mu)\mathcal{B}u, \quad (1.42)$$

(iii) v is a nonzero critical point of $\mathcal{K}(\cdot; \mu)$ if and only if it is an eigenvector of (1.39) corresponding to an eigenvalue λ in $(0, \mu)$ and

$$\langle \mathcal{B}v, v \rangle = \mu - \lambda, \quad (1.43)$$

(iv) if (1.39) has a least eigenvalue λ_1 in $(0, \mu)$, then

$$\gamma(\mu) = \begin{cases} 0 & \text{for } 0 < \mu \leq \lambda_1 \\ -\frac{1}{4}(\mu - \lambda_1)^2 & \text{for } \mu > \lambda_1, \end{cases} \quad (1.44)$$

(v) $\mathcal{K}(\cdot; \mu)$ attains its infimum on H_L . When $\mu > \lambda_1$, it is minimized at an eigenvector of (1.39) corresponding to the least positive eigenvalue λ_1 provided $\mu > \lambda_1$. Otherwise it is minimized at 0.

(vi) If there is a vector e in H_L such that $\langle \mathcal{B}e, e \rangle > 0$, then (1.39) has at least one positive eigenvalue.

PROOF: (i) When \mathcal{B} obeys $(\mathcal{L}4)$ then the last two terms in (1.40) are continuous and weakly continuous. Moreover, $\mathcal{K}(u; \mu) \geq \frac{1}{2}\|u\|_L^2 - \mu^2/4$ for all u in H_L as $\frac{1}{4}x^2 - \frac{\mu}{2}x \geq -\mu^2/4$ for all real x . Hence \mathcal{K} is coercive and its infimum is finite.

(ii) follows from the chain rule as the G -derivative of $\langle \mathcal{B}u, u \rangle$ is $2\mathcal{B}u$.

(iii) v is a nonzero critical point of $\mathcal{K}(\cdot; \mu)$ if and only if it is a nonzero solution of

$$\mathcal{A}u = (\mu - \langle \mathcal{B}u, u \rangle) \mathcal{B}u. \quad (1.45)$$

Thus v is an eigenvector of (1.39) with $\lambda = \mu \langle \mathcal{B}v, v \rangle$. Take inner products of (1.45) with v , then since $\langle \mathcal{A}v, v \rangle > 0$, one has $\lambda(\mu - \lambda) > 0$, or $0 < \lambda < \mu$.

When λ is an eigenvalue of (1.39) lying in $(0, \mu)$ and w is a corresponding eigenfunction, then $\langle \mathcal{B}w, w \rangle = \lambda^{-1} \langle \mathcal{A}w, w \rangle > 0$. Let $v = \tau w$ be the multiple of w which obeys (1.43), then w is a critical point of $\mathcal{K}(\cdot; \mu)$ as it obeys (1.45), so (iii) holds.

(iv) if v is a nonzero critical point of $\mathcal{K}(\cdot; \mu)$, then it obeys (1.45) so

$$\langle \mathcal{A}v, v \rangle = \lambda \langle \mathcal{B}v, v \rangle = \lambda(\mu - \lambda).$$

Thus (iv) holds as

$$\mathcal{K}(v; \mu) = \frac{\lambda}{2}(\mu - \lambda) + \frac{1}{4}(\mu - \lambda)^2 - \frac{\mu}{2}(\mu - \lambda) = -\frac{1}{4}(\mu - \lambda)^2.$$

(v) The existence of a minimizer follows from (i). When $\mu > \lambda_1$, $\gamma(\mu) < 0$ so the minimizer must be a nonzero solution of (1.45) from (iii). Thus it is an eigenvector of (1.39).

(vi) Consider $\phi(t, u) = \mathcal{K}(te; \mu) = \frac{1}{4}t^4b^2 + \frac{1}{2}t^2(c_1 - b\mu)$ where $c_1 = \langle \mathcal{A}e, e \rangle$ and $b = \langle \mathcal{B}e, e \rangle > 0$. Take $\mu > c_1/b$ then $\phi(t, \mu)$ will be negative for t^2 small enough and thus $\gamma(\mu) < 0$. Hence the minimizer of \mathcal{K} on H_L is nonzero and will be an eigenvector of (1.39). Thus (1.39) has at least one positive eigenvalue lying in $(0, c_1/b)$. \square

To find the negative eigenvalues of (1.39) just replace \mathcal{B} with $-\mathcal{B}$ in (1.39) and (1.40). Note that (vi) is sharp as if $\langle \mathcal{B}e, e \rangle \leq 0$ for all e in H_L , then (1.39) cannot have any positive eigenvalues since we have assumed $\langle \mathcal{A}u, u \rangle > 0$ for nonzero u in H_L .

It is worthwhile looking at the problem of finding successive positive eigenvalues and corresponding eigenvectors of (1.39). Let $\{e_j : 1 \leq j \leq J\}$ be a set of eigenvectors of (1.39) corresponding to positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_J, \quad \text{and obeying}$$

$$\langle \mathcal{B}e_j, e_k \rangle = \delta_{jk} \quad \text{for } 1 \leq j, k \leq J. \quad (1.46)$$

Let $W_J = \{u \in H_L : \langle \mathcal{B}u, e_j \rangle = 0 \text{ for } 1 \leq j \leq J\}$. Consider the problem of minimizing \mathcal{K} on W_J and finding

$$\gamma_J(\mu) = \inf_{u \in W_J} \mathcal{K}(u; \mu). \quad (1.47)$$

This solutions of this variational principle will be eigenvectors of \mathcal{A} corresponding to the next smallest eigenvalue of the problem (1.39); whenever they exist.

Theorem 1.7.2 *Assume $(\mathcal{L}1)$ - $(\mathcal{L}4)$ hold. If there exists a vector e in W_J obeying $\langle \mathcal{B}e, e \rangle > 0$, then there is another positive eigenvalue λ_{J+1} and a corresponding eigenvector e_{J+1} of (1.39) with e_{J+1} lying in W_J .*

PROOF: W_J is a closed subspace of H_L and $\mathcal{K}(\cdot; \mu)$ is weakly *l.s.c.* and coercive on H_L so $\mathcal{K}(\cdot; \mu)$ attains its infimum on W_J .

Define ϕ as in the proof of (vi) of the previous theorem. For μ large enough, $\gamma_J(\mu) < 0$ and so the minimizer of $\mathcal{K}(\cdot; \mu)$ on W_J will be an eigenvector corresponding to the least eigenvalue λ_{J+1} of (1.39) whose corresponding eigenfunction lies in W_J . Thus the result holds. \square

This theorem allows us to count exactly the number of linearly independent eigenvectors, corresponding to positive eigenvalues of (1.39). Let $\mathcal{E}_+ = \{u \in H_L : \langle Bu, u \rangle > 0\}$. When V is a subspace of $\mathcal{E}_+ \cup \{0\}$, V is said to be maximal if there is no subspace W of H_L which properly contains V and is a subspace of $\mathcal{E}_+ \cup \{0\}$.

Corollary 1.7.3 *Let V be a maximal subspace of $\mathcal{E}_+ \cup \{0\}$. If the dimension of V is J , then (1.39) has exactly J linearly independent eigenvectors corresponding to positive eigenvalues of (1.39).*

PROOF: This follows directly from the last two theorems. \square

As further corollaries of this result note that this indefinite weighted eigenproblem has infinitely many positive, (or negative), linearly independent eigenvectors provided the maximal subspaces of $\mathcal{E}_+ \cup \{0\}$ (or $\mathcal{E}_- \cup \{0\}$ with $\mathcal{E}_- = \{u \in H_L : \langle Bu, u \rangle < 0\}$) are infinite dimensional. The completeness of the eigenfunctions of (1.39) is obtained by using the usual requirement that this maximal subspace be H_L itself or, alternatively, that $\langle Bv, v \rangle > 0$ for all v in $H_L \setminus \{0\}$.

1.8 LINEAR, SECOND-ORDER, SELF-ADJOINT ELLIPTIC EIGENPROBLEMS

In the previous section, various unconstrained variational principles for finding eigenvalues and eigenvectors of closed, self-adjoint linear operators on Hilbert space were developed. Here these methods will be applied for use for both definite and indefinite weighted eigenproblems for linear, second-order, elliptic boundary value problems. The specific, prototypical examples are eigenvalue problems for the Laplacian or Schroedinger operators. The indefinite case often arise in problems of ecology and population modeling. See Hess and Kato, 1980 and the recent surveys of such problems by Belgacem, 1997 and Cosner, 1990.

Let Ω be an open, bounded, connected set in \mathcal{R}^n with a locally Lipschitz boundary $\partial\Omega$. Consider the problem of finding non-trivial solutions (λ, u) of

$$Au(x) = - \sum_{j,k=1}^n D_j(a_{jk}(x)D_k u) + c(x)u = \lambda b(x)u \quad \text{in } \Omega \quad (1.48)$$

$$\text{subject to} \quad u(x) = 0 \quad \text{on } \partial\Omega. \quad (1.49)$$

Here $D_j = \frac{\partial}{\partial x_j}$ represents partial differentiation with respect to x_j , and we shall require

(E1): Each $a_{jk} : \Omega \rightarrow \mathbb{R}$ is a bounded, Lebesgue measurable function and

$$a_{jk}(x) = a_{kj}(x) \text{ a.e. on } \Omega.$$

(E2): There exists $c_0 > 0$ such that

$$\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq c_0 |\xi|^2 \text{ a.e. on } \Omega, \text{ and for all } \xi \text{ in } \mathbb{R}^n. \quad (1.50)$$

(E3): $c : \Omega \rightarrow [0, \infty)$ and $b : \Omega \rightarrow \mathbb{R}$ are essentially bounded, Lebesgue measurable functions.

(E4): The set $B_+ = \{x \in \Omega : b(x) > 0\}$ has positive Lebesgue measure.

Let $H = L^2(\Omega)$ be the usual real Hilbert space of all square-integrable, Lebesgue measurable, real valued functions defined on Ω . The inner product is

$$\langle u, v \rangle = \int_{\Omega} u(x)v(x) \, dx. \quad (1.51)$$

Under the assumptions (E1)-(E3), the closure of the operator A defined by the left hand side of (1.48)-(1.49) is a self-adjoint, densely defined, linear operator. Its domain is $D_A = H^2(\Omega) \cap H_0^1(\Omega)$; see Brezis, 1983, chapter 9 for definitions of the Sobolev spaces and these properties.

The quadratic form f defined by (1.3), associated with this operator, is

$$f(u) = -\frac{1}{2} \int_{\Omega} u \left[\sum_{j,k} \left(\frac{\partial}{\partial x_j} (a_{jk} \frac{\partial u}{\partial x_k}) - cu \right) \right] dx \quad (1.52)$$

for u in D_A . Upon integrating by parts,

$$f(u) = \frac{1}{2} \int_{\Omega} \left[\sum_{j,k} a_{jk} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} + cu^2 \right] dx. \quad (1.53)$$

In this case H_A will be the Sobolev space $H_0^1(\Omega)$ and the closure \hat{f} of the functional defined by (1.52) is given by (1.53) for all u in $H_0^1(\Omega)$. The dual space H_A^* is $H^{-1}(\Omega)$ as described in Brezis, 1983, chapter 9.

The functional for this problem, analogous to \mathcal{K} in section 1.7, is $\mathcal{G} : H_0^1(\Omega) \times (0, \infty) \rightarrow \mathbb{R}$ with

$$\mathcal{G}(u; \mu) = \frac{1}{2} \int_{\Omega} \left[\sum_{j,k} a_{jk} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} + (c - \mu b)u^2 \right] dx + \frac{1}{4} \left(\int_{\Omega} bu^2 dx \right)^2. \quad (1.54)$$

The variational principle (\mathcal{Q}_μ) is to minimize $\mathcal{G}(\cdot; \mu)$ on $H_0^1(\Omega)$ and to find

$$\alpha(\mu) = \inf_{u \in H_0^1(\Omega)} \mathcal{G}(u; \mu). \quad (1.55)$$

This is an unconstrained problem. Some basic properties of the functional \mathcal{G} and this variational principle will be summarized below. The next theorem parallels theorem 1.7.1. Henceforth all integrals will be Lebesgue integrals over Ω and the domain will often be omitted.

Theorem 1.8.1 *Assume (E1)-(E3) hold and $\mathcal{G}(\cdot; \mu)$ be defined by (1.54). Then*

(i) $\mathcal{G}(\cdot; \mu)$ is continuous, coercive and weakly l.s.c. on $H_0^1(\Omega)$,

(ii) $D\mathcal{G}(\cdot; \mu)$ is Gateaux differentiable on $H_0^1(\Omega)$ with

$$D\mathcal{G}(u, \mu) = \mathcal{A}u + \left(\int bu^2 - \mu \right) bu. \quad (1.56)$$

(iii) *If v is a nonzero critical point of $\mathcal{G}(\cdot; \mu)$, then $v = (\mu - \lambda)^{1/2}e$ where λ is an eigenvalue of (1.48)-(1.49) lying in $(0, \mu)$ and e is a corresponding eigenfunction obeying*

$$\int be^2 = 1 \quad (1.57)$$

(iv) *the nonzero critical values of $\mathcal{G}(\cdot; \mu)$ are $-\frac{1}{4}(\mu - \lambda)^2$ where λ is an eigenvalue of (1.48)-(1.49) in $(0, \mu)$.*

(v) *if λ_1 is the least positive eigenvalue of (1.48)-(1.49) then*

$$\alpha(\mu) = \begin{cases} 0 & \text{when } \mu \leq \lambda_1 \\ -\frac{1}{4}(\mu - \lambda_1)^2 & \text{for } \mu > \lambda_1, \end{cases} \quad (1.58)$$

(vi) $\mathcal{G}(\cdot; \mu)$ attains its infimum on $H_0^1(\Omega)$. When $\mu > \lambda_1$, this infimum is attained at an eigenfunction corresponding to λ_1 ,

(vii) if there is a function w in $H_0^1(\Omega)$ obeying $\int bw^2 > 0$, then (1.48)-(1.49) has at least one positive eigenvalue λ_1 .

PROOF: (i) The operator $\mathcal{A} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is defined by the quadratic form

$$\langle \mathcal{A}u, v \rangle = \int_{\Omega} \left[\sum_{j,k=1}^n a_{jk} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} + cuv \right] dx \quad (1.59)$$

\mathcal{A} is continuous, self-adjoint and positive definite linear operator when (E1)-(E3) hold. Thus the function $g : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by $g(u) = \frac{1}{2} \langle \mathcal{A}u, u \rangle$ is continuous, convex and weakly l.s.c. on $H_0^1(\Omega)$. Moreover from (E1)-(E3) and the assumptions on Ω there exists a $c_1 > 0$ such that

$$g(u) \geq c_1 \|u\|_{1,2}^2 \quad (1.60)$$

where $\|u\|_{1,2}$ is the usual norm on $H_0^1(\Omega)$.

Define $\mathcal{B} : H_0^1(\Omega) \rightarrow L^2(\Omega)$ by $\mathcal{B}u(x) = b(x)u(x)$. \mathcal{B} is a continuous linear operator with $\|\mathcal{B}u\| \leq \|b\|_{\infty} \|u\| \leq \|b\|_{\infty} \|u\|_{1,2}$ where $\|\cdot\|_{\infty}$ is the usual norm on $L^{\infty}(\Omega)$.

Thus $\mathcal{B} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is compact as the imbedding $i : H_0^1(\Omega) \rightarrow L^2(\Omega)$ and its dual $i^* : L^2(\Omega) \rightarrow H^{-1}(\Omega)$ are compact. Hence the functional $\langle \mathcal{B}u, u \rangle$ is weakly continuous on $H_0^1(\Omega)$. The coercivity of \mathcal{G} now follows from (1.60) just as in the proof of 1.7.1.

(ii) follows by a direct computation

(iii) v is a critical point of $\mathcal{G}(\cdot; \mu)$ if and only if it is a solution of

$$\mathcal{A}u = \left(\mu - \int bu^2 \right) bu. \quad (1.61)$$

When v is nonzero then it is a solution of $\mathcal{A}u = \lambda \mathcal{B}u$ with $\int bv^2 = \mu - \lambda$. Since v is $H_0^1(\Omega)$, then v is a (weak) solution of

$$-\sum_{j,k} \frac{\partial}{\partial x_j} \left(a_{jk} \frac{\partial v}{\partial x_k} \right) = (\lambda b - c) v \quad \text{subject to (1.49).}$$

The right hand side here is in L^2 and thus v lies in $H^2(\Omega)$. Hence v is a solution of (1.48)-(1.49) which lies in D_A . The rest of this theorem follows, *mutatis mutandis*, as in theorem 1.7.1. \square

The criterion in (vii) for this problem to have a positive eigenvalue is non-constructive, so the following lemma is helpful.

Lemma 1.8.2 *When (E4) holds, there exists w in $H_0^1(\Omega)$ such that $\int bw^2 > 0$.*

PROOF: When (E4) holds, there exists a $\delta_1 > 0$ such that $B_1 = \{x \in \Omega : b(x) \geq \delta_1\}$ has positive Lebesgue measure. For $0 < \delta \leq \delta_1$, let

$$\chi_\delta(x) = \begin{cases} 1 & \text{if } b(x) \geq \delta \text{ and } d(x, \partial\Omega) \geq \delta \\ 0 & \text{otherwise} \end{cases}$$

Then for $\delta \leq \delta_2$ (say) this function is non-zero on a set of positive measure.

Let ρ_ϵ for $\epsilon > 0$, be the usual regularizing approximate identity with respect to convolution - which will be indicated by \star . Then for $\epsilon < \delta_2$, $\rho_\epsilon \star \chi_\delta$ will be a function in $C_0^\infty(\Omega)$ and $\rho_\epsilon \star \chi_\delta \rightarrow \chi_\delta$ as $\epsilon \rightarrow 0$ in $L^2(\Omega)$. Thus $\lim_{\epsilon \rightarrow 0+} \int b(\rho_\epsilon \star \chi_\delta)^2 = \int b\chi_\delta^2 > 0$. Take $w = \rho_\epsilon \star \chi_\delta$ for ϵ small enough and the lemma holds. \square

Corollary 1.8.3 *When (E1)-(E4) hold, then there is a least positive eigenvalue λ_1 of (1.48)-(1.49). When $\mu > \lambda_1$, then $\mathcal{G}(\cdot; \mu)$ is minimized on $H_0^1(\Omega)$ at an eigenfunction corresponding to this eigenvalue.*

Successive eigenvalues, and corresponding eigenfunctions of (1.48) can be obtained by minimizing $\mathcal{G}(\cdot; \mu)$, for μ large enough, on subspaces on $H_0^1(\Omega)$.

Let $\{e_1, \dots, e_J\}$ be a set of eigenfunctions of (1.48)-(1.49) corresponding to eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_J$ and obeying

$$\int b e_j e_k = \delta_{jk} \text{ for } 1 \leq j, k \leq J. \quad (1.62)$$

Such sets are called *b-orthonormal*. Let $W_J = \{u \in H_0^1(\Omega) : \int b e_j u = 0 \text{ for } 1 \leq j \leq J\}$ and consider the problem $(\mathcal{Q}_{\mu, J})$ of minimizing $\mathcal{G}(\cdot; \mu)$ on W_J and finding

$$\alpha_J(\mu) = \inf_{u \in W_J} \mathcal{G}(u; \mu). \quad (1.63)$$

Theorem 1.8.4 *Assume (E1)-(E4) hold, $\mathcal{G}(\cdot; \mu)$ is defined by (1.54) and W_J as above. If there exists w in W_J obeying $\int b w^2 > 0$, then there exists a least positive eigenvalue λ_{J+1} of (1.48)-(1.49) corresponding to an eigenfunction e_{J+1} in W_J . Moreover (i)*

$$\alpha_J(\mu) = \begin{cases} 0 & \text{if } \mu \leq \lambda_{J+1} \\ -\frac{1}{4}(\mu - \lambda_{J+1})^2 & \text{if } \mu > \lambda_{J+1} \end{cases}$$

and (ii) when $\mu > \lambda_{J+1}$, this infimum is attained at an eigenfunction corresponding to λ_{J+1} .

PROOF: The proof is similar to that of theorem 1.6.1, adapted to this functional. The argument that λ_{J+1} exists is similar to that of (vi) of theorem 1.7.1. \square

Again a Morse theory for this problem can be developed.

Lemma 1.8.5 *Assume (E1)-(E4) hold and $\mathcal{G}(\cdot; \mu)$ is defined by (1.54). Then $\mathcal{DG}(\cdot; \mu)$ is G -differentiable on $H_0^1(\Omega)$ and for any u, h in $H_0^1(\Omega)$*

$$\mathcal{D}^2 \mathcal{G}(u; \mu)h = \mathcal{A}h + \left(\int bu^2 - \mu \right) bh + 2 \left(\int buh \right) bu. \quad (1.64)$$

PROOF: The only nonlinear term in (1.56) is $\mathcal{N}(u) = \left(\int bu^2 \right) bu$. A direct calculation shows that \mathcal{N} is G -differentiable on $H_0^1(\Omega)$ with

$$\mathcal{DN}(u)h = 2 \left(\int buh \right) bu + \left(\int bu^2 \right) bh.$$

This leads to (1.64). \square

The (Hessian) quadratic form associated with the second derivative of this functional is $Q_u : H_0^1(\Omega) \rightarrow \mathbb{R}$ with

$$\begin{aligned} Q_u(h) &= \langle \mathcal{D}^2 \mathcal{G}(u; \mu)h, h \rangle \\ &= \langle \mathcal{A}h, h \rangle + \left(\int bu^2 - \mu \right) \int bh^2 + 2 \left(\int buh \right)^2. \end{aligned} \quad (1.65)$$

This functional fits into the framework described in section 1.3 with $H = L^2(\Omega)$, $X = H_0^1(\Omega)$, and $A = \mathcal{A} + \left(\int bu^2 - \mu \right) \mathcal{B} + 2\mathcal{P}$. Here $\mathcal{P} : H_0^1(\Omega) \rightarrow L^2(\Omega)$ is defined by

$$\mathcal{P}h(x) = \left(\int buh \right) b(x)u(x). \quad (1.66)$$

Take $B = \mathcal{B}$, then the inequality (1.15) follows from (1.60).

Theorem 1.8.6 *Assume (E1)-(E3) hold and v is a non-degenerate critical point of $\mathcal{G}(\cdot; \mu)$ in $H_0^1(\Omega)$, then*

(i) *$v = 0$ is a nondegenerate critical point of $\mathcal{G}(\cdot; \mu)$ if and only if μ is not an eigenvalue of (1.48)-(1.49).*

(ii) *When $v = (\mu - \lambda)^{1/2}e$ with $0 < \lambda < \mu$, then v is nondegenerate if and only if λ is a simple eigenvalue of (1.48)-(1.49).*

PROOF: This follows from (1.64) and theorem 1.3.1 just as was done in the proof of theorem (1.30). \square

To actually evaluate the Morse index of a critical point, stronger assumptions on b are needed. We shall require

(E5): $b(x) > 0$, *a.e.* on Ω and the distinct eigenvalues of (1.48)-(1.49) are ordered.

When (E5) holds, then (A4) holds on $H_0^1(\Omega)$ and the following result holds.

Theorem 1.8.7 *Assume (E1)-(E5) hold and v is a critical point of $\mathcal{G}(\cdot; \mu)$ on $H_0^1(\Omega)$. Then the Morse index of v is finite, and*

(i) *if $v = 0$, then the Morse index of 0 is 0 when $\mu \leq \lambda_1$ and it is*

$$\sum_{\lambda_j < \mu} m(\lambda_j) \text{ when } \mu > \lambda_1 \text{ where } m(\lambda_j) = \dim N(\mathcal{A} - \lambda_j \mathcal{B}).$$

(ii) *if $v = (\mu - \lambda_j)^{1/2}e_j$, then its Morse index is 0 if $j = 1$ and*

$$\sum_{k=1}^{j-1} m(\lambda_k) \quad \text{when } j > 1.$$

PROOF: (i) When $v = 0$, then $Q_0(h) = \langle \mathcal{A}h, h \rangle - \mu \int b h^2$.

Applying theorem 1.3.2, the Morse index of this is finite and equals the number of eigenvalues, counting multiplicity, or $(\mathcal{A}, \mathcal{B})$ which are less than μ .

Thus (i) follows.

(ii) when $v = (\mu - \lambda_j)^{1/2} e_j$, then $\int b v^2 = \mu - \lambda_j$ so

$$Q_v(h) = \langle \mathcal{A}h, h \rangle - \lambda_j \int b h^2 + 2(\mu - \lambda_j) \left(\int b e_j h \right)^2.$$

Applying theorem 1.3.2 again (ii) follows. \square

These results show that the comments at the end of section 1.5 on the behavior of the bifurcation diagram carry over to this problem. In particular when μ increases through an eigenvalue λ_j , a branch of eigenfunctions of (1.48)-(1.49) that is diffeomorphic to an $(m(\lambda_j) - 1)$ -dimensional sphere bifurcates from the zero branch and the Morse index of 0 increases by $m(\lambda_j)$.

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