MATH 6303 **STUDY GUIDE FOR TEST 1** Spring 2008

Know the Definitions. Domain (page 212); associates in a domain R; for $a, b \in R$: $a|b$; a is a unit; a is prime; a is irreducible. Principal Ideal Domain (PID) (page 183); Unique Factorization Domain (UFD) (page 213); greatest common divisor (page 215); relatively prime elements in a UFD (page 215); Euclidean Domain (page 217).

Let F be a field. Then $F[x]$ is a Euclidean Domain (page 217, 3.1 (b)), hence a PID (page 218, 3.2), hence a UFD (page 218, 3.3). Root of $f \in F[x]$ (page 282); subfield of a field (page 168); extension (field) E of F; degree $[E : F]$ (page 285); finite extension (page 285); embedding of a field (page 286). For E an extension of F, $a \in E$ is algebraic over F (page 289); minimal polynomial; E is a algebraic (transcendental) over F (page 290). Finitely generated extension of F (page 292). Algebraic closure of F in an extension E (page 293); embedding *over* a common subfield (page 293). Algebraically closed field; an extension E being an algebraic closure of F (page 295).

Know Examples. Know examples of domains, and of rings that are no domains; of UFDs and domains which are no UFDs; of PIDs and of UFDs which are no PIDs; of Euclidean domains and of a PID which is not a Euclidean Domain (see bottom of page 218).

Know examples of fields and subfields and field extensions; reducible and irreducible polynomials over a field; algebraic and transcendental extensions; finite extensions and infinite extensions of a field; algebraically closed fields and those which are not algebraically closed.

Know the Results. Every irreducible element in a PID is prime (page 213, 1.1); every nonzero nonunit in a UFD is a product of irreducible factors which is unique up to order and associates of the factors (page 214, 1.3); any two elements in a UFD have a g.c.d. which is unique up to unit multiples (page 215, 1.4); every PID is a UFD (page 216); every Euclidean domain is a PID (page 218); polynomial rings over UFDs are UFDs (page 222, 4.3 – this was mentioned in class without proof).

Know the properties of $F[x]$ (page 281). Add to those the following two properties both of which were proved in class:

(vi) The degree formula: If f and g are nonzero polynomials over F, then $\deg(\mathrm{fg}) = \deg \mathrm{f} + \deg \mathrm{g}$.

(vii) The Evaluation homomorphism: If E is an extension field of F and $u \in E$, defining $\varepsilon : F[x] \to E$ by $\varepsilon(f) = f(u)$, $f \in F[x]$, is a ring homomorphism.

A polynomial $f \in F[x]$ of degree at least two which has a root in F is reducible; every reducible $f \in F[x]$ which has degree 2 or 3 has a root in F (page 282). Know the degree formula for fields (2.1, page 285 – we did not prove it in class). Given any nonconstant polynomial f over F, there exists a field containing a subfield isomorphic to F in which f has a root. (page 287, 2.4). If E is an extension field of F and $u \in E$ is algebraic over F, the sub*field* $F(u)$ of E generated by $F \cup \{u\}$ equals the sub*ring* $F[u]$ of E generated by $F \cup \{u\}$, and the set $\{1, u, \ldots, u^{n-1}\}$ is a basis a basis for the vector space $F(u)$ over F where n is the degree of the minimal polynomial of u; in particular, $[F(u) : F] = n$ (page 288, 2.5). Finite field extensions are algebraic (page 290, 3.2). Not every algebraic extension is finite (page 291, 3.4). If $u \in E$ is not algebraic over F, $F(u)$ is not finite over F (page 292, 3.5). A finitely generated extension $E = F(u_1, \ldots, u_n)$ of F with each u_i algebraic over F is finite, hence algebraic over F (page 292, 3.6). The set of all $u \in E$ being algebraic over the subfield F of E is a subfield of E (page 292, 3.7), the largest subfield of E being algebraic over F. Every embedding of an algebraic extension field E of F which fixes all elements of F must be an automorphism of E (page 293, 3.8). Know the equivalent properties characterizing an algebraically closed field (page 295, 4.1). Any two algebraic closures of a field F are F–isomorphic, i.e. there exists a ring isomorphism mapping one to the other fixing all elements of F (page 297, 4.4). Given any field F, there exists an algebraically closed field containing F as a subfield (page 297, 4.5 – we did not prove this in class). Every field has an algebraic closure (page 298, 4.6).

Here is an Example we did not cover (page 299): Let $F = \mathbb{R}$. It is known that $\mathbb{C}$ is algebraically closed. Also, the algebraic closure $\overline{\mathbb{R}}$ of $\mathbb{R}$ in $\mathbb{C}$ is algebraic over $\overline{\mathbb{R}}$, and

$$\mathbb{R} \subseteq \overline{\mathbb{R}} \subseteq \mathbb{C}\,.$$

Since $[\mathbb{C} : \mathbb{R}] = 2$, every field between $\mathbb{R}$ and $\mathbb{C}$ must have dimension 1 or dimension 2 as a vector space over $\mathbb{R}$. Since $\sqrt{-1} \notin \mathbb{R}$, the former is impossible, thus $\overline{\mathbb{R}} = \mathbb{C}$.

Know how to work the homework problems. Save your questions for our review session on Thursday, March 6.

Test 1 is scheduled for Tuesday, March 11 and will address those topics from Chapters 11 and 15 that were covered in class (From Chapter 11, omit pages 221–223, starting with *Definition*, but be aware of 4.3, page 222; from Chapter 15, omit 1.4 through 1.9).