It should be evident that the second part of the proof can be obtained from the first simply by reversing the steps. That is, when each $\Rightarrow$ is replaced by $\Leftarrow$, a valid implication results. In fact, then, we could obtain a proof of both parts by replacing $\Rightarrow$ with $\Leftrightarrow$, where $\Leftrightarrow$ is short for "if and only if." Thus

$$\begin{aligned} x \in A \cap (B \cup C) &\Leftrightarrow x \in A \quad \text{and} \quad x \in (B \cup C) \\ &\Leftrightarrow x \in A, \quad \text{and} \quad x \in B \quad \text{or} \quad x \in C \\ &\Leftrightarrow x \in A \quad \text{and} \quad x \in B, \quad \text{or} \quad x \in A \quad \text{and} \quad x \in C \\ &\Leftrightarrow x \in A \cap B, \quad \text{or} \quad x \in A \cap C \\ &\Leftrightarrow x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

■

Strategy In proving an equality of sets S and T, we can often use the technique of showing that $S \subseteq T$ and then check to see whether the steps are reversible. In many cases, the steps are indeed reversible, and we obtain the other part of the proof easily. However, this method should not obscure the fact that there are still two parts to the argument: $S \subseteq T$ and $T \subseteq S$.

There are some interesting relations between complements and unions or intersections. For example, it is true that

$$(A \cap B)' = A' \cup B'.$$

This statement is one of two that are known as **De Morgan's† Laws**. De Morgan's other law is the statement that

$$(A \cup B)' = A' \cap B'.$$

Stated somewhat loosely in words, the first law says that the complement of an intersection is the union of the individual complements. The second similarly says that the complement of a union is the intersection of the individual complements.

Exercises 1.1

True or False

Label each of the following statements as either true or false.

1. Two sets are equal if and only if they contain exactly the same elements.
2. If A is a subset of B and B is a subset of A, then A and B are equal.
3. The empty set is a subset of every set except itself.
4. $A - A = \emptyset$ for all sets A.
5. $A \cup A = A \cap A$ for all sets A.

---

†Augustus De Morgan (1806–1871) coined the term mathematical induction and is responsible for rigorously defining the concept. Not only does he have laws of logic bearing his name but also the headquarters of the London Mathematical Society and a crater on the moon.

6. $A \subset A$ for all sets A.
7. $\{a, b\} = \{b, a\}$
8. $\{a, b\} = \{b, a, b\}$
9. $A - B = C - B$ implies $A = C$, for all sets A, B, and C.
10. $A - B = A - C$ implies $B = C$, for all sets A, B, and C.

Exercises

1. For each set A, describe A by indicating a property that is a qualification for membership in A.
   - **a.** $A = \{0, 2, 4, 6, 8, 10\}$
   - **b.** $A = \{1, -1\}$
   - **c.** $A = \{-1, -2, -3, \ldots\}$
   - **d.** $A = \{1, 4, 9, 16, 25, \ldots\}$

2. Decide whether or not each statement is true for $A = \{2, 7, 11\}$ and $B = \{1, 2, 9, 10, 11\}$. Sec. 3.1, #37–39
   - **a.** $2 \subseteq A$
   - **b.** $\{11, 2, 7\} \subseteq A$
   - **c.** $2 = A \cap B$
   - **d.** $\{7, 11\} \in A$
   - **e.** $A \subseteq B$
   - **f.** $\{7, 11, 2\} = A$

3. Decide whether or not each statement is true, where A and B are arbitrary sets.
   - **a.** $B \cup A \subseteq A$
   - **b.** $B \cap A \subseteq A \cup B$
   - **c.** $\emptyset \subseteq A$
   - **d.** $0 \in \emptyset$
   - **e.** $\emptyset \in \{\emptyset\}$
   - **f.** $\emptyset \subseteq \{\emptyset\}$
   - **g.** $\{\emptyset\} \subseteq \emptyset$
   - **h.** $\{\emptyset\} = \emptyset$
   - **i.** $\emptyset \in \emptyset$
   - **j.** $\emptyset \subseteq \emptyset$

4. Decide whether or not each of the following is true for all sets A, B, and C. Sec. 2.2, #33–36
   - **a.** $A \cap A' = \emptyset$
   - **b.** $A \cap \emptyset = A \cup \emptyset$
   - **c.** $A \cap (B \cup C) = A \cup (B \cap C)$
   - **d.** $A \cup (B' \cap C') = A \cup (B \cup C)'$
   - **e.** $A \cup (B \cap C) = (A \cup B) \cap C$
   - **f.** $(A \cap B) \cup C = A \cap (B \cup C)$
   - **g.** $A \cup (B \cap C) = (A \cap C) \cup (B \cap C)$
   - **h.** $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$

5. Evaluate each of the following sets, where

$$U = \{0, 1, 2, 3, \ldots, 10\}$$
$$A = \{0, 1, 2, 3, 4, 5\}$$
$$B = \{0, 2, 4, 6, 8, 10\}$$
$$C = \{2, 3, 5, 7\}.$$

   - **a.** $A \cup B$
   - **b.** $A \cap C$
   - **c.** $A' \cup B$
   - **d.** $A \cap B \cap C$
   - **e.** $A' \cap B \cap C$
   - **f.** $A \cup (B \cap C)$
   - **g.** $A \cap (B \cup C)$
   - **h.** $(A \cup B')'$
   - **i.** $A - B$
   - **j.** $B - A$
   - **k.** $A - (B - C)$
   - **l.** $C - (B - A)$
   - **m.** $(A - B) \cap (C - B)$
   - **n.** $(A - B) \cap (A - C)$

6. Write each of the following as either A, A', U, or $\emptyset$, where A is an arbitrary subset of the universal set U.

   a. $A \cap A$ **b.** $A \cup A$
   c. $A \cap A'$ **d.** $A \cup A'$
   e. $A \cup \emptyset$ **f.** $A \cap \emptyset$
   g. $A \cap U$ **h.** $A \cup U$
   i. $U \cup A'$ **j.** $A - \emptyset$
   k. $\emptyset'$ **l.** U'
   m. $(A')'$ **n.** $\emptyset - A$

7. Write out the power set, $\mathcal{P}(A)$, for each set A.

   a. $A = \{a\}$ **b.** $A = \{0, 1\}$
   Sec. 3.1, #37–39 ⋖
   c. $A = \{a, b, c\}$ **d.** $A = \{1, 2, 3, 4\}$
   e. $A = \{1, \{1\}\}$ **f.** $A = \{\{1\}\}$
   g. $A = \{\emptyset\}$ **h.** $A = \{\emptyset, \{\emptyset\}\}$

8. Describe two partitions of each of the following sets.

   a. $\{x \mid x$ is an integer$\}$ **b.** $\{a, b, c, d\}$
   c. $\{1, 5, 9, 11, 15\}$ **d.** $\{x \mid x$ is a complex number$\}$

9. Write out all the different partitions of the given set A.

   a. $A = \{1, 2, 3\}$ **b.** $A = \{1, 2, 3, 4\}$

10. Suppose the set A has n elements where $n \in \mathbf{Z}^+$.

    a. How many elements does the power set $\mathcal{P}(A)$ have?
    Sec. 2.2, #33–36 ⋖
    b. If $0 \le k \le n$, how many elements of the power set $\mathcal{P}(A)$ contain exactly k elements?

11. State the most general conditions on the subsets A and B of U under which the given equality holds.

    a. $A \cap B = A$ **b.** $A \cup B' = A$
    c. $A \cup B = A$ **d.** $A \cap B' = A$
    e. $A \cap B = U$ **f.** $A' \cap B' = \emptyset$
    g. $A \cup \emptyset = U$ **h.** $A' \cap U = \emptyset$

12. Let $\mathbf{Z}$ denote the set of all integers, and let

$$A = \{x \mid x = 3p - 2 \text{ for some } p \in \mathbf{Z}\}$$
$$B = \{x \mid x = 3q + 1 \text{ for some } q \in \mathbf{Z}\}.$$

    Prove that $A = B$.

13. Let $\mathbf{Z}$ denote the set of all integers, and let

$$C = \{x \mid x = 3r - 1 \text{ for some } r \in \mathbf{Z}\}$$
$$D = \{x \mid x = 3s + 2 \text{ for some } s \in \mathbf{Z}\}.$$

    Prove that $C = D$.

In Exercises 14–33, prove each statement.

14. $A \cap B \subseteq A \cup B$

15. $(A')' = A$

16. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

17. $A \subseteq B$ if and only if $B' \subseteq A'$.

18. $A \cup (B \cup C) = (A \cup B) \cup C$

19. $(A \cup B)' = A' \cap B'$

20. $(A \cap B)' = A' \cup B'$

21. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

22. $A \cap (A' \cup B) = A \cap B$

23. $A \cup (A' \cap B) = A \cup B$

24. $A \cup (A \cap B) = A \cap (A \cup B)$

25. If $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

26. If $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

27. $B - A = B \cap A'$

28. $A \cap (B - A) = \emptyset$

29. $A \cup (B - A) = A \cup B$

30. $(A \cup B) - C = (A - C) \cup (B - C)$

31. $(A - B) \cup (A \cap B) = A$

32. $A \subseteq B$ if and only if $A \cup B = B$.

33. $A \subseteq B$ if and only if $A \cap B = A$.

34. Prove or disprove that $A \cup B = A \cup C$ implies $B = C$.

35. Prove or disprove that $A \cap B = A \cap C$ implies $B = C$.

36. Prove or disprove that $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

37. Prove or disprove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

38. Prove or disprove that $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.

39. Express $(A \cup B) - (A \cap B)$ in terms of unions and intersections that involve A, A', B, and B'.

40. Let the operation of addition be defined on subsets A and B of U by $A + B = (A \cup B) - (A \cap B)$. Use a Venn diagram with labeled regions to illustrate each of the following statements.

a. $A + B = (A - B) \cup (B - A)$

b. $A + (B + C) = (A + B) + C$

c. $A \cap (B + C) = (A \cap B) + (A \cap C)$.

41. Let the operation of addition be as defined in Exercise 40. Prove each of the following statements.

a. $A + A = \emptyset$

b. $A + \emptyset = A$

1.2 Mappings

The concept of a function is fundamental to nearly all areas of mathematics. The term *function* is the one most widely used for the concept that we have in mind, but it has become traditional to use the terms *mapping* and *transformation* in algebra. It is likely that these words are used because they express an intuitive feel for the association between the elements involved. The basic idea is that correspondences of a certain type exist between

Exercises 1.2

True or False

Label each of the following statements as either true or false.

1. $A \times A = A$, for every set A.
2. $A \times \emptyset = \emptyset$, for every set A.
3. Let $f: A \to B$ where A and B are nonempty. Then $f^{-1}(f(S)) = S$ for every subset S of A.
4. Let $f: A \to B$ where A and B are nonempty. Then $f(f^{-1}(T)) = T$ for every subset T of B.
5. Let $f: A \to B$. Then $f(A) = B$ for all nonempty sets A and B.
6. Every bijection is both one-to-one and onto.
7. A mapping is onto if and only if its codomain and range are equal.
8. Let $g: A \to A$ and $f: A \to A$. Then $(f \circ g)(a) = (g \circ f)(a)$ for every a in A.
9. Composition of mappings is an associative operation.

Exercises

1. For the given sets, form the indicated Cartesian product.

   a. $A \times B; A = \{a, b\}, B = \{0, 1\}$
   b. $B \times A; A = \{a, b\}, B = \{0, 1\}$
   c. $A \times B; A = \{2, 4, 6, 8\}, B = \{2\}$
   d. $B \times A; A = \{1, 5, 9\}, B = \{-1, 1\}$
   e. $B \times A; A = B = \{1, 2, 3\}$

2. For each of the following mappings, state the domain, the codomain, and the range, where $f: \mathbf{E} \to \mathbf{Z}$.

   a. $f(x) = x/2, x \in \mathbf{E}$ **b.** $f(x) = x, x \in \mathbf{E}$
   c. $f(x) = |x|, x \in \mathbf{E}$ **d.** $f(x) = x + 1, x \in \mathbf{E}$

3. For each of the following mappings, write out $f(S)$ and $f^{-1}(T)$ for the given S and T, where $f: \mathbf{Z} \to \mathbf{Z}$.

   a. $f(x) = |x|; S = \mathbf{Z} - \mathbf{E}, T = \{1, 3, 4\}$
   b. $f(x) = \begin{cases} x + 1 & \text{if } x \text{ is even} \\ x & \text{if } x \text{ is odd;} \end{cases} \quad S = \{0, 1, 5, 9\}, T = \mathbf{Z} - \mathbf{E}$
   c. $f(x) = x^2; S = \{-2, -1, 0, 1, 2\}, T = \{2, 7, 11\}$
   d. $f(x) = |x| - x; S = T = \{-7, -1, 0, 2, 4\}$

4. For each of the following mappings $f: \mathbf{Z} \to \mathbf{Z}$, determine whether the mapping is onto and whether it is one-to-one. Justify all negative answers.

   a. $f(x) = 2x$ **b.** $f(x) = 3x$
   c. $f(x) = x + 3$ **d.** $f(x) = x^3$
   e. $f(x) = |x|$ **f.** $f(x) = x - |x|$

g. $f(x) = \begin{cases} x & \text{if } x \text{ is even} \\ 2x - 1 & \text{if } x \text{ is odd} \end{cases}$ **h.** $f(x) = \begin{cases} x & \text{if } x \text{ is even} \\ x - 1 & \text{if } x \text{ is odd} \end{cases}$

i. $f(x) = \begin{cases} x & \text{if } x \text{ is even} \\ \dfrac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$ **j.** $f(x) = \begin{cases} x - 1 & \text{if } x \text{ is even} \\ 2x & \text{if } x \text{ is odd} \end{cases}$

5. For each of the following mappings $f\colon \mathbf{R} \to \mathbf{R}$, determine whether the mapping is onto and whether it is one-to-one. Justify all negative answers. (Compare these results with the corresponding parts of Exercise 4.)

a. $f(x) = 2x$ **b.** $f(x) = 3x$

c. $f(x) = x + 3$ **d.** $f(x) = x^3$

e. $f(x) = |x|$ **f.** $f(x) = x - |x|$

6. For the given subsets A and B of $\mathbf{Z}$, let $f(x) = 2x$ and determine whether $f\colon A \to B$ is onto and whether it is one-to-one. Justify all negative answers.

a. $A = \mathbf{Z}$, $B = \mathbf{E}$ **b.** $A = \mathbf{E}$, $B = \mathbf{E}$

7. For the given subsets A and B of $\mathbf{Z}$, let $f(x) = |x|$ and determine whether $f\colon A \to B$ is onto and whether it is one-to-one. Justify all negative answers.

a. $A = \mathbf{Z}$, $B = \mathbf{Z}^+ \cup \{0\}$ **b.** $A = \mathbf{Z}^+$, $B = \mathbf{Z}$

c. $A = \mathbf{Z}^+$, $B = \mathbf{Z}^+$ **d.** $A = \mathbf{Z} - \{0\}$, $B = \mathbf{Z}^+$

8. For the given subsets A and B of $\mathbf{Z}$, let $f(x) = |x + 4|$ and determine whether $f\colon A \to B$ is onto and whether it is one-to-one. Justify all negative answers.

a. $A = \mathbf{Z}$, $B = \mathbf{Z}$ **b.** $A = \mathbf{Z}^+$, $B = \mathbf{Z}^+$

9. For the given subsets A and B of $\mathbf{Z}$, let $f(x) = 2^x$ and determine whether $f\colon A \to B$ is onto and whether it is one-to-one. Justify all negative answers.

a. $A = \mathbf{Z}^+$, $B = \mathbf{Z}$ **b.** $A = \mathbf{Z}^+$, $B = \mathbf{Z}^+ \cap \mathbf{E}$

10. For each of the following parts, give an example of a mapping from $\mathbf{E}$ to $\mathbf{E}$ that satisfies the given conditions.

a. one-to-one and onto **b.** one-to-one and not onto

c. onto and not one-to-one **d.** not one-to-one and not onto

11. For the given $f\colon \mathbf{Z} \to \mathbf{Z}$, determine whether f is onto and whether it is one-to-one. Prove that your conclusions are correct.

a. $f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ 0 & \text{if } x \text{ is odd} \end{cases}$ **b.** $f(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 2x & \text{if } x \text{ is odd} \end{cases}$

c. $f(x) = \begin{cases} 2x + 1 & \text{if } x \text{ is even} \\ \dfrac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$ **d.** $f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ \dfrac{x-3}{2} & \text{if } x \text{ is odd} \end{cases}$

e. $f(x) = \begin{cases} 3x & \text{if } x \text{ is even} \\ 2x & \text{if } x \text{ is odd} \end{cases}$ **f.** $f(x) = \begin{cases} 2x - 1 & \text{if } x \text{ is even} \\ 2x & \text{if } x \text{ is odd} \end{cases}$

12. Let $A = \mathbf{R} - \{0\}$ and $B = \mathbf{R}$. For the given $f: A \to B$, determine whether f is onto and whether it is one-to-one. Prove that your decisions are correct.

a. $f(x) = \dfrac{x-1}{x}$ **b.** $f(x) = \dfrac{2x-1}{x}$

c. $f(x) = \dfrac{x}{x^2+1}$ **d.** $f(x) = \dfrac{2x-1}{x^2+1}$

13. For the given $f: A \to B$, determine whether f is onto and whether it is one-to-one. Prove that your conclusions are correct.

a. $A = \mathbf{Z} \times \mathbf{Z}$, $B = \mathbf{Z} \times \mathbf{Z}$, $f(x, y) = (y, x)$

b. $A = \mathbf{Z} \times \mathbf{Z}$, $B = \mathbf{Z}$, $f(x, y) = x + y$

c. $A = \mathbf{Z} \times \mathbf{Z}$, $B = \mathbf{Z}$, $f(x, y) = x$

d. $A = \mathbf{Z}$, $B = \mathbf{Z} \times \mathbf{Z}$, $f(x) = (x, 1)$

e. $A = \mathbf{Z}^+ \times \mathbf{Z}^+$, $B = \mathbf{Q}$, $f(x, y) = x/y$

f. $A = \mathbf{R} \times \mathbf{R}$, $B = \mathbf{R}$, $f(x, y) = 2^{x+y}$

14. Let $f: \mathbf{Z} \to \{-1, 1\}$ be given by $f(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ -1 & \text{if } x \text{ is odd.} \end{cases}$

a. Prove or disprove that f is onto.

b. Prove or disprove that f is one-to-one.

c. Prove or disprove that $f(x_1 + x_2) = f(x_1)f(x_2)$.

d. Prove or disprove that $f(x_1x_2) = f(x_1)f(x_2)$.

15. a. Show that the mapping f given in Example 2 is neither onto nor one-to-one.

b. For this mapping f, show that if $S = \{1, 2\}$, then $f^{-1}(f(S)) \neq S$.

c. For this same f and $T = \{4, 9\}$, show that $f(f^{-1}(T)) \neq T$.

16. Let $g: \mathbf{Z} \to \mathbf{Z}$ be given by $g(x) = \begin{cases} x & \text{if } x \text{ is even} \\ \dfrac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$

a. For $S = \{3, 4\}$, find $g(S)$ and $g^{-1}(g(S))$.

b. For $T = \{5, 6\}$, find $g^{-1}(T)$ and $g(g^{-1}(T))$.

17. Let $f: \mathbf{Z} \to \mathbf{Z}$ be given by $f(x) = \begin{cases} 2x - 1 & \text{if } x \text{ is even} \\ 2x & \text{if } x \text{ is odd.} \end{cases}$

a. For $S = \{0, 1, 2\}$, find $f(S)$ and $f^{-1}(f(S))$.

b. For $T = \{-1, 1, 4\}$, find $f^{-1}(T)$ and $f(f^{-1}(T))$.

18. Let $f: \mathbf{Z} \to \mathbf{Z}$ and $g: \mathbf{Z} \to \mathbf{Z}$ be defined as follows. In each case, compute $(f \circ g)(x)$ for arbitrary $x \in \mathbf{Z}$.

a. $f(x) = 2x$, $g(x) = \begin{cases} x & \text{if } x \text{ is even} \\ 2x - 1 & \text{if } x \text{ is odd} \end{cases}$

b. $f(x) = 2x$, $g(x) = x^3$

c. $f(x) = x + |x|$, $g(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ -x & \text{if } x \text{ is odd} \end{cases}$

d. $f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ x + 1 & \text{if } x \text{ is odd} \end{cases}$ $\quad g(x) = \begin{cases} x - 1 & \text{if } x \text{ is even} \\ 2x & \text{if } x \text{ is odd} \end{cases}$

e. $f(x) = x^2, g(x) = x - |x|$

19. Let f and g be defined in the various parts of Exercise 18. In each part, compute $(g \circ f)(x)$ for arbitrary $x \in \mathbf{Z}$.

In Exercises 20–22, suppose m and n are positive integers, A is a set with exactly m elements, and B is a set with exactly n elements.

20. How many mappings are there from A to B?

21. If $m = n$, how many one-to-one correspondences are there from A to B?

22. If $m \leq n$, how many one-to-one mappings are there from A to B?

23. Let a and b be constant integers with $a \neq 0$, and let the mapping $f\colon \mathbf{Z} \to \mathbf{Z}$ be defined by $f(x) = ax + b$.

a. Prove that f is one-to-one.

b. Prove that f is onto if and only if $a = 1$ or $a = -1$.

24. Let $f\colon A \to B$, where A and B are nonempty.

a. Prove that $f(S_1 \cup S_2) = f(S_1) \cup f(S_2)$ for all subsets S_1 and S_2 of A.

b. Prove that $f(S_1 \cap S_2) \subseteq f(S_1) \cap f(S_2)$ for all subsets S_1 and S_2 of A.

c. Give an example where there are subsets S_1 and S_2 of A such that

$$f(S_1 \cap S_2) \neq f(S_1) \cap f(S_2).$$

d. Prove that $f(S_1) - f(S_2) \subseteq f(S_1 - S_2)$ for all subsets S_1 and S_2 of A.

e. Give an example where there are subsets S_1 and S_2 of A such that

$$f(S_1) - f(S_2) \neq f(S_1 - S_2).$$

25. Let $f\colon A \to B$, where A and B are nonempty, and let T_1 and T_2 be subsets of B.

a. Prove that $f^{-1}(T_1 \cup T_2) = f^{-1}(T_1) \cup f^{-1}(T_2)$.

b. Prove that $f^{-1}(T_1 \cap T_2) = f^{-1}(T_1) \cap f^{-1}(T_2)$.

c. Prove that $f^{-1}(T_1) - f^{-1}(T_2) = f^{-1}(T_1 - T_2)$.

d. Prove that if $T_1 \subseteq T_2$, then $f^{-1}(T_1) \subseteq f^{-1}(T_2)$.

26. Let $g\colon A \to B$ and $f\colon B \to C$. Prove that $(f \circ g)^{-1}(T) = g^{-1}(f^{-1}(T))$ for any subset T of C.

Strategy Exercise 13 of this section requests a proof that the inverse of an element with respect to an associative binary operation is unique. A standard way to prove the uniqueness of an entity is to assume that two such entities exist and then prove the two to be equal.

Example 11 Each element $x \in \mathbf{Z}$ has a two-sided inverse $(-x + 2) \in \mathbf{Z}$ with respect to the binary operation $*$ given by

$$x * y = x + y - 1, \qquad (x, y) \in \mathbf{Z} \times \mathbf{Z},$$

since

$$x * (-x + 2) = (-x + 2) * x = -x + 2 + x - 1 = 1 = e.$$ ■

Exercises 1.4

True or False

Label each of the following statements as either true or false.

1. If a binary operation on a nonempty set A is commutative, then an identity element will exist in A.
2. If $*$ is a binary operation on a nonempty set A, then A is closed with respect to $*$.
3. Let $A = \{a, b, c\}$. The power set $\mathcal{P}(A)$ is closed with respect to the binary operation $\cap$ of forming intersections.
4. Let $A = \{a, b, c\}$. The empty set $\emptyset$ is the identity element in $\mathcal{P}(A)$ with respect to the binary operation $\cap$.
5. Let $A = \{a, b, c\}$. The power set $\mathcal{P}(A)$ is closed with respect to the binary operation $\cup$ of forming unions.
6. Let $A = \{a, b, c\}$. The empty set $\emptyset$ is the identity element in $\mathcal{P}(A)$ with respect to the binary operation $\cup$.
7. Any binary operation defined on a set containing a single element is commutative and associative.
8. An identity and inverses exist in a set containing a single element upon which a binary operation is defined.
9. The set of all bijections from A to A is closed with respect to the binary operation of composition defined on the set of all mappings from A to A. Sec. 1.6, #8 ◁

Exercises

1. Decide whether the given set B is closed with respect to the binary operation defined on the set of integers $\mathbf{Z}$. If B is not closed, exhibit elements $x \in B$ and $y \in B$, such that $x * y \notin B$.

   a. $x * y = xy, \quad B = \{-1, -2, -3, \ldots\}$

   b. $x * y = x - y, \quad B = \mathbf{Z}^+$

c. $x * y = x^2 + y^2$, $B = \mathbf{Z}^+$

d. $x * y = \operatorname{sgn} x + \operatorname{sgn} y$, $B = \{-2, -1, 0, 1, 2\}$ where $\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$

e. $x * y = |x| - |y|$, $B = \mathbf{Z}^+$

f. $x * y = x + xy$, $B = \mathbf{Z}^+$

g. $x * y = xy - x - y$, B is the set of all odd integers.

h. $x * y = x^y$, B is the set of positive odd integers.

2. In each part following, a rule is given that determines a binary operation $*$ on the set $\mathbf{Z}$ of all integers. Determine in each case whether the operation is commutative or associative and whether there is an identity element. Also find the inverse of each invertible element.

a. $x * y = x + xy$

b. $x * y = x$

c. $x * y = x + 2y$

d. $x * y = 3(x + y)$

e. $x * y = 3xy$

f. $x * y = x - y$

g. $x * y = x + xy + y$

h. $x * y = x + y + 3$

i. $x * y = x - y + 1$

j. $x * y = x + xy + y - 2$

k. $x * y = |x| - |y|$

l. $x * y = |x - y|$

m. $x * y = x^y$ for $x, y \in \mathbf{Z}^+$

n. $x * y = 2^{xy}$ for $x, y \in \mathbf{Z}^+$

3. Let S be a set of three elements given by $S = \{A, B, C\}$. In the following table, all of the elements of S are listed in a row at the top and in a column at the left. The result $x * y$ is found in the row that starts with x at the left and in the column that has y at the top. For example, $B * C = C$ and $C * B = A$.

$*$	A	B	C
A	C	A	B
B	A	B	C
C	B	A	C

a. Is the binary operation $*$ commutative? Why?

b. Determine whether there is an identity element in S for $*$.

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c. If there is an identity element, which elements have inverses?

4. Let S be the set of three elements given by $S = \{A, B, C\}$ with the following table.

$*$	A	B	C
A	A	B	C
B	B	C	A
C	C	A	B