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J. S. MAC NERNEY: *A Personal Memory*

John Sheridan Mac Nerney was a founding member of the editorial board of the Houston Journal of Mathematics.

He was born in New York City on January 10, 1923 and died June 2, 1979 in Houston. He attended Trinity College from 1939 to 1941. From the University of Texas in Austin, he received his B.A. degree with highest honors in 1948 and his Ph.D. degree in Mathematics in 1951 under the supervision of Professor H. S. Wall. He had worked as a Vibration Analyst at United Aircraft Corporation in East Hartford, Connecticut from 1941 to 1943, and had served in the United States Army Air Force from 1943 to 1946. He taught at Northwestern University (1951-52), the University of North Carolina (1952-67), and the University of Houston (1967-79).

Professor Mac Nerney was a member of the American Association for the Advancement of Science, the American Mathematical Society, the Mathematics Association of America, the North Carolina Academy of Sciences, the Elisha Mitchell Scientific Society, Circolo Matematico di Palermo, Phi Beta Kappa, and Sigma Xi. He was president of the North Carolina chapter of Sigma Xi, 1966-67. He was listed in American Men of Science and Who's Who in the South and Southwest from which much of the above data was obtained.

John Mac Nerney was a mathematician, a teacher, and a friend. I have the highest regard for him in all three categories. His mathematical interests were different from mine, so I shall leave as an exercise for the reader to outline the highlights of his contributions to Mathematical Analysis.

I was Mac's colleague at the University of North Carolina during the academic year 1964-65, and again at the University of Houston from 1967 until his death. Most of the happy memories I have from the year in Chapel Hill are of the friendship of me and my wife, Kathie, and Mac and his wife, the lovely Kathleen Mary O'Connor Mac Nerney, whom he married December 8, 1945.

It was Mac and Kathleen who helped us find and move into a house in Chapel

Hill. I remember Kathleen scrubbing the bathroom of that house from floor to ceiling. They lined up a pediatrician for our daughter, Virginia, and an obstetrician at the University Hospital for Kathie. When our second daughter, Carolyn, was born in November, the only visitors the mother and baby were allowed were the father and two sets of grandparents. We listed Mac and Kathleen as one of those sets.

Once, while we were in North Carolina, Kathie's father sent us a case of Ranch Style beans - a Texas delight not obtainable in Chapel Hill - which we shared with the Mac Nerneys. When Kathie and our children preceded me home from an 18 month stay in Australia in 1973, Mac and Kathleen welcomed them at the Houston Airport with a one gallon can of Ranch Style beans.

As a colleague at Chapel Hill, Mac was the man who stumbled over the ropes with me. Mac's comment when I proved a theorem and then found that Burton Jones had already done it was that I was lucky. After all, he pointed out, I had proved a good theorem; I knew that as fine a mathematician as Burton Jones was interested in it; and I didn't have to write it up for publication. One evening I devised what I thought was an exceedingly clever argument which seemed to prove something I wanted to know. My elation, however, turned first to deflation, when I noticed that if the argument were correct, it would also settle the continuum hypothesis, and then to frustration, when I could not find the error that I knew had to be there. The next afternoon, Mac consented to listen to my argument, which he did until I reached a point at which I found an error. And then we traded places for me to hear the argument he had worked out the night before. As I recall, that one was a proof.

One of the stories Mac liked to tell was of a time when, as a graduate student at Texas, he was in his office thinking about a problem. His friend Pat Porcelli came into his office and sat in a chair. After a couple of hours of complete silence, Pat stood up, commented that it had been a very productive afternoon, and left.

It is easy to paint a portrait of a man as a character. It is hard to paint a portrait of a man of character. John Sheridan Mac Nerney was a good man.

Howard Cook
Houston 1980

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FINITELY ADDITIVE SET FUNCTIONS

J. S. Mac Nerney

CONTENTS

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FINITELY ADDITIVE SET FUNCTIONS

I. ORDER-CHARACTERIZATION OF A PRE-RING OF SUBSETS OF A SET

J. S. Mac Nerney*

ABSTRACT. Suppose that $\{E,\leqslant\}$ is an upper semi-lattice D, which is an upper extension of the nondegenerate partially ordered set R without a least element. It is shown that the following statements (1) and (2) are equivalent. (1) There exists a function γ from R onto a collection Q of subsets of some set such that (a) if u is an element of R and Y is a finite subset of R then $u \leqslant \sup_D Y$ if, and only if, $\gamma(u)$ is covered by the γ-image of Y, and (b) if G is a finite collection of members of Q then there is a collection M of mutually exclusive members of Q such that each set in the collection G is filled up by a finite subcollection of M. (2) If G is a finite subset of R then there exists a subset M of R such that (i) if X is a finite subset of M and y is an element of M which does not belong to X then there is no element t of R such that $t \leqslant \sup_D X$ and $t \leqslant y$, and (ii) each element of G is the supremum in D of a finite subset of M. Proof that (1) is a consequence of (2) is effected in terms of (A) the set R'' to which P belongs if, and only if, P is a subset of R which has, and is maximal with respect to having, the property that if Y is a finite subset of P then there is an element u of R such that, for each element w of Y, $u \leqslant w$, and (B) the function γ from R such that if v belongs to R then $\gamma(v)$ is the subset of R'' to which P belongs if, and only if, v belongs to P. A pre-ring is a collection Q of subsets of a set such that the condition (1,b) is satisfied.

Introduction. The reader is invited to consider, as a central theme in much that follows, the proposition that if G is a finite collection of (closed and bounded) number intervals then there is a collection M of nonoverlapping number intervals such that each interval in G is filled up by a finite subcollection of M.

Suppose that the ordered pair $\{E,\leqslant\}$ is a partially ordered system D which is an upper semi-lattice, and is an *upper extension* (in J. Schmidt's sense [17]) of the nondegenerate (*i.e.*, having more than one element) partially ordered set R: R is a

*Presented to the American Mathematical Society on March 8, 1974.

nondegenerate subset of E, the (implicit) partial ordering of R is the intersection with R $\times$ R of the partial ordering $\leqslant$ of the set E, and each element of E is the supremum in D ($\sup_D$) of a subset of R. Inasmuch as the present author would find it inconvenient to refer to an empty set and does not do so, at least in the present context, this initial supposition precludes a least element of E in D unless there is an element o of R such that, for each element x of R, $o \leqslant x$ (see [17, page 40] for relevant technical comment; there may be a reader who will find it convenient to supply one of the implicitly intended phrases non-empty and non-void in each instance of current reference to a set or a collection or a family). It should be noted that $\leqslant$ is a partial ordering in the sense described by G. Birkhoff [5], rather than in the sense described by N. Dunford and J. T. Schwartz [8, page 4]; Birkhoff [5, page 20] calls the latter type of relation a quasi-ordering. Apparently, therefore, it is appropriate here to specify that, if each of x, y, and z is an element of E, (i) $x \leqslant x$, (ii) if $x \leqslant y$ and $y \leqslant x$ then y is x, (iii) if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$, and (iv) if G is a finite subset of E then $\sup_D G$ is an element v of E such that if u is in G then $u \leqslant v$ and, if w is an element of E such that if u is in G then $u \leqslant w$, $v \leqslant w$.

If G is a collection each member of which is a set then G is said to *fill up* H provided H is the set G* (R. L. Moore's terminology and notation [15] for the sum of all the sets in the collection G in case G is nondegenerate, and for the only member of G in the alternative case); as usual [13, 15], such a collection G is said to *cover* H provided H is a subset of G*. In the case that R is a collection of subsets of a set L and $\leqslant$ has the meaning "is a subset of," one upper extension $\{E,\leqslant\}$ of R, which is an upper semi-lattice, is the *additive extension* of R (T. H. Hildebrandt's terminology [12]), wherein H is an element of E only in case H is a subset of L which is filled up by a finite subset of R. In this case it has been shown by J. von Neumann [22] that if R is a *half-ring* of subsets of L then the additive extension of R is a ring of subsets of L. One may recall that the essence of von Neumann's argument [22, page 85 ff.] is a proof that if G is a finite set of members of R then there is a collection M of mutually exclusive members of R such that each set in the collection G is filled up by a finite subcollection of M; similar argument leads to the same conclusion about R provided only that R is a *semi-ring* as defined by P. R. Halmos [9, page 22] (such an argument is indicated by A. C. Zaanen [23, page 26]). The central idea of these arguments is termed *refinement pre-ring* [6] or *pre-ring* [7] by W. M. Bogdanowicz.

DEFINITION. The statement that the collection Q of subsets of the set K is a *pre-ring* means that if G is a finite collection of members of Q then there is a collection M of mutually exclusive members of Q such that each set belonging to the collection G is filled up by a finite subcollection of M.

The present author is led to this idea from consideration of integrals based on a subdivision-refinement process. If F is a family of collections of subsets of a set K such that (i) each member of F is a collection of mutually exclusive sets and (ii) if M_1 and M_2 are members of F then there is a member M_3 of F such that each set belonging to M_1 or to M_2 is filled up by a finite subcollection of M_3, it may be shown that F* is a pre-ring of subsets of K. Conversely, if Q is a pre-ring of subsets of the set K and F is the family of which M is a member only in case M is a finite collection of mutually exclusive elements of Q, it is clear that the family F has the foregoing properties (i) and (ii).

It is also clear that if Q is a pre-ring of subsets of K then the additive extension of Q is a ring V such that each member of V either belongs to Q or is filled up by a finite collection of mutually exclusive members of Q. Moreover, in this case, a finitely additive function from Q to a set of numbers has only one finitely additive extension to V. Indeed, one description of the condition that the collection V of subsets of K be a *ring* is the following: if H is a finite collection of members of V then H* belongs to V and there is a collection M of mutually exclusive members of V such that each set in the collection H is filled up by a finite subcollection of M. It should be noted that this notion of a *ring of subsets of a set* is the one frequently arising in treatments of measure theory [9, 12, 22, 23], rather than that cited by Birkhoff [5, page 12]; Hildebrandt [12, page 146 ff.] calls the latter an additive and multiplicative class of sets.

Now, here is a description of the Central Problem for which one solution is provided in the present report.

CENTRAL PROBLEM. Find a necessary and sufficient condition on the set R, relatively to D, that there should exist a function γ from R onto a pre-ring of subsets of some set such that if u is an element of R and Y is a finite subset of R then $u \leqslant \sup_D Y$ only in case $\gamma(u)$ is covered by the γ-image of the set Y.

SOLUTION. It is shown that the following is such a necessary and sufficient condition: if G is a finite subset of R then there is a subset M of R such that

(1) if X is a finite subset of M and y is a member of M which does not belong to X then there is no element t of R such that $t \leqslant \sup_D X$ and $t \leqslant y$, and

(2) each element of G is the supremum in D of a finite subset of M.

Geometric Perspectives. In the context in which R is a pre-ring of subsets of a set L, it is natural to define a *partitioning* of a member K of R, as is done by Halmos [9, page 31] for the case that R is a semi-ring, to be a finite collection of mutually exclusive members of R filling up K.

In a topological context, however, R. H. Bing [1] and E. E. Moise [14] have been led to the notion of a *partitioning* of a continuous curve L as a finite collection G of mutually exclusive connected open sets such that G* is dense in the set L. In the case that G is a *regular partitioning* of L (each member of G being the interior of its closure [3]), there is the "equivalent" collection M of closures of members of G: M fills up L and no interior point of a member of M belongs to any other set in the collection M. A primitive instance, of course, is the case that L is a (closed and bounded) number interval and M is a finite collection of nonoverlapping subintervals of L filling up L; this instance, and higher dimensional cases, occur in discussions of the concept of "an additive function of intervals" (*e.g.*, in Hildebrandt [12]). The popular replacement of intervals with left-closed intervals [9] or with right-closed intervals [23] may be thought of as an informal description of such a function γ as is mentioned in the Central Problem of the present report.

There is also the notion of a *brick partitioning* G of a continuous curve L (the elements of the regular partitioning G are further required to be uniformly locally connected, as is the interior of the sum of the closures of each pair of elements of G): it is known [2; 3, Theorem 10] that each continuous curve has a decreasing sequence of brick partitionings. The results of Bing and E. E. Floyd [4] implicitly draw attention to the collection R of all elements of the terms of some decreasing sequence of brick partitionings of a continuous curve L, and to the upper extension $\{E,\leqslant\}$ of R, in which E is the collection to which K belongs only in case K is the interior of the sum of the closures of the elements of some finite subcollection of R, with $\leqslant$ having the meaning "is a subset of" as in the case of the additive extension of R.

In an investigation of (finitely additive) integrals, J. A. Reneke [16] has found convenient the following postulate, among others, concerning a collection R of subsets of a set L: there exists a function $\emptyset$ from R such that if v is a member of R then $\emptyset(v)$ is an element of v which belongs to no other member of any finite subcollection M of R, containing v, such that no member of R lies in two members of M. A central part is then played, in Reneke's investigation, by the family F of all such finite collections M of "relatively prime" members of R; it is further postulated there [16] that if A and B are members of R such that some member of R lies in both of them then there is a member M of F filling up B, with a subset filling up the common part of A and B, such that if v is a member of M not lying in A then no member of R lies both in v and in A. A *partitioning* of a member K of R is, in that context, a member of F which fills up K. By reasoning as indicated in the Proof of [16, Theorem 2.1], it may be proved that if G is a finite subcollection of R then there is a member M of F such that each member of G is filled up by a subcollection of M. In one application of Reneke's principal results [16, page 106 ff.], it seems essential that the members of an element of F not be required to be mutually exclusive.

In consequence of the postulates indicated in the preceding paragraph, each member M of the family F has this property: if X is a subcollection of M and y is an element of M which does not belong to X then no member of R lies both in X* and in y. Suppose, on the contrary, that X is a subcollection of the member M of F and y is an element of M which does not belong to X and t is an element of R which lies both in X* and in y. There is a member N of F such that if z is t or z belongs to X then z is filled up by a subcollection of N; let s be an element of N lying in t. Now, there is an element z of X such that s lies in z, since, otherwise, $\emptyset(s)$ would belong to an element of N different from s. This involves a contradiction, since the element s of R lies in both the elements y and z of M.

Algebraic Perspectives. There is a connection between present results and M. H. Stone's celebrated Representation Theorem for Boolean Rings [19, 20, 21, *et.seq.*] (J.

Schmidt [18] has further references; J. L. Kelley [13, pages 81-83 and 168-169] has encapsulated the pertinent portion of Stone's results). Although the Central Problem is solved here independently of Stone's Representation Theorem, it may be appropriate here to indicate that connection. Suppose that V is a ring of subsets of the set L (in the sense previously indicated), and E_0 is the subset of 2^L (the set of all functions from L to the set of which the numbers 0 and 1 are the only elements) to which x belongs only in case either x is the zero-function θ on L or there is a member g of V such that x(t) is 1 or 0 accordingly as the element t of L does or does not belong to g; let $\leqslant_0$ be the subset of $E_0 \times E_0$ to which $\{x,y\}$ belongs only in case it is true that, for each element t of L, either x(t) is y(t) or x(t) = 0 and y(t) = 1. The ordered pair $\{E_0,\leqslant_0\}$ is a distributive [5, page 12] and relatively complemented [5, page 16] lattice with least element θ. The relevant portion of Stone's Representation Theorem is that every distributive and relatively complemented lattice with a zero-element, and at least two other elements, arises this way - in the sense of lattice-isomorphisms [5, page 24].

Suppose, now, that $\{E_0,\leqslant_0\}$ is a lattice C which is distributive and relatively complemented, with zero-element θ and at least two other elements in the set E_0. Consider the upper semi-lattice $D = \{E,\leqslant\}$, where E is the set of all elements of E_0 different from θ and $\leqslant$ is the intersection with $E \times E$ of the partial ordering $\leqslant_0$ of E_0. There are two properties of D which can be established directly (and independently of Stone's Theorem, *supra*): (1) If M is a subset of E and there are not two elements x and y of M such that, for some t in E, $t \leqslant x$ and $t \leqslant y$, then, if X is a finite subset of M and y is a member of M which does not belong to X, there is no element t of E such that $t \leqslant \sup_D X$ and $t \leqslant y$; (2) if G is a finite set of elements of E then there is a finite set M of elements of E such that each element of G is the supremum in D of a subset of M and there are not two elements x and y of M such that, for some t in E, $t \leqslant x$ and $t \leqslant y$. From the foregoing considerations, upon requiring R to be all of E in the Central Problem and in the indicated Solution, one may see that the present results provide an internal characterization of all such semi-lattices D.

The descriptive term "internal" is used here in contradistinction to such a theorem, for example, as that which Kelley [13, page 150] attributes to Alexandroff and in which a locally compact Hausdorff space is characterized as a space S, with the relative topology, obtained from a nondegenerate and compact Hausdorff space S_0 by omitting a single point from S_0.

Finally, it may be noted that (in the sense of lattice-duality) a similar specialization of the present results provides an internal characterization of lower semi-lattices with a least element which are obtained from Boolean lattices (distributive and complemented lattices with zero- and unit-elements) by omitting the unit-element. If $\{E,\leqslant\}$ is a lower semi-lattice A with a zero-element then, in order that A should be of this type, it is necessary and sufficient that: if G is a finite subset of E then there exists a subset M of E such that

(1) if X is a finite subset of M and y is a member of M which does not belong to X then there is no element t of E such that $\inf_A X \leqslant t$ and $y \leqslant t$, and

(2) each element of G is the infimum in A of a finite subset of M.

No further attention is called, in the present report, to similarly dual results.

Necessity of the Condition. The initial supposition from the Introduction is hereby invoked: the ordered pair $\{E,\leqslant\}$ is a partially ordered system D which is an upper semi-lattice, and is an upper extension of the nondegenerate partially ordered set R.

Here is a notational device which serves to preclude ambiguity in case there is a subset of R which is itself a member of R. If γ is a relation with initial set (or domain) R then the γ-image function, denoted by $\gamma^{\to}$, is the function to which the ordered pair $\{U,H\}$ belongs only in case U is a subset of R and H is the set to which t belongs only in case there is an element s of U such that the ordered pair $\{s,t\}$ belongs to γ: $H = \gamma^{\to}(U)$, the γ-image of the set U.

THEOREM 0. *If* γ *is a function from* R *onto a pre-ring of subsets of the set* L *such that, if* u *is an element of* R *and* Y *is a finite subset of* R, $u \leqslant \sup_D Y$ *only in case* $\gamma(u)$ *is a subset of* $\gamma^{\to}(Y)^*$ *then the following statements are true:*

(1) *if* M *is a subset of* R *then, in order that no element of* L *should belong to two members of* $\gamma^{\to}(M)$, *it is necessary and sufficient that if* X *is a finite subset of* M *and* y *is an element of* M *which does not belong to* X *then there is no element* t *of* R *such that* $t \leqslant \sup_D X$ *and* $t \leqslant y$, *and*

(2) *if* P *is a subset of* R, *then, in order that it be true that if* Y *is a finite subset of* P *then there is an element of* L *which belongs to every member of* $\gamma^{\rightarrow}(Y)$, *it is necessary and sufficient that if* Y *is a finite subset of* P *then there is an element* u *of* R *such that, for each element* w *of* Y, $u \leqslant w$.

PROOF. With the observation that, under the indicated hypothesis, γ is a reversible transformation, the proof is accomplished in four steps.

STEP 1a: Suppose M is a subset of R such that no element of L belongs to two members of $\gamma^{\rightarrow}(M)$, X is a finite subset of M, and y is an element of M which does not belong to X. Suppose that there is an element t of R such that $t \leqslant \sup_D X$ and $t \leqslant y$, so that $\gamma(t)$ is a subset both of $\gamma^{\rightarrow}(X)^*$ and of $\gamma(y)$, and let p be an element of $\gamma(t)$. Since p belongs to $\gamma^{\rightarrow}(X)^*$, there is an element u of X such that p belongs to $\gamma(u)$. Since y does not belong to X and γ is reversible, the element p of L belongs to both $\gamma(u)$ and $\gamma(h)$. This involves a contradiction.

STEP 1b: Suppose M is a subset of R such that if X is a finite subset of M and y is a member of M which does not belong to X then there is no element t of R such that $t \leqslant \sup_D X$ and $t \leqslant y$. Suppose u and w are elements of M and p is an element of L which belongs to both $\gamma(u)$ and $\gamma(w)$. Since $\gamma^{\rightarrow}(R)$ is a pre-ring, there is a subset Z of R such that $\gamma^{\rightarrow}(Z)$ is a collection of mutually exclusive sets and each of $\gamma(u)$ and $\gamma(w)$ is filled up by a finite subcollection of $\gamma^{\rightarrow}(Z)$. Let v be a member of Z such that p belongs to $\gamma(v)$: since p belongs to no member of $\gamma^{\rightarrow}(Z)$ different from $\gamma(v)$, $\gamma(v)$ is a subset of both $\gamma(u)$ and $\gamma(w)$. Hence v is an element of R such that $v \leqslant u$ and $v \leqslant w$. This involves a contradiction.

STEP 2a: Suppose P is a subset of R such that if Y is a finite subset of P then there is an element of L which belongs to every member of $\gamma^{\rightarrow}(Y)$, and Y is a finite set of elements of P. Let p be an element of L which belongs to every member of $\gamma^{\rightarrow}(Y)$, and Z be a subset of R such that $\gamma^{\rightarrow}(Z)$ is a collection of mutually exclusive sets and each member of $\gamma^{\rightarrow}(Y)$ is filled up by a finite subcollection of $\gamma^{\rightarrow}(Z)$. Let u be a member of Z such that p belongs to $\gamma(u)$: since p belongs to no member of $\gamma^{\rightarrow}(Z)$ different from $\gamma(u)$, $\gamma(u)$ is a subset of each set in the collection $\gamma^{\rightarrow}(Y)$. Hence, for each member w of Y, $u \leqslant w$.

STEP 2b: If Y is a finite set of elements of R and u is an element of R such that, for each element w of Y, $u \leqslant w$ then each element of $\gamma(u)$ is an element of L which belongs to every set in the collection $\gamma^{\to}(Y)$.

COROLLARY. *The stipulated condition on* R, *relatively to* D, *is necessary for the existence of a function* γ *from* R *onto a pre-ring of subsets of some set such that if* u *is an element of* R *and* Y *is a finite subset of* R *then* $u \leqslant \sup_D Y$ *only in case it is true that* $\gamma(u)$ *is a subset of* $\gamma^{\to}(Y)^*$.

PROOF. Suppose that γ is such a function from R onto the pre-ring Q of subsets of the set L, and G is a finite set of elements of R. As observed in the Proof of Theorem 0, γ is a reversible transformation. Since Q is a pre-ring, there exists a set M of elements of R such that $\gamma^{\to}(M)$ is a collection of mutually exclusive members of Q and each member of $\gamma^{\to}(G)$ is filled up by a finite subcollection of $\gamma^{\to}(M)$. By Theorem 0 (1), if X is a finite subset of M and y is an element of M which does not belong to X then there is no element t of R such that $t \leqslant \sup_D X$ and $t \leqslant y$. Suppose that K is an element of G, and let Z be a finite subset of M such that $\gamma(K)$ is filled up by $\gamma^{\to}(Z)$. Since $\gamma(K)$ is the set $\gamma^{\to}(Z)^*$, it is true that $K \leqslant \sup_D Z$. If x is an element of Z then $\gamma(x)$ is a subset of $\gamma^{\to}(Z)^*$, which is $\gamma(K)$, so that $x \leqslant K$. Hence $\sup_D Z \leqslant K$, so that K is the supremum in D of the set Z. This establishes the Corollary.

Sufficiency of the Condition. The initial supposition from the Introduction is again invoked: the ordered pair $\{E,\leqslant\}$ is a partially ordered system D which is an upper semi-lattice, and is an upper extension of the nondegenerate partially ordered set R. There are two types of subsets of R, as indicated in the statement of Theorem 0, to which attention is now called.

DEFINITIONS. A *type*-1 *setset of* R is a subset M of R such that if X is a finite subset of M and y is an element of M which does not belong to X then there is no element t of R such that $t \leqslant \sup_D X$ and $t \leqslant y$. A *type*-2 *subset of* R is a subset P of R such that if Y is a finite subset of P then there is an element u of R such that, for each element w of Y, $u \leqslant w$. If J is one of the integers 1 and 2 then a *full* type-J subset of R is a type-J subset of R which is not a proper subset of any type-J subset of R.

It is clear that, if J is one of the integers 1 and 2, each degenerate subset of R is a type-J subset of R and every subset of any type-J subset of R is itself a type-J subset of R. Moreover, by the familiar Maximality Principle (as formulated by M. Zorn [24] or, dually, by R. L. Moore [15, Theorem 39] - this is Theorem 121 in the 1932 Edition of [15]), if J is one of the integers 1 and 2 then each type-J subset of R is a subset of a full type-J subset of R. Now, the stipulated condition on R, relatively to D, is assumed hereinafter as follows.

SUBDIVISION AXIOM. *If* G *is a finite subset of* R, *there is a type*-1 *subset* M *of* R *such that each element of* G *is the supremum in* D *of a finite subset of* M.

Let R″ denote the collection to which P belongs only in case P is a full type-2 subset of R, and γ be the function to which the ordered pair $\{u,h\}$ belongs only in case u is an element of R and h is the set to which P belongs only in case P is a member of R″ to which u belongs. It may be noted that the assertion, that each type-2 subset of R is a subset of a full type-2 subset of R, has the following interpretation: if G is a type-2 subset of R then there is an element of R″ which belongs to every set in the collection $\gamma^{\rightarrow}(G)$. It is to be shown that $\gamma^{\rightarrow}(R)$ is a pre-ring of subsets of R″, and that if u is an element of R and Y is a finite subset of R then $u \leqslant \sup_D Y$ only in case $\gamma(u)$ is a subset of $\gamma^{\rightarrow}(Y)^*$. To this end, here is a sequence of nine Theorems based on the Subdivision Axiom.

THEOREM 1. *No type*-1 *subset of* R *has two subsets* X *and* Y *such that* X *is finite and* $\sup_D X = \sup_D Y$.

PROOF. Suppose, on the contrary, that M is a type-1 subset of R, X is a finite subset of M, Y is another subset of M, and $\sup_D X = \sup_D Y$. If Y is finite then there is an element v of one of the sets X and Y such that, if Z is the other one of the sets X and Y, $v \leqslant \sup_D Z$ but v does not belong to Z; since Z is a finite subset of M, this involves a contradiction. Therefore, Y is not finite and so there is an element w of Y which does not belong to X. Now, w is an element of M and $w \leqslant \sup_D X$. This involves a contradiction.

THEOREM 2. *Suppose* G *is a finite set of elements of* R, M *is a type*-1 *subset of* R, *and each element of* G *is the supremum in* D *of a finite subset of* M. *Then the set* W, *to which* u *belongs only in case* u *is an element of* M *and there is an element* h *of* G *such that* $u \leqslant h$, *is finite and* $\sup_D W = \sup_D G$.

PROOF. By Theorem 1 no element of G is the supremum in D of two subsets of M and, since G is finite, W is finite. Let f be the function to which $\{h,k\}$ belongs only in case h is an element of G and k is the subset of M to which the element u of M belongs only in case $u \leqslant h$. Since $f^{\to}(G)$ is W and if h is an element of G then $h = \sup_D f(h)$, it follows that $\sup_D G = \sup_D W$.

THEOREM 3. *If* M *is a subset of* R *such that if* u *and* v *are elements of* M *then there is no element* t *of* R *such that* $t \leqslant u$ *and* $t \leqslant v$, *then* M *is a type-*1 *subset of* R.

PROOF. Suppose that the subset M of R is not a type-1 subset of R. Let X be a finite subset of M and y be an element of M which does not belong to X and t be an element of R such that $t \leqslant \sup_D X$ and $t \leqslant y$. With reference to the Subdivision Axiom and Theorems 1 and 2, let W be a finite type-1 subset of R such that $\sup_D W = \sup_D X$ and if s is t or s belongs to X then s is the supremum in D of a subset of W. It follows from Theorems 1 and 2 that, if r is an element of W, there is an element u of X such that $r \leqslant u$. Let q be an element of W such that $q \leqslant t$, and u be an element of X such that $q \leqslant u$. Now, u and y are elements of M and q is an element of R such that $q \leqslant u$ and $q \leqslant y$.

THEOREM 4. *The subset* M *of* R *is a type-*1 *subset of* R *only in case there is no element of* R″ *which belongs to two members of* $\gamma^{\to}(M)$.

PROOF. Suppose that M is a subset of R. It is clear from the Definitions that, if M is a type-1 subset of R, no two members of M belong to any type-2 subset of R so that no element of R″ belongs to two members of $\gamma^{\to}(M)$. If no element of R″ belongs to two members of $\gamma^{\to}(M)$ then no two members of M belong to any type-2 subset of R so that, by Theorem 3, M is a type-1 subset of R.

THEOREM 5. *If* P *is an element of* R″ *and* W *is a finite type-*1 *subset of* R *such that* $\sup_D W$ *belongs to* P, *then only one element of* W *belongs to* P.

PROOF. Suppose, on the contrary, that u belongs to the member P of R″ and W is a finite type-1 subset of R such that $u = \sup_D W$ and it is not true that only one element of W belongs to P. Since (from the Definitions) no two elements of any type-1 subset of R can both belong to some type-2 subset of R, there is no element of W which belongs to P. If Q is a finite subset of P to which u belongs then (1) there is an element t of R such that if n belongs to Q then $t \leqslant n$, (2) by the Subdivision Axiom and Theorem 2 there is a finite type-1 subset X of R such that $u = \sup_D X$ and if s is t or s belongs to W then s is the supremum in D of a subset of X, (3) it follows from Theorem 2 that if v belongs to X then there is an element y of W such that $v \leqslant y$, and therefore (4) there is an element y of W such that, for some element v of X, $v \leqslant t$ and $v \leqslant y$. Accordingly, there is a function g such that if Q is a finite subset of P then g(Q) is the set to which y belongs only in case y belongs to W and there is an element v of R such that if n belongs to Q then $v \leqslant n$ and $v \leqslant y$. If Q_1 and Q_2 are finite subsets of P and y belongs to $g(Q_1+Q_2)$ then there is an element v of R such that if n belongs to Q_1 or to Q_2 then $v \leqslant n$ and $v \leqslant y$, so that y belongs to $g(Q_1)$ and to $g(Q_2)$. Since the set W is finite, there is a finite subset Q_0 of P such that, for each finite subset Q of P, $g(Q_0)$ is a subset of g(Q). Let z be an element of $g(Q_0)$. Since z belongs to W, z does not belong to P; if Q is a finite subset of P then z belongs to g(Q) so that there is an element v of R such that if n belongs to Q then $v \leqslant n$ and $v \leqslant z$. Since P is a full type-2 subset of R, this involves a contradiction.

THEOREM 6. *If* u *is an element of* R *then, for each finite type-*1 *subset* W *of* R *such that* $u = \sup_D W$, $\gamma(u)$ *is the set* $\gamma^{\to}(W)^*$.

PROOF. Suppose that u is an element of R and W is a finite type-1 subset of R such that $u = \sup_D W$. It is clear from the Definitions that, if t belongs to the full type-2 subset P of R and $t \leqslant u$, u must belong to P. Hence, $\gamma^{\to}(W)^*$ is a subset of $\gamma(u)$. By Theorem 5, if P belongs to $\gamma(u)$ then there is only one set in the collection $\gamma^{\to}(W)$ to which P belongs. Hence $\gamma(u)$ is a subset of $\gamma^{\to}(W)^*$.

THEOREM 7. *If* u *is an element of* R *and* Y *is a finite subset of* R *such that* $u \leqslant \sup_D Y$, *then* $\gamma(u)$ *is a subset of* $\gamma^{\to}(Y)^*$.

PROOF. Suppose that u is an element of R and Y is a finite subset of R such that $u \leqslant \sup_D Y$. With reference to the Subdivision Axiom and Theorems 1 and 2, let V be a finite type-1 subset of R such that $\sup_D V = \sup_D Y$ and if s is u or s belongs to Y then s is the supremum in D of a subset of V. It follows from Theorems 1 and 2 that if t belongs to V then there is an element x of the set Y such that $t \leqslant x$ and that there is only one subset W of V such that $u = \sup_D W$. If t is an element of W then, since there is an element x of Y such that $t \leqslant x$, $\gamma(t)$ is a subset of $\gamma^{\to}(Y)^*$. Now, $\gamma(u)$ is a subset of $\gamma^{\to}(Y)^*$ since, by Theorem 6, $\gamma(u)$ is the set $\gamma^{\to}(W)^*$.

THEOREM 8. *If* u *is an element of* R *and* Y *is a finite subset of* R *such that* $\gamma(u)$ *is a subset of* $\gamma^{\to}(Y)^*$, *then* $u \leqslant \sup_D Y$.

PROOF. Suppose that u is an element of R and Y is a finite subset of R and it is not true that $u \leqslant \sup_D Y$. Let G be the sum of the set Y and the set of which u is the only element. With reference to the Subdivision Axiom and Theorems 1 and 2, let V be a finite type-1 subset of R such that $\sup_D V = \sup_D G$ and each element of G is the supremum in D of a subset of V. By Theorem 2, $\sup_D Y$ is the supremum in D of the set W to which s belongs only in case s is an element of V and there is an element t of Y such that $s \leqslant t$. It follows from Theorem 6 that $\vec{\gamma}(Y)^*$ is $\vec{\gamma}(W)^*$. Since it is not true that $u \leqslant \sup_D W$, W is not V: let z be an element of V which does not belong to W, and P be a full type-2 subset of R to which z belongs. By Theorem 2 there is an element h of G such that $z \leqslant h$: it follows that h is u, so that P belongs to $\gamma(u)$. Since z does not belong to W, it follows from Theorem 4 that P does not belong to any set in the collection $\vec{\gamma}(W)$. Therefore, $\gamma(u)$ is not a subset of $\vec{\gamma}(Y)^*$.

THEOREM 9. *The collection* $\vec{\gamma}(R)$ *is a pre-ring of subsets of the set* R''.

PROOF. Suppose that G is a finite set of elements of R. By the Subdivision Axiom, there is a type-1 subset M of R such that each element of G is the supremum in D of a finite subset of M. Since G is nondegenerate, it follows that M is nondegenerate. Since, by Theorem 8, the transformation γ is reversible, it follows from Theorem 4 that $\vec{\gamma}(M)$ is a collection of mutually exclusive sets. It follows from Theorem 6 that each set in the collection $\vec{\gamma}(G)$ is filled up by a finite subcollection of $\vec{\gamma}(M)$. Therefore, $\vec{\gamma}(R)$ is a pre-ring of subsets of R''.

Realizations. Throughout this section, R is understood to be a collection of subsets of a set L and the implicit partial ordering has the meaning "lies in" or "is a subset of," and the upper semi-lattice D is understood to be the additive extension of R so that if X is a subcollection of R then the assertion that $H = \sup_D X$ may be replaced by the assertion that X fills up H (*cf.* Introduction, third paragraph). Now, E is the collection to which H belongs only in case H is a subset of L which is filled up by a finite subcollection of R.

The *Definitions* from the preceding section take the following form: a type-1 subcollection of R is a subset M of R such that if X is a finite subset of M then no element of R lies both in X* and in a member of M which does not belong to X; a type-2 subcollection of R is a subset P of R such that if Y is a finite set of members of

P then some element of R lies in all the sets in the collection Y. The *Subdivision Axiom* takes the following form: if G is a finite subcollection of R then there is a type-1 subcollection M of R such that each set belonging to G is filled up by a finite subcollection of M.

It can not be proved that if Q is a collection of subsets of L such that the additive extension of Q is a ring then Q is a pre-ring; this could not be proved even if it were further stipulated that the additive extension of Q be an *algebra* (a ring to which L itself belongs, [9, 12]) and that the common part of each two intersecting members of Q should belong to Q. Consider the following Example.

EXAMPLE 1. Let L be the right-closed number interval (0,1], and Q be the minimal collection of subsets of L determined as follows. Both (0,2/3] and (1/3,1] belong to Q; if $0 \leqslant a < b \leqslant 1$ and both (a,a+2b/3] and (2a+b/3,b] belong to Q then all five of the following sets belong to Q:

(a,7a+2b/9], (8a+b/9,2a+b/3], (2a+b/3,a+2b/3], (a+2b/3,a+8b/9], and (2a+7b/9,b].

Clearly the additive extension of Q is an algebra of subsets of L. If u and v are intersecting members of Q neither of which is a subset of the other then uv belongs to Q, but no one of the three sets u-uv, v-uv, and u+v either belongs to Q or is filled up by a finite collection of mutually exclusive members of Q.

That Theorem 3 is not a consequence of the Definitions, independently of the Subdivision Axiom, may be seen by considering the following Example.

EXAMPLE 2. Let L be the real line, and R be the collection of all number intervals [s,t] such that s is an integer and t is s+2. Clearly there are not two members u and v of R such that some member of R lies in both u and v. Each member of R, however, lies in the sum of two other members of R.

It follows from the Subdivision Axiom (with the help of Theorem 5) that if P is a full type-2 subcollection of R then P is a filter-base, *i.e.*, that if u and v are elements of the member P of R″ then there is some element of P which lies in both u and v. That this is not a consequence of the Definitions, independently of the Subdivision Axiom, may be seen from the following Example.

EXAMPLE 3. Let L be any infinite set, R be the collection of all degenerate subsets of L together with all complements (in L) of degenerate subsets of L, and P be the collection of all nondegenerate elements of R. Clearly P is a full type-2 subcollection of R. There are, however, no two elements of the collection P such that some element of P lies in both of them. (This Example was called to the attention of the present author by J. A. Schatz in a conversation which took place on May 8, 1973.)

The following Theorem may be proved on the basis of the Subdivision Axiom, with the help of Theorems 7, 8, and 9.

THEOREM 10. *If* u *and* v *are elements of* R *such that some element of* R *lies in both of them and* W *is a finite type-*1 *subcollection of* R *of which some subset fills up* u *and some subset fills up* v, (i) *there is a subset* X *of* W *such that* X* *lies in both* u *and* v *and each element of* R *lying in both* u *and* v *lies in* X*, *and* (ii) *if* v *is not a subset of* u *then there is a subset* Y *of* W *such that* Y* *lies in* v, *no element of* R *lies both in* u *and in* Y*, *and if* t *is an element of* R *lying in* v *such that no element of* R *lies both in* u *and in* t *then* t *lies in* Y*.

It can not be proved on the basis of the Subdivision Axiom, however, that if u and v are elements of R such that some element of R lies in both of them then some subcollection of R fills up the common part of u and v, nor that if u is a proper subset of v then some element of R lies in v-u, nor that even if some element of R does lie in v-u then v-u is filled up by some subset of R. Indeed, none of these propositions could be proved even if it were further stipulated that L itself should belong to the collection R. Consider the following Example.

EXAMPLE 4. Let L be the interval [0,4] of real numbers, and R be the collection consisting of L itself together with the six subsets of L enumerated as follows: t_0 is the interval [0,3], t_1 is the interval [0,1] together with the number 2, t_2 is the half-open interval [1,2), t_3 is the half-open interval (2,3], t_4 is the interval [3,4] together with the number 2, t_5 is the interval [1,4]. Consider the collection M of which the elements are the sets t_1, t_2, t_3, and t_4: it may be shown that M is a type-1 subcollection of R filling up L. There are three-element subcollections A and B of M filling up t_0 and t_5, respectively. The common part X of A and B is the set of which t_2 and t_3 are the only elements, but the common part of t_0 and t_5 is the interval [1,3]: X* is the sum of the sets [1,2) and (2,3], but no element of R which contains the number 2 is a subset of the interval [1,3]. Moreover, t_0 is a proper subset of L but no element of R lies in L-t_0, and t_1 is a proper subset of t_0 but t_3 is the only element of R which lies in t_0-t_1.

Contrary to what might be expected from the instances cited in the section on Geometric Perspectives, it can not be proved on the basis of the Subdivision Axiom that there is a function $\emptyset$ from R such that if v is an element of R then $\emptyset$(v) is an element of L which belongs to v but does not belong to any other set in any type-1 subcollection of R containing v. Consider the following Example.

EXAMPLE 5. Let L be the interval [0,1] of real numbers, and R be the collection of all subsets of L having positive (Lebesgue) measure. This may be shown: a subcollection M of R is a type-1 subcollection of R provided there are not two elements u and v of M such that some element of R lies in both u and v.

There are cases in which the pre-ring $\gamma^{\rightarrow}(R)$ of subsets of R″ has a simple realization. One such case is the primitive instance cited earlier (Geometric Perspectives, second paragraph). Consider the following final Example.

EXAMPLE 6. Let L be the real line, and R be the collection of all number intervals. If P is a member of R″ then there exists a number c such that either P consists of all [a,b] such that $a \leqslant c < b$ or P consists of all [a,b] such that $a < c \leqslant b$. Let L″ be the subset of L X L to which $\{c,m\}$ belongs only in case $m^2 = 1$, with the familiar lexicographic ordering: $\{c,m\} < \{d,n\}$ only in case either c is d and $m < n$ or $c < d$. Let δ be a function from R″ into L″ such that if P is in R″ then $\delta(P)$ is $\{c,1\}$ or $\{c,-1\}$, accordingly as P consists of all [a,b] such that $a \leqslant c < b$ or of all [a,b] such that $a < c \leqslant b$. Now, δ is a reversible transformation from R″ onto L″ and if [c,d] is an element of R then the δ-image of the set $\gamma([c,d])$ is the L″-interval $[\{c,1\},\{d,-1\}]$.

Summary. Suppose that R is a collection of subsets of the set L. It seems that the idea of a nonoverlapping subcollection of R is adequately encompassed by the idea of a type-1 subcollection of R. Accordingly, the following Definitions seem to be appropriate.

DEFINITIONS. The subcollection M of R is *nonoverlapping relatively to* R provided that if X is a finite subcollection of M then no member of R which is covered by X lies in any member of M which does not belong to X. A function f from R to an additive Abelian semigroup is R-*additive* provided that if M is a finite subcollection of R which is nonoverlapping relatively to R and M* belongs to R then $f(M^*) = \Sigma_{u \text{ in } M} f(u)$.

It may be noted, in passing, that (i) one might say that the subcollection M of R is nonoverlapping relatively to the collection T of subsets of L provided that if X is a finite subcollection of M then no member of T which is covered by X lies in any member of M which does not belong to X, and (ii) to say that the collection M of members of R is nonoverlapping relatively to the collection L′ of all degenerate subsets of L, in the sense (i), would be equivalent to saying that M is a collection of mutually exclusion members of R.

The Solution given for the Central Problem, in the present report, may be interpreted as an assertion that the following Subdivision Axiom is a provision for the existence of R-additive functions from R to the real numbers and for such functions to be endowed with the usual properties of finitely additive functions. Indeed, it is a consequence of that Solution that this Subdivision Axiom is a necessary and sufficient condition on the collection R *relatively to its additive extension* for there to exist a function γ from R onto a pre-ring of subsets of some set such that, if u is a set in R and Y is a finite subcollection of R, u is covered by the collection Y only in case $\gamma(u)$ is covered by the γ-image of Y.

SUBDIVISION AXIOM. *If* G *is a finite subcollection of* R *then there exists a subcollection* M *of* R *which is nonoverlapping relatively to* R *such that each set belonging to the collection* G *is filled up by a finite subcollection of* M.

It may be noted that, in order that this Axiom should be satisfied, it is necessary and sufficient that there should be at least one *subdivision-refinement process* for R, *i.e.*, at least one family F such that (i) each member of F is a subcollection of R which is nonoverlapping relatively to R, (ii) if M_1 and M_2 are members of F then there is a member M_3 of F such that each set belonging to M_1 or to M_2 is filled up by a finite subcollection of M_3, and (iii) each set belonging to R is filled up by a finite subcollection of F*. Moreover, if F is such a subdivision-refinement process for R and γ is the function indicated in the section entitled Sufficiency of the Condition, the γ-image of each nondegenerate member of F is a collection of mutually exclusive sets, and the family $\gamma^{\leftrightarrow}(F)$ is itself a subdivision-refinement process for the γ-image of the collection R.

Prospectus. In a second report, it will be assumed that R is a pre-ring of subsets of a set L, filling up L, and a new representation (*cf.* Hildebrandt [10, 11] or Dunford and Schwartz [8, page 392 ff.]) will be given for the dual of a normed linear space $\{S,\|\cdot\|\}$ such that: S is the space to which f belongs only in case f is a finitely additive function from R to the (real or complex) numbers and there exists a nonnegative number b such that if M is a finite collection of mutually exclusive members of R then $\Sigma_{t \text{ in } M}|f(t)| \leqslant b$, in which case $\|f\|$ is the least such b. That analysis will be presented in the somewhat more general context wherein the members f of S are functions from R to a complete (real or complex) inner product space $\{Y,\langle\cdot,\cdot\rangle\}$, with norm $[\![\cdot]\!]$ corresponding to the inner product function $\langle\cdot,\cdot\rangle$ and the preceding inequalities replaced by $\Sigma_{t \text{ in } M}[\![f(t)]\!] \leqslant b$. Representations are given for the space C of all continuous linear transformations in the space $\{S,\|\cdot\|\}$, for the space D of all continuous linear transformations from $\{S,\|\cdot\|\}$ to the scalars, and for the space E of all continuous linear transformations from $\{S,\|\cdot\|\}$ to $\{Y,[\![\cdot]\!]\}$. Each of these representations is a linear isomorphism, is an isometry (with respect to the usual norm), and is determined by integrals based on the general subdivision-refinement process F, to which M belongs only in case M is a finite subcollection of R and no element of L belongs to two sets in M.

REFERENCES

1. R. H. Bing, *Partitioning a set,* Bull. Amer. Math. Soc., 55(1949), 1101-1110.
2. ———, *Complementary domains of continuous curves,* Fund. Math., 36(1949), 303-318.
3. ———, *Partitioning continuous curves,* Bull. Amer. Math. Soc., 58(1952), 536-556.
4. R. H. Bing and E. E. Floyd, *Coverings with connected intersections,* Trans. Amer. Math. Soc., 69(1950), 387-391.
5. G. Birkhoff, *Lattice Theory,* Amer. Math. Soc. Colloq. Publ., 25(1967), revised, Providence.
6. W. M. Bogdanowicz, *Lebesgue-Bochner type spaces of group-valued summable functions and abstract integrals,* Queen's Papers Pure Appl. Math., 10(1967), 205-208.
7. W. M. Bogdanowicz and J. N. Welch, *Integration generated by projective volumes over Hilbert spaces,* Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 19(1971), 49-57.
8. N. Dunford and J. T. Schwartz, *Linear Operators Part I,* Interscience, New York, 1958.
9. P. R. Halmos, *Measure Theory,* Van Nostrand, New York, 1950.
10. T. H. Hildebrandt, *On bounded linear functional operations,* Trans. Amer. Math. Soc., 36(1934), 868-875.

11. ———, *Linear operations on functions of bounded variation,* Bull. Amer. Math. Soc., 44(1938), 75.
12. ———, *Introduction to the Theory of Integration,* Academic Press, New York, 1963.
13. J. L. Kelley, *General Topology,* Van Nostrand, New York, 1955.
14. E. E. Moise, *Grille decomposition and convexification theorems for compact metric locally connected continua,* Bull. Amer. Math. Soc., 55(1949), 1111-1121.
15. R. L. Moore, *Foundations of Point Set Theory,* Amer. Math. Soc., Colloq. Publ., 13 (revised), Providence, 1962.
16. J. A. Reneke, *On the existence and representation of integrals,* Ill. J. Math., 14(1970), 99-112.
17. J. Schmidt, *Universal and internal properties of some extensions of partially ordered sets,* J. reine angew. Math., 253(1972), 28-42.
18. ———, *Boolean duality extended,* in Theory of Sets and Topology (in honour of Felix Hausdorff), 435-453, Deutscher Verlag der Wissenschaften, Berlin, 1972.
19. M. H. Stone, *The theory of representations for Boolean algebras,* Trans. Amer. Math. Soc., 40(1936), 37-111.
20. ———, *Application of the theory of Boolean rings to general topology,* Trans. Amer. Math. Soc., 41(1937), 375-481.
21. ———, *The representation of Boolean algebras,* Bull. Amer. Math. Soc., 44(1938), 807-815.
22. J. von Neumann, *Functional Operators Volume I,* Annals of Math. Studies, 21(1950), Princeton University Press, Princeton.
23. A. C. Zaanen, *Integration,* Interscience, New York, 1967.
24. M. Zorn, *A remark on method in transfinite algebra,* Bull. Amer. Math. Soc., 41(1935), 667-670.

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FINITELY ADDITIVE SET FUNCTIONS
II. LINEAR OPERATIONS ON A SPACE OF FUNCTIONS OF BOUNDED VARIATION

J. S. Mac Nerney*

ABSTRACT. Let S be the space of all functions of bounded variation on $[0,1]$ which are anchored at 0, S^+ be the set of all real nondecreasing functions in S, and, for each t in $[0,1]$ and f in S, $P_t f$ be the function h in S such that h(u) is f(u) or f(t) accordingly as $u \leqslant t$ or $u \geqslant t$. The equations $A(\lambda)(\alpha)(t) = \lambda(P_t\alpha)$, for λ in the dual D of S and α in S^+ and t in $[0,1]$, define a linear isomorphism A from D onto the set of all functions g from S^+ into S such that (1) there is a $b \geqslant 0$ such that if α is in S^+ and $0 \leqslant u < v \leqslant 1$ then $|g(\alpha)(v)-g(\alpha)(u)| \leqslant b[\alpha(v)-\alpha(u)]$ {the least such b is the norm of the member $A^{-1}(g)$ of D} and (2) if α and β are in S^+ and there is a $c > 0$ such that $[\alpha(v)-\alpha(u)] \leqslant c[\beta(v)-\beta(u)]$ for $0 \leqslant u < v \leqslant 1$ then $g(\alpha)(t) = \int_0^t [dg(\beta)d\alpha]/d\beta$ for each t in $(0,1]$. If $g = A(\lambda)$ and f is in S then $\lambda(f)$ is an integral in this sense: for each α in S^+ such that Hellinger's (subdivision-refinement) integral $\int_0^1 |df|^2/d\alpha$ exists, $\lambda(f) = \int_0^1 [dg(\alpha)df]/d\alpha$. All this remains true in case, from the beginning, all the functions in S are further required to be right-continuous at each number between 0 and 1. These, and related results about representation of linear operations, are presented in the somewhat more general context wherein S is a space of finitely additive set functions from a pre-ring R to a complete inner product space Y, and the norm of a function h in S is the total variation of h relatively to the usual norm on Y. There are also, then, representations of the space E of all continuous linear functions from S to the evaluation-space Y of S: E, with the standard norm, is shown to have the additional natural structure of a B*-algebra with an identity.

Introduction. The reader is invited to consider, as primitive instances of the present situation, the following two possible cases: (1) L is the real line (*i.e.*, the set of all real numbers) and R is the collection of all right-closed intervals of real numbers, and (2) L is the set of all nonnegative integers and R is the collection of all degenerate subsets (*i.e.*, one-element subsets) of L.

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Initially, in this report, it is supposed that R is a *pre-ring of subsets* of a set L [1, 2, 15] filling up L, *i.e.*, that R is a collection of subsets of the set L, filling up L, such that if G is a finite collection of members of R then there is a collection M of mutually exclusive members of R such that each set belonging to the collection G is filled up by a finite subcollection of M; the letter F stands for the family of all finite subcollections M of R such that no element of L belongs to two sets in M. If $\{X,|\cdot|\}$ is a normed linear space, h is a function from R to X, and K is a subset of L which is filled up by some subcollection of R then the statement that $T = \int_{K/F} h$ (with respect to the norm $|\cdot|$), the *integral over* K *relatively to* F *of the function* h, means that T is in X and, if ϵ is a positive number, there is a member M of F filling up a subset of K such that, if W is a member of F filling up a subset of K and each set in M is filled up by a subcollection of W, $|T - \Sigma_{v \text{ in } W} h(v)| < \epsilon$. This is a slight extension of the usual notion of a subdivision-refinement integral, or σ-integral, wherein it would be assumed that some member of F actually fills up the set K (as, *e.g.*, by T. H. Hildebrandt [8, page 27 ff.] and A. Kolmogoroff [11, page 682 ff.]). If X is the real line or the complex plane, $|\cdot|$ is understood to be the absolute value or modulus function and the parenthetical phrase involving the norm $|\cdot|$ is implicit.

The ordered pair $\{Y,\langle\cdot,\cdot\rangle\}$ is supposed to be a nondegenerate complete (real or complex) inner product space, the norm corresponding to the inner product function $\langle\cdot,\cdot\rangle$ is denoted by $[\![\cdot]\!]$, and the phrase "the scalars" refers to the real line or to the complex plane accordingly as $\{Y,\langle\cdot,\cdot\rangle\}$ is a real or a complex space. Elementary properties of such spaces (as in M. H. Stone [24] and J. von Neumann [25, 26]) are used without explicit reference. The letter j denotes a conjugation in $\{Y,\langle\cdot,\cdot\rangle\}$, as defined by Stone [24, page 357]: j is a transformation from Y to Y such that j^2 is the identity function on Y and $\langle j\xi,j\eta\rangle = \langle\eta,\xi\rangle$ for every ξ and η in Y. The set of all linear transformations from Y to Y is denoted by L(Y); if B is a member of L(Y) which is continuous (with respect to $[\![\cdot]\!]$) then B^* denotes the adjoint of B with respect to the inner product $\langle\cdot,\cdot\rangle$, so that if $\{\xi,\eta\}$ is in $Y \times Y$ then $\langle\xi,B^*\eta\rangle = \langle B\xi,\eta\rangle$. If G is a function from R to L(Y) and ξ is in Y, $G\cdot\xi$ is the function from R such that $(G\cdot\xi)(t) = G(t)\xi$ for t in R. It may be noted that the equations $\Psi(\eta)(\xi) = \langle\xi,j\eta\rangle = \langle\eta,j\xi\rangle$, for ξ and η in Y, would define a linear isomorphism Ψ from Y onto the space of all linear functions from Y to the scalars, continuous with respect to $\llbracket\cdot\rrbracket$, and Ψ would be an isometry with respect to the usual norm on the Ψ-image of Y.

Let S_0 be the family consisting of all functions f from R to Y such that (i) f is *finitely additive* in the sense that if the member M of F fills up the member u of R then $\Sigma_{t \text{ in } M} f(t) = f(u)$ and (ii) f is *of bounded variation* in the sense that there is a nonnegative number b such that $\Sigma_{t \text{ in } M} \llbracket f(t) \rrbracket \leqslant b$ for every M in F: the least such number b is the *total variation of* f and is denoted by $\|f\|$. From the completeness of Y with respect to $\llbracket\cdot\rrbracket$, it is clear that S_0, coupled with the function $\|\cdot\|$, is a linear normed complete space (a space of type B, a Banach space); in case Y is finite dimensional, it is linearly homeomorphic to a space sometimes [3, page 160] denoted by $ba(L,R_A,Y)$, the points of which are finitely additive extensions of functions in S_0 to the ring R_A which is generated by R. Such extensions to R_A, although available, are of only peripheral interest here. Attention will be drawn to linear operations on a certain type of subspace of S_0.

Of central interest are the following three functions: (1) the function P from R such that, for each t in R, P_t is the function from S_0 to S_0 such that if f is in S_0 then, for each u in R, $P_t f(u)$ is 0 or $\Sigma_{v \text{ in } M} f(v)$ accordingly as u does not intersect t or M is a member of F filling up the common part of u and t, (2) the function V from S_0 such that if f is in S_0 then Vf is the function from R such that, for each t in R, $Vf(t) = \|P_t f\|$ (if f is in S_0 and ξ is in Y, $Vf\cdot\xi$ denotes the function from R to Y such that $(Vf\cdot\xi)(t) = \|P_t f\|\xi$ for each t in R), and (3) the function J from S_0 to S_0 such that $(Jf)(t) = j(f(t))$ for each f in S_0 and t in R. The following formulas, valid for $\{u,t\}$ in R X R and f in S_0 and ξ in Y, may be noted: $P_u P_t = P_t P_u$, $\|P_t Vf\cdot\xi\| = \llbracket\xi\rrbracket \int_{t/F} \llbracket f \rrbracket$, and $\|Jf\| = \|f\|$.

Suppose, now, that S is a nondegenerate linear subspace of S_0, closed with respect to the norm $\|\cdot\|$, such that if t is in R and f is in S and ξ is in Y then the function $P_t Vf\cdot\xi$ belongs to S. It may be noted that S_0, itself, is such a linear subspace S. Here is a description of the Central Problem for which some solutions are provided in the present report.

CENTRAL PROBLEM. Find an isometrically isomorphic representation, which is determined by integration over L relatively to F, for each of the following:

(1) the linear space D consisting of all continuous linear functions from the space $\{S,\|\cdot\|\}$ to the scalars - the norm of the member λ of D is the least nonnegative number b such that if f is in S then $|\lambda(f)| \leq b\|f\|$,

(2) the linear space E consisting of all continuous linear functions from the space $\{S,\|\cdot\|\}$ to the space $\{Y,[\![\cdot]\!]\}$ - the norm of the member μ of E is the least nonnegative number b such that if f is in S then $[\![\mu(f)]\!] \leq b\|f\|$, and

(3) a linear space C(S,X) consisting of all continuous linear transformations (normed in the usual way) from $\{S,\|\cdot\|\}$ to a linear normed complete space $\{X,|\cdot|\}$ of functions from some set R_0 into Y with this property: if s is in R_0 then there is a positive number p such that, for every member g of X, $[\![g(s)]\!] \leq p|g|$.

In connection with the discovery by F. Riesz [21] concerning the dual of the space of all continuous (real or complex) functions on the unit interval, there seems to be some special interest in the aforementioned linear space D, even when it arises subject to the following (admissible) conditions: (i) R consists of all subsets t of [0,1] such that t is one of the types [0,p], (p,q], and (q,1], for numbers p and q such that $0 < p < q < 1$, (ii) the space Y is one-dimensional, and (iii) the subspace S consists of all functions f in S_0 such that, if $\epsilon > 0$ and $0 < p < 1$, there is a number r in (p,1] such that if q is a number in (p,r] then $|f((p,q])| < \epsilon$ (*cf.* Chapter III of Riesz and Sz.-Nagy [22], concerning the connection between S and the dual of a space of continuous functions). There is T. H. Hildebrandt's representation [6,7] for D, under the conditions that the space Y is one-dimensional and S is S_0, but there [7] the total variation norm is replaced by the supremum norm on the finitely additive extensions of members of S_0 to the ring R_A generated by R (*cf.* footnote on page 374 and remarks on pages 392-393 of Dunford and Schwartz [3]): Hildebrandt's representation is determined by Stieltjes-type integration over R_A relatively to the family of all finite collections of mutually exclusive subsets of R_A filling up R_A. There are, also, R. D. Mauldin's contributions [16, 17] to the theory of the space D. In Mauldin's departure [17] beyond scalar-valued measures, hypotheses on the space $\{Y,[\![\cdot]\!]\}$ are relaxed from those of the present treatment but countably additive extensions are assumed for the members of S (as in [16]), and questions of cardinality persist. The investigation reported here has been independent of Mauldin's work but points of contact occur in use of Hellinger-type integrals for recovery of functionals.

Description of Solutions. Let S^+ be the V-image of S and H be a function from S^+ such that, for each α in S^+, H_α is the family to which f belongs only in case f is a finitely additive function from R to Y and there is a finitely additive function h from R to the nonnegative numbers such that $\int_{L/F} h$ exists and $\|f(t)\|^2 \leqslant \alpha(t)h(t)$ for each t in R - so that f is in S_0 and $\|f\|^2 \leqslant \int_{L/F} \alpha \int_{L/F} h$ (by Schwarz's inequality). It is clear that if f is in S then Vf is a member α of S^+ such that f belongs to H_α.

It is shown that there exists a function Q from S^+, opposite to Solutions of the Central Problem, such that

(1) if α is in S^+ then H_α is a linear subspace of S, Q_α is an inner product for H_α such that the space $\{H_\alpha, Q_\alpha\}$ is complete, if t is in R then the restriction of P_t to H_α is a Q_α-orthogonal projection in $\{H_\alpha, Q_\alpha\}$, and the restriction of J to H_α is a conjugation in $\{H_\alpha, Q_\alpha\}$,

(2) there is a function π from the subset of $S^+ \times S^+$ to which $\{\alpha,\beta\}$ belongs only in case H_α is a subset of H_β, in which case $\pi(\alpha,\beta)$ is a function from H_β to H_α to which $\{g,h\}$ belongs only in case $Q_\alpha(f,h) = Q_\beta(f,g)$ for each f in H_α, and

(3) the ordered triple $\{H,Q,\pi\}$ determines an inverse limit system in the sense that if each of α, β, and γ is in S^+ then (i) If H_α is a subset of H_β then $\pi(\alpha,\beta)$ is a continuous linear transformation from $\{H_\beta, Q_\beta\}$ to $\{H_\alpha, Q_\alpha\}$, (ii) if H_α is a subset of H_β and H_β is a subset of H_γ then $\pi(\alpha,\gamma)$ is the composite transformation $\pi(\alpha,\beta)\pi(\beta,\gamma)$, and (iii) if H_α is H_β then $\pi(\beta,\alpha)$ is the inverse of $\pi(\alpha,\beta)$.

Such a function Q from S^+ is provided by a variant of an integral which was introduced by E. Hellinger [4], and extended by J. Radon [18]: the variant is

$$Q_\alpha(f,g) = \int_{L/F} \langle f,g\rangle/\alpha \text{ for each } \alpha \text{ in } S^+ \text{ and } \{f,g\} \text{ in } H_\alpha \times H_\alpha,$$

with Hellinger's notational convention to the effect that, for each set t in the collection R, $\frac{\langle f,g\rangle}{\alpha}(t)$ is 0 or $\frac{\langle f(t),g(t)\rangle}{\alpha(t)}$ accordingly as $\alpha(t)$ is 0 or not. It is shown that if each of α and β is in S^+ then H_α is a subset of H_β only in case there is a nonnegative number c such that $\alpha(t) \leqslant c\,\beta(t)$ for each t in R, in which case the transformation $\pi(\alpha,\beta)$ is given by the formulas

$$\langle(\pi(\alpha,\beta)g)(t),\xi\rangle = \int_{t/F} \langle g,\alpha\cdot\xi\rangle/\beta \text{ for g in } H_\beta, \text{ t in R, and } \xi \text{ in Y.}$$

Moreover, if each of α and β is in S^+ then $H_{\alpha+\beta}$ is the vector sum of H_α and H_β, and there is a member γ of S^+ such that H_γ is the common part of H_α and H_β, and Q_γ is $Q_\alpha+Q_\beta$ on $H_\gamma \times H_\gamma$. The degenerate space $\{H_0,Q_0\}$ corresponds to the zero member of S^+: omission of $\{H_0,Q_0\}$ would entail awkwardness of description here, inasmuch as it is shown (Theorem 10) that the H-image of S^+ is a distributive lattice (relatively to the relation "is a subset of") with least element H_0.

Consistently with standard usage (*e.g.*, by J. L. Kelley, I. Namioka, *et al.* [10, page 11]), the *inverse limit space* determined by the ordered triple $\{H,Q,\pi\}$ is the linear space to which g belongs only in case g is a function from S^+ such that, for each α in S^+, $g(\alpha)$ is a function belonging to H_α and if β is a member of S^+ such that H_α is a subset of H_β then $g(\alpha) = \pi(\alpha,\beta)g(\beta)$: inv-lim-$\{H,Q,\pi\}$ denotes this space. It may be noted that there has been no prior assertion of the existence of a non-zero point in this inverse limit space.

REPRESENTATION OF D. The equations $\langle A(\lambda)(\alpha)(t),j\xi\rangle = \lambda(P_t\alpha\cdot\xi)$, for λ in D and α in S^+ and t in R and ξ in Y, define a linear isomorphism A from D onto the subspace of inv-lim-$\{H,Q,\pi\}$ to which the point g of inv-lim-$\{H,Q,\pi\}$ belongs only in case there is a nonnegative number b such that, for each α in S^+ and t in R, $\|g(\alpha)(t)\| \leqslant b\,\alpha(t)$, in which case the norm of the member $A^{-1}(g)$ of D is the least such number b. If the ordered pair $\{\lambda,g\}$ belongs to A and f is in S then $\lambda(f)$ is an integral over L relatively to F in the following sense: for each α in S^+ such that f belongs to H_α, $\lambda(f) = \int_{L/F} \langle g(\alpha),Jf\rangle/\alpha$.

Now, let INV-LIM-$\{H,Q,\pi\}$ denote the linear space to which G belongs only in case G is a function from S^+ such that, for each α in S^+, $G(\alpha)$ is a finitely additive function from R to L(Y) and, if ξ is in Y, $G(\alpha)\cdot\xi$ belongs to H_α and if β is a member of S^+ such that H_α is a subset of H_β then $G(\alpha)\cdot\xi = \pi(\alpha,\beta)(G(\beta)\cdot\xi)$.

REPRESENTATION OF E. The equations $\omega(\mu)(\alpha)(t)\xi = \mu(P_t\alpha\cdot\xi)$, for μ in E and α in S^+ and t in R and ξ in Y, define a linear isomorphism ω from E onto the subspace of INV-LIM-$\{H,Q,\pi\}$ to which the point G of INV-LIM-$\{H,Q,\pi\}$ belongs only in case there is a nonnegative number b such that, for each α in S^+ and t in R and ξ in Y, $\|G(\alpha)(t)\xi\| \leqslant b\,\alpha(t)\|\xi\|$, in which case the norm of the member $\omega^{-1}(G)$ of E is the least such number b. If the ordered pair $\{\mu,G\}$ belongs to ω and f is in S then $\mu(f)$

is an integral over L relatively to F in the following sense: for each α in S^+ such that f belongs to H_α, $\mu(f) = \int_{L/F} G(\alpha)\cdot f/\alpha$ with respect to $[\![\cdot]\!]$.

In the preceding Representation, Hellinger's notational convention persists to the effect that, for each set t in the collection R, $\frac{G(\alpha)\cdot f}{\alpha}(t)$ is the point 0 or $G(\alpha)(t)f(t)/\alpha(t)$ in Y accordingly as $\alpha(t)$ is the number 0 or not. Moreover, it is shown that if G is in the ω-image of E then so is the function G′, defined by: $G'(\alpha)(t) = G(\alpha)(t)^*$ for α in S^+ and t in R. Hence, there is a natural norm-preserving involution in E, to which the ordered pair $\{\mu,\mu'\}$ belongs only in case μ is in E and $\mu' = \omega^{-1}(\omega(\mu)')$, *i.e.*, $\langle\mu(P_t\alpha\cdot\xi),\eta\rangle = \langle\xi,\mu'(P_t\alpha\cdot\eta)\rangle$ for α in S^+ and t in R and $\{\xi,\eta\}$ in $Y \times Y$.

Now, let N be the function from S^+ such that, for each α in S^+, N_α is the norm for H_α corresponding to the inner product Q_α - so that $N_\alpha(f) = Q_\alpha(f,f)^{1/2}$ for each f in H_α. Let C be the space of all continuous linear transformations in $\{S,\|\cdot\|\}$, normed in the usual manner: the norm of the member B of C is the least nonnegative number b such that if f is a member of S then $\|Bf\| \leqslant b\|f\|$.

REPRESENTATION OF E IN C. The equations $(\zeta(\mu)f)(t) = \mu(P_tf)$, for μ in E and f in S and t in R, define an isometric linear isomorphism ζ from E onto the subspace of C to which the member B of C belongs only in case, for each t in R and f in S, $B(P_tf) = P_t(Bf)$. In order that the linear transformation B from S into S should belong to the ζ-image of E, it is necessary and sufficient that (i) for each t in R and f in S, $B(P_tf) =,P_t(Bf)$, (ii) for each α in S^+, B should map H_α into H_α, and (iii) there should exist a nonnegative number b such that, for each α in S^+ and f in H_α, $N_\alpha(Bf) \leqslant b\, N_\alpha(f)$, in which case the norm of the member $\zeta^{-1}(B)$ of E is the least such number b. If the ordered pair $\{\mu,B\}$ belongs to ζ and f is in S then $\mu(f) = \int_{L/F} Bf$ with respect to $[\![\cdot]\!]$.

Each of the foregoing integral representations is effected by the existence of a function Π from S^+ such that, for each α in S^+, Π_α is a function from F such that if M is in F then $\Pi_\alpha(M)$ is an orthogonal projection in the space $\{H_\alpha,Q_\alpha\}$ with the property that if each of f and g is a member of H_α then

$$Q_\alpha(f - \Pi_\alpha(M)f,g - \Pi_\alpha(M)g) = Q_\alpha(f,g) - \Sigma_{t \text{ in } M}\frac{\langle f,g\rangle}{\alpha}(t);$$

there are, of course, the associated inequalities (for all such α, M, and f)

$$\|f - \Pi_\alpha(M)f\|^2 \leqslant N_\alpha(f - \Pi_\alpha(M)f)^2 \int_{L/F}\alpha.$$

In terms of the Representation ζ, there is a natural multiplication defined in E: $\mu_1 \cdot \mu_2 = \zeta^{-1}(\zeta(\mu_1)\zeta(\mu_2))$ for $\{\mu_1,\mu_2\}$ in E $\times$ E. The identity element e of E, for this multiplication, is given by: $e(f) = \int_{L/F} f$ for f in S. It is shown that if μ is in E then, for each α in S^+, the restriction to H_α of $\zeta(\mu')$ is the adjoint (with respect to Q_α) of the restriction to H_α of $\zeta(\mu)$. Let $\{X,(\cdot,\cdot)\}$ be the direct sum over S^+ of the spaces $\{H_\alpha,Q_\alpha\}$: X is the linear space to which f belongs only in case f is a function from S^+ such that, for each β in S^+, f_β is a member of H_β and there is a positive number p such that $\Sigma_{\alpha \text{ in } \sigma} N_\alpha(f_\alpha)^2 \leqslant p$ for each finite subset σ of S^+, and $(f,g) = \Sigma_\alpha Q_\alpha(f_\alpha,g_\alpha)$ for $\{f,g\}$ in X $\times$ X. Now, it is clear from the aforementioned facts about ζ and ω that the equations

$$(Z(\mu)f,g) = \Sigma_\alpha Q_\alpha(\zeta(\mu)f_\alpha,g_\alpha), \text{ for } \mu \text{ in E and } \{f,g\} \text{ in } X \times X,$$

define an isometric involution-preserving algebra-isomorphism Z from E onto what is sometimes [20, 23] called a B*-algebra of continuous linear transformations in the space $\{X,(\cdot,\cdot)\}$. Identification of the Z-image of E in the algebra A_0 of all continuous linear transformations in $\{X,(\cdot,\cdot)\}$ may be made by considering: the algebra A_1 of all members B of A_0 with a representation Ψ such that

$$(Bf,g) = \Sigma_\alpha Q_\alpha(\Psi(B)_\alpha f_\alpha,g_\alpha) \text{ for } \{f,g\} \text{ in } X \times X,$$

where, for each α in S^+, $\Psi(B)_\alpha$ is a continuous linear transformation in $\{H_\alpha,Q_\alpha\}$ and there is a positive number p such that $N_\alpha(\Psi(B)_\alpha h) \leqslant p\, N_\alpha(h)$ for each α in S^+ and h in H_α; the algebra A_2 of all members B of A_1 such that if α is in S^+ and t is in R and h is in H_α then $\Psi(B)_\alpha P_t h = P_t \Psi(B)_\alpha h$; and, finally, the algebra A_3 of all members B of A_2 such that if α and β are members of S^+ such that H_α is a subset of H_β then $\Psi(B)_\alpha$ is the restriction to H_α of the transformation $\Psi(B)_\beta$. It is shown (Theorem 25) that the Z-image A_3 of E is weakly closed in the algebra A_0.

It is the aforementioned family of orthogonal projections $\Pi_\alpha(M)$, for α in S^+ and M in F, which makes available the general representation (Theorem 20) for any such space C(S,X) as is indicated in the statement of the Central Problem. This latter representation Ω, defined in terms of INV-LIM-$\{H,Q,\pi\}$, may be viewed as an extension of the representation ω of the space E.

The Inverse Limit System.

THEOREM 1. *Suppose* f *is a finitely additive function from* R *to* Y, α *is in* S^+, *and if* v *is a member of* R *such that* $\alpha(v) = 0$ *then* $f(v) = 0$. *Then, if* M *and* W *are members of* F *such that each set in* M *is filled up by a subset of* W,

$$\Sigma_{s \text{ in } M} \|f(s)\|^2/\alpha(s) \leqslant \Sigma_{t \text{ in } W} \|f(t)\|^2/\alpha(t).$$

PROOF. It follows from Schwarz's inequality as applied to finite sums, together with Hellinger's notational convention to the effect that $\|f(v)\|^2/\alpha(v)$ be interpreted as the number 0 in case $\alpha(v) = 0$, that if U is a member of F then

$$\|\Sigma_{v \text{ in } U} f(v)\|^2 \leqslant (\Sigma_{v \text{ in } U} \|f(v)\|)^2 \leqslant \Sigma_{v \text{ in } U} \alpha(v)\ \Sigma_{t \text{ in } U} \|f(t)\|^2/\alpha(t).$$

Hence, the conclusion is a consequence of the finitely additive character of f.

THEOREM 2. *If* f *is a finitely additive function from* R *to* Y *and* α *is in* S^+ *and* b *is a nonnegative number then the following three statements are equivalent:*

(1) *there is a finitely additive function* h *from* R *to a set of nonnegative numbers such that* $\int_{L/F} h \leqslant b$ *and, for each* t *in* R, $\|f(t)\|^2 \leqslant \alpha(t)h(t)$,

(2) *if* M *is a member of* F *then, for each function* x *from* M *to* Y,

$$|\Sigma_{u \text{ in } M} \langle f(u), x(u)\rangle|^2 \leqslant b\ \Sigma_{u \text{ in } M} \alpha(u)\|x(u)\|^2, \textit{ and}$$

(3) *if* v *is a member of* R *such that* $\alpha(v) = 0$ *then* $f(v) = 0$ *and, for each member* M *of* F, $\Sigma_{u \text{ in } M} \|f(u)\|^2/\alpha(u) \leqslant b$.

PROOF. If the statement (3) is true then it is a consequence of Theorem 1 that the equations $h(t) = \int_{t/F} \|f\|^2/\alpha$, for t in R, define a finitely additive function h from R which fulfills the conditions given in the statement (1).

If, now, the statement (1) is true then, for each member M of F and each function x from M to Y,

$$|\Sigma_{u \text{ in } M} \langle f(u), x(u)\rangle| \leqslant \Sigma_{u \text{ in } M} \{h(u)\alpha(u)\}^{1/2} \|x(u)\|,$$

so that the statement (2) is a consequence of Schwarz's inequality.

If, finally, the statement (2) is true then (i) it is clear that if v is a member of R such that $\alpha(v) = 0$ then $f(v) = 0$, and (ii) if M is a member of F and x is the function defined by $x(u) = 0$ or $f(u)/\alpha(u)$ for u in M, accordingly as $\alpha(u)$ is 0 or not, then the inequality indicated in the statement (3) is apparent.

THEOREM 3. *If* α *is in* S^+ *then* (1) H_α *is a linear subspace of* S_0, (2) *there is a* *norm* N_α *for* H_α *such that if* f *is in* H_α *then* $N_\alpha(f)^2 = \int_{L/F} \|f\|^2/\alpha$, (3) *if* f *is in* H_α *then* $\|f\|^2 \leq N_\alpha(f)^2 \int_{L/F} \alpha$, *and* (4) H_α *is complete with respect to* N_α.

Theorem 3 may be proved as a consequence of Theorems 1 and 2, with the help of the observations that, for each α in S^+, (i) Theorem 2 provides additional characterizations of the family H_α and (ii) if f is in H_α then $N_\alpha(f)$ is the square root of the least nonnegative number b such that one of the three numbered statements indicated in Theorem 2 is true.

THEOREM 4. *If* α *is in* S^+ *then the family* U_α, *to which* g *belongs only in case there is a member* M *of* F *and a function* x *from* M *to* Y *such that* g *is the function* $\Sigma_{u \text{ in } M} P_u \alpha \cdot x(u)$, *is a linear subspace of* H_α.

PROOF. It follows from the definition of the function P that, if u is in R and W is a member of F filling up u and f is in S_0, $P_u f = \Sigma_{t \text{ in } W} P_t f$. It is clear that, if t is in R and α is in S^+ and ξ is in Y, $\|(P_t \alpha \cdot \xi)(u)\| \leq \alpha(u)\|\xi\|$ for each u in R so that the function $P_t \alpha \cdot \xi$ belongs to the family H_α.

Suppose α is in S^+. It is clear, from the linearity of H_α, that U_α is a subset of H_α. Suppose M is a member of F, x is a function from M to Y, and W is a member of F such that each set in M is filled up by a subcollection of W. Let K be a function from M such that if u is in M then K(u) is the subset of W to which the element t of W belongs only in case t lies in u. There is a function z from W to Y such that (i) if the member t of W lies in the member u of M then z(t) is x(u) and (ii) if the member t of W does not lie in any member of M then z(t) is 0. If u is in M then K(u) is a member of F filling up u; hence

$$\Sigma_{u \text{ in } M} P_u \alpha \cdot x(u) = \Sigma_{u \text{ in } M} \Sigma_{t \text{ in } K(u)} P_t \alpha \cdot x(u) = \Sigma_{t \text{ in } W} P_t \alpha \cdot z(t).$$

The assertion of the Theorem follows, with the help of the fact that if M_1 and M_2 are members of F then there is a member W of F such that each set in M_1 or M_2 is filled up by a subcollection of W.

THEOREM 5. *If* α *is in* S^+ *and* t *is in* R *then each of* J *and* P_t *maps* H_α *into* H_α *and, for each* f *in* H_α, $N_\alpha(Jf) = N_\alpha(f)$ *and* $N_\alpha(P_t f)^2 = \int_{t/F} \|f\|^2/\alpha$.

PROOF. Suppose α is in S^+. The assertions concerning the function J are immediate consequences of the definitions since $\|Jf(t)\| = \|f(t)\|$ for each f in H_α and each t in R. Suppose f is in H_α, h is such a function from R as is indicated in the statement (1) of Theorem 2 with $b = N_\alpha(f)^2$, and t is in R. If u is in R and M is a member of F filling up the common part of u and t,

$$\|P_t f(u)\|^2 \leqslant (\Sigma_{v \text{ in } M}\|f(v)\|)^2 \leqslant (\Sigma_{v \text{ in } M}\{\alpha(v)h(v)\}^{1/2})^2 \leqslant \alpha(u)h(u).$$

Hence, $P_t f$ belongs to H_α and $N_\alpha(P_t f) \leqslant N_\alpha(f)$. The indicated integral formula for $N_\alpha(P_t f)^2$ may be verified by considering members of F having subcollections filling up t, in conjunction with the formula for N_α indicated in Theorem 3.

THEOREM 6. *If* α *is in* S^+, *then* (1) *there is a function* Q_α *from* $H_\alpha \times H_\alpha$ *such that* $Q_\alpha(f,g) = \int_{L/F}\langle f,g\rangle/\alpha$ *for each* $\{f,g\}$ *in* $H_\alpha \times H_\alpha$, (2) Q_α *is an inner product for* H_α *to which* N_α *is the corresponding norm,* (3) *the restriction of* J *to* H_α *is a conjugation in* $\{H_\alpha,Q_\alpha\}$, *and* (4) *for each* $\{f,g\}$ *in* $H_\alpha \times H_\alpha$ *and* t *in* R

$$Q_\alpha(P_t f,g) = \int_{t/F}\langle f,g\rangle/\alpha = Q_\alpha(f,P_t g),$$

so that the restriction of P_t *to* H_α *is a* Q_α*-orthogonal projection in* $\{H_\alpha,Q_\alpha\}$.

PROOF. Suppose α is in S^+. The existence of the function Q_α from $H_\alpha \times H_\alpha$, as indicated in (1), is a simple consequence of the following equations:

$$\Sigma_{u \text{ in } M}\|f(u)+g(u)\|^2/\alpha(u) - \Sigma_{u \text{ in } M}\|f(u)-g(u)\|^2/\alpha(u) = 4 \text{ Re } \Sigma_{u \text{ in } M}\langle f(u),g(u)\rangle/\alpha(u)$$

for $\{f,g\}$ in $H_\alpha \times H_\alpha$ and each M in the family F, with the customary notational convention (*cf.* Theorem 1) in case there is a member u of M such that $\alpha(u) = 0$. It is similarly clear that if f is in H_α then $Q_\alpha(f,f) = N_\alpha(f)^2$, and that Q_α is an inner product for H_α, so that (2) is true. Moreover, since J maps H_α into H_α and J^2 is the identity on S_0 and, for each $\{f,g\}$ in $H_\alpha \times H_\alpha$ and M in F,

$$\Sigma_{u \text{ in } M}\langle jf(u),jg(u)\rangle/\alpha(u) = \Sigma_{u \text{ in } M}\langle g(u),f(u)\rangle/\alpha(u),$$

it follows that the restriction of J to H_α is a conjugation in $\{H_\alpha,Q_\alpha\}$. Now, let t be a member of R. It is clear that $P_t^2 = P_t$ on S_0 and, by Theorem 5, P_t maps H_α into H_α. If $\{f,g\}$ is in $H_\alpha \times H_\alpha$ then the indicated integral formula for $Q_\alpha(P_t f,g)$, and that for $Q_\alpha(f,P_t g)$, may be verified by considering members of F having subcollections filling up t, in conjunction with the formula for Q_α which is given in (1). Thus, the restriction of P_t to H_α is Hermitian with respect to the inner product Q_α, and so is a Q_α-orthogonal projection of H_α onto a closed linear subspace of $\{H_\alpha,Q_\alpha\}$.

THEOREM 7. *If* α *is in* S^+ *then* (1) *if* f *is in* H_α *and* t *is in* R *and* ξ *is in* Y *then*

$\langle f(t),\xi\rangle = Q_\alpha(f, P_t\alpha\cdot\xi)$, (2) *the family* U_α *(as described in Theorem 4) is a dense linear subspace of* $\{H_\alpha, Q_\alpha\}$, *and* (3) H_α *is a linear subspace of* S.

PROOF. Suppose α is in S^+. It should be recalled that if t is in R and ξ is in Y then $P_t\alpha\cdot\xi$ is a member of S. Since S is a linear subspace of S_0, U_α is a subset of S; by Theorem 4, U_α is a linear subspace of H_α.

If f is in H_α and t is in R and ξ is in Y then, for every member W of the family F filling up the set t,

$$\langle f(t),\xi\rangle = \Sigma_{u \text{ in } W}\langle f(u),\xi\rangle = \Sigma_{u \text{ in } W}\langle f(u),(\alpha\cdot\xi)(u)\rangle/\alpha(u),$$

so that, in accordance with assertion (4) of Theorem 6,

$$\langle f(t),\xi\rangle = \int_{t/F}\langle f,\alpha\cdot\xi\rangle/\alpha = Q_\alpha(f, P_t\alpha\cdot\xi).$$

This establishes assertion (1). Since the space $\{H_\alpha, Q_\alpha\}$ is complete, if U_α were not dense in this space then there would be a non-zero member f of H_α belonging to the Q_α-orthogonal complement (in H_α) of U_α - this would involve a contradiction to (1). Hence, assertion (2) is true.

Suppose, now, that f is a member of H_α which does not belong to S. If g is a member of the family U_α then, by the assertion (3) of Theorem 3,

$$\|f - g\|^2 \leqslant N_\alpha(f - g)^2 \int_{L/F}\alpha.$$

Since U_α is dense in $\{H_\alpha, Q_\alpha\}$, and S is closed with respect to the norm $\|\cdot\|$, this involves a contradiction.

THEOREM 8. *Suppose that* α *is in* S^+ *and, for each* M *in* F, $\Pi_\alpha(M)$ *is the function from* H_α *determined as follows: if* f *is in* H_α *and* x *is a function from* M *to* Y *such that, for each* t *in* M, x(t) *is* 0 *or* $f(t)/\alpha(t)$ *accordingly as* $\alpha(t)$ *is* 0 *or not, then* $\Pi_\alpha(M)f = \Sigma_{t \text{ in } M} P_t\alpha\cdot x(t)$. *Then*

(1) *if* M *is in* F, $\Pi_\alpha(M)$ *is the* Q_α*-orthogonal projection from* H_α *onto the subset of* U_α *(cf. Theorem 4) to which the member* g *of* U_α *belongs only in case there is a function* x *from* M *to* Y *such that* $g = \Sigma_{t \text{ in } M} P_t\alpha\cdot x(t)$, *and*

(2) *if* $\{f,g\}$ *is in* $H_\alpha \times H_\alpha$ *and* M *is in* F,

$$Q_\alpha(f - \Pi_\alpha(M)f, g - \Pi_\alpha(M)g) = Q_\alpha(f,g) - \Sigma_{t \text{ in } M}\frac{\langle f,g\rangle}{\alpha}(t).$$

PROOF. For each M in F, let $U_\alpha(M)$ be the subset of U_α indicated in the assertion (1). If M is in F then, for each function x from M to Y, it follows from Theorem 7 that

$$N_\alpha(\Sigma_{t \text{ in } M} P_t \alpha \cdot x(t))^2 = \Sigma_{t \text{ in } M} \alpha(t) \| x(t) \|^2,$$

whence $U_\alpha(M)$ is a closed linear subspace of $\{H_\alpha, Q_\alpha\}$; moreover, for each such M and x, if f is in H_α then by Theorem 7, for each v in M and η in Y,

$$Q_\alpha(f - \Sigma_{t \text{ in } M} P_t \alpha \cdot x(t), P_v \alpha \cdot \eta) = \langle f(v) - \alpha(v)x(v), \eta \rangle.$$

This establishes the assertion (1). Suppose, now, that $\{f,g\}$ is in $H_\alpha \times H_\alpha$ and M is in F: it follows from assertion (1) that

$$Q_\alpha(f - \Pi_\alpha(M)f, g - \Pi_\alpha(M)g) = Q_\alpha(f,g) - Q_\alpha(\Pi_\alpha(M)f, \Pi_\alpha(M)g).$$

If each of x and y is a function from M to Y such that, for each t in M,

$$x(t) = 0 \text{ or } f(t)/\alpha(t) \text{ and } y(t) = 0 \text{ or } g(t)/\alpha(t)$$

accordingly as $\alpha(t)$ is 0 or not, then (again by Theorem 7)

$$Q_\alpha(\Pi_\alpha(M)f, \Pi_\alpha(M)g) = \Sigma_{t \text{ in } M} \alpha(t) \langle x(t), y(t) \rangle = \Sigma_{t \text{ in } M} \frac{\langle f,g \rangle}{\alpha}(t).$$

THEOREM 9. *If each of* α *and* β *is in* S^+ *then, in order that* H_α *should be a subset of* H_β, *it is necessary and sufficient that there be a nonnegative number* c *such that* $\alpha(t) \leqslant c\,\boldsymbol{\beta}(t)$ *for each* t *in* R, *in which case* $\pi(\alpha,\beta)$ *is a continuous linear transformation from* $\{H_\beta, Q_\beta\}$ *to* $\{H_\alpha, Q_\alpha\}$ *given by the formlas*

$$\langle (\pi(\alpha,\beta)g)(t), \xi \rangle = \int_{t/F} \langle g, \alpha \cdot \xi \rangle / \beta \textit{ for } g \textit{ in } H_\beta, \ t \textit{ in } R, \textit{ and } \xi \textit{ in } Y.$$

PROOF. Suppose each of α and β belongs to S^+. It is clear from Theorem 2 that the indicated condition is sufficient for H_α to be a subset of H_β. Suppose, now, that H_α is a subset of H_β. By Theorem 7, if f is in H_α and t is in R and ξ is in Y then $Q_\alpha(f, P_t \alpha \cdot \xi) = \langle f(t), \xi \rangle = Q_\beta(f, P_t \beta \cdot \xi)$. Therefore, if M is in F and x is a function from M to Y then, for each f in H_α,

$$Q_\alpha(f, \Sigma_{t \text{ in } M} P_t \alpha \cdot x(t)) = Q_\beta(f, \Sigma_{t \text{ in } M} P_t \beta \cdot x(t)).$$

Since, by Theorems 4 and 7, the family U_β is a dense linear subspace of $\{H_\beta, Q_\beta\}$, it follows that $\{H_\alpha, Q_\alpha\}$ is continuously included in $\{H_\beta, Q_\beta\}$, *i.e.*, that the identity transformation on H_α is a continuous linear transformation from $\{H_\alpha, Q_\alpha\}$ into $\{H_\beta, Q_\beta\}$. Hence, the transformation $\pi(\alpha,\beta)$, to which $\{g,h\}$ belongs only in case g is in H_β and h is in H_α and $Q_\alpha(f,g) = Q_\beta(f,g)$ for each f in H_α, is a continuous linear transformation from $\{H_\beta,Q_\beta\}$ into $\{H_\alpha,Q_\alpha\}$. Thus, there exists a nonnegative number c such that

$$Q_\alpha(\pi(\alpha,\beta)g,\pi(\alpha,\beta)g) \leqslant c\, Q_\beta(g,g) \text{ for each g in } H_\beta,$$

and, if t is in R and ξ is in Y, the ordered pair $\{P_t\beta\cdot\xi,P_t\alpha\cdot\xi\}$ belongs to $\pi(\alpha,\beta)$ so that $\alpha(t)\|\xi\|^2 = Q_\alpha(P_t\alpha\cdot\xi,P_t\alpha\cdot\xi) \leqslant c\, Q_\beta(P_t\beta\cdot\xi,P_t\beta\cdot\xi) = c\,\beta(t)\|\xi\|^2$, whence it follows that $\alpha(t) \leqslant c\,\beta(t)$; finally, if g is in H_β and t is in R and ξ is in Y,

$$\langle(\pi(\alpha,\beta)g)(t),\xi\rangle = Q_\alpha(\pi(\alpha,\beta)g,P_t\alpha\cdot\xi) = Q_\beta(g,P_t\alpha\cdot\xi) = \int_{t/F}\langle g,\alpha\cdot\xi\rangle/\beta\ ,$$

the latter formula being justified on the basis of Theorem 6. For the continuous inclusion of $\{H_\alpha,Q_\alpha\}$ in $\{H_\beta,Q_\beta\}$, one has von Neumann's extension (see Stone's footnote [24, page iv]) of the Hellinger-Toeplitz Theorem (*cf.* Rudin [23, page 110]).

THEOREM 10. *If each of* α *and* β *is in* S^+, *the following statements are true:*

(1) $H_{\alpha+\beta}$ *is the vector sum of* H_α *and* H_β, *to which* h *belongs only in case there is a member* $\{f,g\}$ *of* $H_\alpha \times H_\beta$ *such that* $f + g = h$,

(2) *the formulas* $\gamma(t) = \int_{t/F}\alpha\beta/(\alpha+\beta)$ *for* t *in* R, *define a member* γ *of* S^+ *such that* H_γ *is the common part* $H_\alpha H_\beta$ *of* H_α *and* H_β *and* $Q_\gamma = Q_\alpha + Q_\beta$ *on* $H_\gamma \times H_\gamma$, *and*

(3) *for every member* γ *of* S^+, *the common part of* H_α *and* $H_{\beta+\gamma}$ *is the vector sum of* $H_\alpha H_\beta$ *and* $H_\alpha H_\gamma$.

PROOF. Supposing that each of α and β is in S^+, one sees from the linearity of S that $\alpha+\beta$ belongs to S^+. It follows from Theorem 9 that each of H_α and H_β is a subset of $H_{\alpha+\beta}$; moreover, if h is in $H_{\alpha+\beta}$ then the formulas

$$f = \pi(\alpha,\alpha+\beta)h \text{ and } g = \pi(\beta,\alpha+\beta)h$$

define a member $\{f,g\}$ of $H_\alpha \times H_\beta$ such that $f+g = \pi(\alpha+\beta,\alpha+\beta)h = h$. Hence, the statement (1) is true.

Now, by the type of reasoning employed in the Proof of Theorem 6, here are formulas for the function γ which are equivalent to those indicated in (2):

$$\gamma(t) = \tfrac{1}{4}[\alpha(t)+\beta(t)] - \tfrac{1}{4}\int_{t/F}(\alpha-\beta)^2/(\alpha+\beta) \text{ for each t in R.}$$

Hence, the indicated formulas define a finitely additive function γ from R to the nonnegative numbers. Moreover, by Theorem 9, for each ξ in Y

$$\gamma\cdot\xi = \pi(\alpha,\alpha+\beta)(\beta\cdot\xi) = \pi(\beta,\alpha+\beta)(\alpha\cdot\xi),$$

so that $\gamma\cdot\xi$ belongs to H_α and to H_β, γ belongs to S^+, and H_γ is a subset of the common part $H_\alpha H_\beta$ of H_α and H_β. Clearly, $Q_\alpha+Q_\beta$ is an inner product Q' for $H_\alpha H_\beta$ such that the space $\{H_\alpha H_\beta, Q'\}$ is complete. Moreover, if f is in $H_\alpha H_\beta$ then

$$Q'(f,P_t\gamma\cdot\xi) = Q_\alpha(P_t f,\pi(\alpha,\alpha+\beta)(\beta\cdot\xi)) + Q_\beta(P_t f,\pi(\beta,\alpha+\beta)(\alpha\cdot\xi))$$

$$= Q_{\alpha+\beta}(P_t f,\beta\cdot\xi) + Q_{\alpha+\beta}(P_t f,\alpha\cdot\xi)$$

$$= Q_{\alpha+\beta}(f,P_t(\alpha+\beta)\cdot\xi) = \langle f(t),\xi\rangle$$

for every t in R and ξ in Y; hence, if M is in F and x is a function from M to Y,

$$|\Sigma_{t \text{ in } M}\langle f(t),x(t)\rangle|^2 \leqslant Q'(f,f)\Sigma_{t \text{ in } M}\gamma(t)\|x(t)\|^2,$$

so that, by Theorem 2, f belongs to H_γ and $Q_\gamma(f,f) \leqslant Q'(f,f)$: thus, H_γ is $H_\alpha H_\beta$. Since (by the foregoing secondary description of γ) $[\alpha(t)+\beta(t)]\gamma(t) \leqslant \alpha(t)\beta(t)$ for each t in R, it follows that if f is in H_γ then, for each M in F,

$$\Sigma_{u \text{ in } M}\frac{\|f\|^2}{\alpha}(u) + \Sigma_{u \text{ in } M}\frac{\|f\|^2}{\beta}(u) \leqslant \Sigma_{u \text{ in } M}\frac{\|f\|^2}{\gamma}(u),$$

so that $N_\alpha(f)^2 + N_\beta(f)^2 \leqslant N_\gamma(f)^2$. Therefore, $Q_\alpha(f,f) + Q_\beta(f,f) = Q_\gamma(f,f)$ for every f in H_γ. Now, by the type of reasoning indicated in the first part of the Proof of Theorem 6, $Q_\alpha + Q_\beta = Q_\gamma$ on $H_\gamma \times H_\gamma$. Therefore (2) is true.

Apropos of the statement (3), now, let γ be any member of S^+. By (1), $H_{\beta+\gamma}$ is the vector sum of H_β and H_γ; hence, the vector sum of $H_\alpha H_\beta$ and $H_\alpha H_\gamma$ is a subset of $H_\alpha H_{\beta+\gamma}$. By (2), there is a member δ of S^+ such that, for each t in R,

$$\delta(t) = \int_{t/F}\alpha\cdot(\beta+\gamma)/(\alpha+\beta+\gamma) \leqslant \int_{t/F}\alpha\beta/(\alpha+\beta) + \int_{t/F}\alpha\gamma/(\alpha+\gamma)$$

and H_δ is $H_\alpha H_{\beta+\gamma}$; by (1) and (2) and Theorem 9, H_δ is a subset of the vector sum of $H_\alpha H_\beta$ and $H_\alpha H_\gamma$. This completes the Proof of Theorem 10.

Representation of Linear Operations. It should be recalled that inv-lim-$\{H,Q,\pi\}$ denotes the linear space to which g belongs only in case g is a function from S^+ such that, for each α in S^+, $g(\alpha)$ is a member of H_α and if β is a member of S^+ such that H_α is a subset of H_β then $g(\alpha) = \pi(\alpha,\beta)g(\beta)$; and INV-LIM-$\{H,Q,\pi\}$ denotes the linear space to which G belongs only in case (i) G is a function from S^+ to a set of finitely additive functions from R to L(Y) and (ii) if η is in Y then there is a member g of the space inv-lim-$\{H,Q,\pi\}$ such that $g(\alpha) = G(\alpha)\cdot\eta$ for every α in S^+. One may note that, by Theorem 9, if B is in L(Y) then there is a member G of INV-LIM-$\{H,Q,\pi\}$ such that, for each α in S^+ and t in R and ξ in Y, $G(\alpha)(t)\xi = \alpha(t)B\xi$.

THEOREM 11. *If α is in S^+ and g is a finitely additive function from R to Y and b is a nonnegative number, then the following two statements are equivalent:*

(1) *if t is in R then $|g(t)| \leqslant b\,\alpha(t)$, and*

(2) *g belongs to H_α and, for each f in H_α, $|Q_\alpha(g,f)| \leqslant b\|f\|$.*

PROOF. Suppose α is in S^+ and g is a finitely additive function from R to Y and b is a nonnegative number. If the statement (2) is true and t is in R then

$$|\langle g(t),\xi\rangle| = |Q_\alpha(g,P_t\alpha\cdot\xi)| \leqslant b\|P_t\alpha\cdot\xi\| = b\,\alpha(t)|\xi|$$

for each ξ in Y, so that $|g(t)| \leqslant b\,\alpha(t)$.

Suppose that if t is in R then $|g(t)| \leqslant b\,\alpha(t)$. It follows from Theorems 2 and 3 that g belongs to H_α and $N_\alpha(g)^2 \leqslant b^2 \int_{L/F}\alpha$. If f is in H_α then

$$\left|\Sigma_{u \text{ in } M}\frac{\langle g,f\rangle}{\alpha}(u)\right| \leqslant \Sigma_{u \text{ in } M} b|f(u)| \leqslant b\|f\|$$

for each M in F, so that $|Q_\alpha(g,f)| \leqslant b\|f\|$.

THEOREM 12. *The equations $\langle A(\lambda)(\alpha)(t),j\xi\rangle = \lambda(P_t\alpha\cdot\lambda)$, for λ in D and α in S^+ and t in R and ξ in Y, define a linear isomorphism A from the space D onto the subspace of* inv-lim-$\{H,Q,\pi\}$ *to which the point g of* inv-lim-$\{H,Q,\pi\}$ *belongs only in case there is a nonnegative number b such that, for each α in S^+ and t in R, $|g(\alpha)(t)| \leqslant b\,\alpha(t)$, in which case the norm of the member $A^{-1}(g)$ of D is the least such number b. If the ordered pair $\{\lambda,g\}$ belongs to A and f is in S then $\lambda(f)$ is an integral over L relatively to F in the following sense: for each α in S^+ such that f belongs to H_α, $\lambda(f) = \int_{L/F}\langle g(\alpha),Jf\rangle/\alpha$.*

PROOF. Suppose that b is the norm of the member λ of D. It is clear that the equations $\langle g(\alpha)(t),j\xi\rangle = \lambda(P_t\alpha\cdot\xi)$, for α in S^+ and t in R and ξ in Y, define a function g from S^+ to a set of finitely additive functions from R to Y, and that (for each such α, t, and ξ) $|\langle g(\alpha)(t),j\xi\rangle| \leqslant b\,\alpha(t)|\xi|$; by Theorem 11, if α is in S^+, $g(\alpha)$ belongs to H_α and $|Q_\alpha(g,f)| \leqslant b\|f\|$ for each f in H_α. If α is in S^+ and M is in F then, for each f in H_α,

$$\Sigma_{u \text{ in } M}\frac{\langle g(\alpha),Jf\rangle}{\alpha}(u) = \lambda(\Pi_\alpha(M)f)$$

so that, by Theorems 3 and 8, $\lambda(f) = Q_\alpha(g(\alpha),Jf)$. If α and β are members of S^+ such that H_α is a subset of H_β then, for each f in H_α,

$$Q_\alpha(g(\alpha),f) = \lambda(Jf) = Q_\beta(g(\beta),f)$$

so that g belongs to inv-lim-$\{H,Q,\pi\}$. All other allegations involved in Theorem 12 may be established by similar appeals to preceding Theorems, with the help of the fact that the H-image of S^+ fills up the space S (*cf.* Theorem 7, and remarks accompanying the initial description of the function H).

THEOREM 13. *Suppose each of* α *and* β *is in* S^+, $m_{\alpha\beta}$ *is the set to which* Γ *belongs only if case* Γ *is a function from* $R \times R$ *such that* (i) *if* t *is in* R *then each of* $\Gamma(\cdot,t)$ *and* $\Gamma(t,\cdot)$ *is a finitely additive function from* R *to* L(Y) *and* (ii) *there is a nonnegative number* b *such that if* M *is a member of* F *and each of* x *and* y *is a function from* M *to* Y *then*

$$|\Sigma_{\{u,v\} \text{ in } M \times M}\langle x(u),\Gamma(u,v)y(v)\rangle|^2 \leqslant b^2\Sigma_{u \text{ in } M}\alpha(u)\|x(u)\|^2\Sigma_{v \text{ in } M}\beta(v)\|y(v)\|^2,$$

and $T_{\alpha\beta}$ *is the set to which* B *belongs only in case* B *is a continuous linear transformation from* $\{H_\beta,Q_\beta\}$ *to* $\{H_\alpha,Q_\alpha\}$. *Then the equations*

$$\Phi(B)(u,v)\xi = B(P_v\beta\cdot\xi)(u), \textit{ for } B \textit{ in } T_{\alpha\beta} \textit{ and } \{u,v\} \textit{ in } R \times R \textit{ and } \xi \textit{ in } Y,$$

define a reversible linear transformation Φ *from* $T_{\alpha\beta}$ *onto* $m_{\alpha\beta}$, *such that if the ordered pair* $\{B,\Gamma\}$ *belongs to* Φ *then*

(1) *in order that the nonnegative number* b *should satisfy the condition (ii) it is necessary and sufficient that, for each member* f *of* H_β, $N_\alpha(Bf) \leqslant b\, N_\beta(f)$,

(2) *for each member* f *of* H_β, *and each* t *in* R *and* η *in* Y, *the function* $\Gamma(t,\cdot)^*\eta$ *belongs to* H_β *and* $\langle Bf(t),\eta\rangle = Q_\beta(f,\Gamma(t,\cdot)^*\eta)$, *and*

(3) *for each* f *in* H_β, Bf *is an integral over* L *relatively to* F *in the following sense: the function* h *from* R *to a set of functions from* R *to* Y, *such that if* t *is in* R *then* h(t) *is the constant* 0 *or the function* $\Gamma(\cdot,t)f(t)/\beta(t)$ *accordingly as* $\beta(t)$ *is the number* 0 *or not, maps* R *into* H_α *and* $Bf = \int_{L/F} h$ *with respect to* N_α.

PROOF. Suppose B is a member of $T_{\alpha\beta}$, and let k be the least nonnegative number b such that if g is in H_β then $N_\alpha(Bg) \leqslant b\, N_\beta(g)$. It is clear that there is a function Γ from $R \times R$ to L(Y) such that

$$\Gamma(u,v)\xi = B(P_v\beta\cdot\xi)(u) \text{ for each } \{u,v\} \text{ in } R \times R \text{ and } \xi \text{ in } Y,$$

and that if t is in R then each of $\Gamma(\cdot,t)$ and $\Gamma(t,\cdot)$ is finitely additive. If M is in F, each of x and y is a function from M to Y,

$$f = \Sigma_{u \text{ in } M} P_u \alpha \cdot x(u), \text{ and } g = \Sigma_{v \text{ in } M} P_v \beta \cdot y(v),$$

then $\{f,g\}$ is in $U_\alpha \times U_\beta$ (*cf.* Theorem 4) and it follows from Theorem 7 that

$$Q_\alpha(f,Bg) = \Sigma_{\{u,v\} \text{ in } M \times M} \langle x(u),\Gamma(u,v)y(v)\rangle,$$

$$N_\alpha(f)^2 = \Sigma_{u \text{ in } M} \alpha(u) \| x(u) \|^2, \text{ and}$$

$$N_\beta(g)^2 = \Sigma_{v \text{ in } M} \beta(v) \| y(v) \|^2:$$

hence, the condition (ii) is satisfied with b the number k, and Γ belongs to $m_{\alpha\beta}$. Now, (a) it follows from the pattern of argument indicated in the Proof of Theorem 4 that if $\{f,g\}$ is in $U_\alpha \times U_\beta$ then there is a member M of F, a function x from M to Y, and a function y from M to Y such that $\{f,g\}$ is determined by the foregoing formulas, and (b) by Theorem 7, U_α is dense in $\{H_\alpha,Q_\alpha\}$ and U_β is dense in $\{H_\beta,Q_\beta\}$: hence, k is the least nonnegative number b such that the condition (ii) holds. Let A denote the $\{Q_\alpha,Q_\beta\}$-adjoint of B, so that A is a continuous linear transformation from $\{H_\alpha,Q_\alpha\}$ to $\{H_\beta,Q_\beta\}$ and

$$Q_\alpha(f,Bg) = Q_\beta(Af,g) \text{ for each } \{f,g\} \text{ in } H_\alpha \times H_\beta.$$

If t is in R and η is in Y then by Theorem 7, for each u in R and ξ in Y,

$$\langle\Gamma(t,u)\xi,\eta\rangle = Q_\alpha(B(P_u\beta\cdot\xi),P_t\alpha\cdot\eta) = Q_\beta(P_u\beta\cdot\xi,A(P_t\alpha\cdot\eta)) = \langle\xi,A(P_t\alpha\cdot\eta)(u)\rangle,$$

so that $\Gamma(t,u)^*\eta = A(P_t\alpha\cdot\eta)(u)$; therefore, if t is in R and η is in Y, the function $\Gamma(t,\cdot)^*\eta$ belongs to H_β and, for each g in H_β,

$$\langle Bg(t),\eta\rangle = Q_\alpha(Bg,P_t\alpha\cdot\eta) = Q_\beta(g,\Gamma(t,\cdot)^*\eta) = \int_{L/F} \langle\Gamma(t,\cdot)g,\eta\rangle/\beta.$$

Suppose, now, that Γ belongs to $m_{\alpha\beta}$ and that k is the least nonnegative number b such that the condition (ii) holds. In consequence of Theorems 2, 4, and 7, the equations

$$B_0(\Sigma_{v \text{ in } M} P_v\beta\cdot x(v)) = \Sigma_{v \text{ in } M} \Gamma(\cdot,v)x(v),$$

for members M of F and functions x from M to Y, define a linear transformation B_0 from U_β into H_α such that $N_\alpha(B_0 g) \leqslant k\, N_\beta(g)$ for each g in U_β. Inasmuch as U_β is dense in $\{H_\beta,Q_\beta\}$ (by Theorem 7), it follows that there is only one member B of $T_{\alpha\beta}$ of which B_0 is a subset, and that if f is in H_β then $N_\alpha(Bf) \leqslant k\, N_\beta(f)$.

The foregoing arguments suffice to establish all but assertion (3) of this Theorem;

(3) is a consequence of Theorem 8, since $\Sigma_{t \text{ in } M} h(t) = B(\Pi_\beta(M)f)$ for f in H_β and h the indicated function from R (to H_α) and M in the family F.

THEOREM 14. *Suppose each of* α *and* β *is in* S^+, $m_{\alpha\beta}(P)$ *is the set to which* G *belongs only in case* (i) G *is a finitely additive function from* R *to* L(Y) *and* (ii) *there is a nonnegative number* b *such that*

$$|\langle\xi, G(t)\eta\rangle|^2 \leqslant b^2\alpha(t)\beta(t)\|\xi\|^2\|\eta\|^2 \textit{ for each } t \textit{ in } R \textit{ and } \{\xi,\eta\} \textit{ in } Y \times Y,$$

and $T_{\alpha\beta}(P)$ *is the set to which* B *belongs only in case* B *is a continuous linear transformation from the space* $\{H_\beta, Q_\beta\}$ *to the space* $\{H_\alpha, Q_\alpha\}$ *such that, for each* t *in* R *and* f *in* H_β, $B(P_t f) = P_t(Bf)$. *Then the equations*

$$\Psi(B)(t)\xi = B(\beta\cdot\xi)(t), \textit{ for } B \textit{ in } T_{\alpha\beta}(P) \textit{ and } t \textit{ in } R \textit{ and } \xi \textit{ in } Y,$$

define a reversible linear transformation Ψ *from* $T_{\alpha\beta}(P)$ *onto* $m_{\alpha\beta}(P)$ *such that, if the ordered pair* $\{B,G\}$ *belongs to* Ψ *then*

(1) *in order that the nonnegative number* b *should satisfy the condition (ii) it is necessary and sufficient that, for each member* f *of* H_β, $N_\alpha(Bf) \leqslant b\, N_\beta(f)$,

(2) *for each member* f *of* H_β, *and each* t *in* R *and* η *in* Y, *the function* $G^*\eta$ *belongs to* H_β *and* $\langle Bf(t),\eta\rangle = Q_\beta(P_t f, G^*\eta)$, *and*

(3) *in case* α *is* β, *in order that the nonnegative number* b *should satisfy the condition (ii) it is necessary and sufficient that, if* f *is in* H_β, $\|Bf\| \leqslant b\,\|f\|$.

PROOF. Suppose B is a member of $T_{\alpha\beta}(P)$, and let k be the least nonnegative number b such that if f is in H_β then $N_\alpha(Bf) \leqslant b\, N_\beta(f)$. It follows from Theorem 3 that if ξ is in Y then $\beta\cdot\xi$ is in H_β and $N_\beta(\beta\cdot\xi)^2 = \|\xi\|^2\int_{L/F}\beta$: hence, there is a finitely additive function G from R to L(Y) such that

$$G(t)\xi = B(\beta\cdot\xi)(t) \text{ for each } t \text{ in } R \text{ and } \xi \text{ in } Y,$$

and that if ξ is in Y then the function $G\cdot\xi$ belongs to H_α. With Φ the function as described in Theorem 13, let $\Gamma = \Phi(B)$: if $\{u,v\}$ is in $R \times R$ and ξ is in Y,

$$\Gamma(u,v)\xi = B(P_v\beta\cdot\xi)(u) = P_v(B(\beta\cdot\xi))(u) = P_v(G\cdot\xi)(u);$$

hence, if M is in F and each of x and y is a function from M to Y,

$$\Sigma_{\{u,v\} \text{ in } M \times M}\langle x(u),\Gamma(u,v)y(v)\rangle = \Sigma_{u \text{ in } M}\langle x(u),G(u)y(u)\rangle,$$

so that, by Theorem 13, the condition (ii) of the present Theorem is satisfied with b the number k. Therefore G belongs to $m_{\alpha\beta}(P)$. From the implicit symmetry of the aforementioned condition (ii), if t is in R and η is in Y then the function $G^*\eta$ belongs to H_β and $\Gamma(t,\cdot)^*\eta = P_t(G^*\eta)$: the assertion (2) follows from the assertion (4) of Theorem 6 and the assertion (2) of Theorem 13.

Suppose, now, that G belongs to $m_{\alpha\beta}(P)$ and that k is the least nonnegative number b such that the condition (ii) holds. If η is in Y then, for each t in R,

$$\|G(t)\eta\|^2 \leqslant k^2\alpha(t)\beta(t)\|\eta\|^2 \text{ and } \|G(t)^*\eta\|^2 \leqslant k^2\alpha(t)\beta(t)\|\eta\|^2,$$

so that the function $G\cdot\eta$ belongs to H_α and the function $G^*\eta$ belongs to H_β: hence, there is a function Γ from $R \times R$ to $L(Y)$ such that

$$\Gamma(u,v)\xi = P_v(G\cdot\xi)(u) \text{ for each } \{u,v\} \text{ in } R \times R \text{ and } \xi \text{ in } Y.$$

It is clear that if t is in R then eqch of $\Gamma(\cdot,t)$ and $\Gamma(t,\cdot)$ is finitely additive and, for each η in Y, $\Gamma(t,\cdot)^*\eta = P_t(G^*\eta)$. Moreover, if M is in F and each of x and y is a function from M to Y, then

$$\begin{aligned}&|\Sigma_{\{u,v\} \text{ in } M \times M}\langle x(u),\Gamma(u,v)y(v)\rangle|^2\\ &= |\Sigma_{u \text{ in } M}\langle x(u),G(u)y(u)\rangle|^2\\ &\leqslant k^2\Sigma_{u \text{ in } M}\alpha(u)\|x(u)\|^2\Sigma_{v \text{ in } M}\beta(v)\|y(v)\|^2,\end{aligned}$$

so that the condition (ii) of Theorem 13 is satisfied with b the number k; hence, the function Γ belongs to the set $m_{\alpha\beta}$. With Φ the function as described in Theorem 13, let $B = \Phi^{-1}(\Gamma)$. If f is in H_β then, by Theorems 6 and 13,

$$N_\alpha(Bf) \leqslant k\, N_\beta(f) \text{ and } \langle Bf(t),\eta\rangle = Q_\beta(P_tf,G^*\eta) \text{ for t in R and } \eta \text{ in Y:}$$

it follows that, if f is in H_β and u is in R, $B(P_uf) = P_u(Bf)$ so that B belongs to $T_{\alpha\beta}(P)$. If M is in F and x is a function from M to Y then

$$B(\Sigma_{u \text{ in } M}P_u\beta\cdot x(u)) = \Sigma_{u \text{ in } M}P_u(G\cdot x(u)):$$

thus, the reversibility of Ψ follows from the density of U_β in the space $\{H_\beta,Q_\beta\}$.

Suppose, finally, that α is β and the ordered pair $\{B,G\}$ belongs to Ψ. If b is a nonnegative number such that the condition (ii) holds then, for each f in H_β and t in R and η in Y, $\|G(t)^*\eta\| \leqslant b\,\beta(t)\|\eta\|$ so that, by Theorem 11,

$$|\langle Bf(t),\eta\rangle| = |Q_\beta(P_tf,G^*\eta)| \leqslant b\|P_tf\|\,\|\eta\|,$$

whence $|||Bf(t)||| \leqslant b\|P_t f\|$: therefore $\|Bf\| \leqslant b\|f\|$. Suppose, then, that b is a nonnegative number such that if f is in H_β then $\|Bf\| \leqslant b\|f\|$. If t is in R and η is in Y then

$$|||G(t)\eta||| = |||B(\beta\cdot\eta)(t)||| \leqslant \|P_t(B(\beta\cdot\eta))\| = \|B(P_t\beta\cdot\eta)\| \leqslant b\|P_t\beta\cdot\eta\| = b\,\beta(t)|||\eta|||,$$

so that the number b satisfies the condition (ii). This completes the proof.

THEOREM 15. *Suppose β is in S^+, and the sets $m_{\beta\beta}(P)$, $T_{\beta\beta}(P)$, and Ψ are as described in Theorem 14 (with $\alpha = \beta$). The following statements are true:*

(1) *if $\{B,G\}$ belongs to Ψ and A is the adjoint of B with respect to Q_β, so that $Q_\beta(f,Ag) = Q_\beta(Bf,g)$ for each $\{f,g\}$ in $H_\beta \times H_\beta$, then $\{A,G^*\}$ belongs to Ψ,*

(2) *if α is a member of S^+ such that H_α is a subset of H_β and G is in $m_{\beta\beta}(P)$ and K is a function from R to L(Y) such that $K\cdot\eta = \pi(\alpha,\beta)(G\cdot\eta)$ for each η in Y, then K belongs to the set $m_{\alpha\alpha}(P)$ and $K^*\xi = \pi(\alpha,\beta)(G^*\xi)$ for each ξ in Y, and*

(3) *if each of $\{B_1,G'\}$ and $\{B_2,G''\}$ belongs to Ψ and $G = \Psi(B_1B_2)$ then, for each t in R and η in Y, $G(t)\eta = \int_{t/F} G'G''\eta/\beta$ with respect to the norm $|||\cdot|||$.*

PROOF. Suppose $\{B,G\}$ belongs to Ψ and A is the adjoint of B with respect to Q_β: if t is in R and $\{f,g\}$ is in $H_\beta \times H_\beta$ then, as justified by Theorem 6,

$$Q_\beta(A(P_tf),g) = Q_\beta(P_tf,Bg) = Q_\beta(f,P_t(Bg))$$
$$= Q_\beta(f,B(P_tg)) = Q_\beta(Af,P_tg) = Q_\beta(P_t(Af),g).$$

Therefore A belongs to $T_{\beta\beta}(P)$. If t is in R and $\{\xi,\eta\}$ is in $Y \times Y$ then, with computations justified by Theorems 6 and 7,

$$\langle (G^*\xi)(t),\eta\rangle = \langle G(t)^*\xi,\eta\rangle = \langle \xi,G(t)\eta\rangle = \langle \xi,B(\beta\cdot\eta)(t)\rangle$$
$$= Q_\beta(P_t\beta\cdot\xi,B(\beta\cdot\eta)) = Q_\beta(A(P_t\beta\cdot\xi),\beta\cdot\eta)$$
$$= Q_\beta(P_t(A(\beta\cdot\xi)),\beta\cdot\eta) = Q_\beta(A(\beta\cdot\xi),P_t\beta\cdot\eta) = \langle A(\beta\cdot\xi)(t),\eta\rangle.$$

Hence, for each ξ in Y, $G^*\xi = A(\beta\cdot\xi)$ so that $\{A,G^*\}$ belongs to Ψ.

Suppose α is a member of S^+ such that H_α is a subset of H_β, G is in $m_{\beta\beta}(P)$, and K is a function from R to L(Y) such that $K\cdot\eta = \pi(\alpha,\beta)(G\cdot\eta)$ for each η in Y. Let b be a nonnegative number such that (*cf.* Theorem 14)

$$|||G(t)\eta||| \leqslant b\,\beta(t)|||\eta||| \text{ for each t in R and } \eta \text{ in Y}.$$

If t is in R and $\{\xi,\eta\}$ is in $Y \times Y$ then by Theorems 6 and 9

$$\langle\xi,K(t)\eta\rangle = \langle\xi,\pi(\alpha,\beta)(G\cdot\eta)(t)\rangle = Q_\beta(P_t\alpha\cdot\xi,G\cdot\eta),$$

so that, by Theorem 11,

$$|\langle\xi,K(t)\eta\rangle| \leqslant \|P_t\alpha\cdot\xi\|b\|\eta\| = b\,\alpha(t)\|\xi\|\,\|\eta\|.$$

Therefore, the function K (clearly finitely additive) belongs to $m_{\alpha\alpha}(P)$. Now, if t is in R and $\{\xi,\eta\}$ is in $Y \times Y$ then, from the formulas in Theorem 9,

$$\langle(K^*\xi)(t),\eta\rangle = \langle\xi,K(t)\eta\rangle = \int_{t/F}\langle\alpha\cdot\xi,G\cdot\eta\rangle/\beta = \int_{t/F}\langle G^*\xi,\alpha\cdot\eta\rangle/\beta,$$

whence $K^*\xi = \pi(\alpha,\beta)(G^*\xi)$.

Finally, the assertion (3) is justified by Theorems 13 and 14, with the help of the fact that if t is in R and f is in H_β then $\|f(t)\| \leqslant \beta(t)^{1/2}N_\beta(f)$.

THEOREM 16. *Suppose that* B *is a linear transformation from* S *into* S_0 *which is continuous with respect to the norm* $\|\cdot\|$, *and that if* t *is in* R *and* f *is in* S *then* $B(P_tf) = P_t(Bf)$. *Then* (i) *if* α *is in* S^+ *then* B *maps* H_α *into* H_α, *so that* B *maps* S *into* S, *and* (ii) *if* b *is a nonnegative number then the following two conditions are equivalent:*

(1) *if* f *is in* S *then* $\|Bf\| \leqslant b\|f\|$, *and*

(2) *if* α *is in* S^+ *and* f *is in* H_α *then* $N_\alpha(Bf) \leqslant b\,N_\alpha(f)$.

PROOF. Suppose that b is a nonnegative number such that if f is in S then $\|Bf\| \leqslant b\|f\|$. It may be noted that, if t is in R and f is in S,

$$\|P_tBf\| = \|BP_tf\| \leqslant b\,\|P_tf\|.$$

Suppose, now, that α is in S^+ and f is a member of H_α: by repeated application of Theorems 2, 3, and 5, it follows that if t is in R then

$$\|Bf(t)\|^2 \leqslant \|P_tBf\|^2 = \|BP_tf\|^2 \leqslant b^2\|P_tf\|^2 \leqslant b^2\alpha(t)\,N_\alpha(P_tf)^2,$$

and, from this, that Bf belongs to H_α and $N_\alpha(Bf) \leqslant b\,N_\alpha(f)$. Therefore, if α is in S^+ then B maps H_α into H_α and condition (1) implies condition (2). That (1) is implied by (2), is a consequence of the terminal assertion in Theorem 14, and the fact that if f is in S then, for some α in S^+, f belongs to the set H_α.

THEOREM 17. *The equations* $(\zeta(\mu)f)(t) = \mu(P_tf)$, *for* μ *in* E *and* f *in* S *and* t *in* R, *define a linear isomorphism* ζ *from the space* E *onto the collection of all continuous linear transformations* B *in* $\{S,\|\cdot\|\}$ *with the property that if* t *is in* R *and* f *is in* S *then* $B(P_t f) = P_t(Bf)$. *If the ordered pair* $\{\mu, B\}$ *belongs to* ζ *and* b *is a nonnegative number, then* (i) *the following three conditions are equivalent:*

(1) *if* f *is in* S *then* $|\mu(f)| \leqslant b\|f\|$,

(2) *if* f *is in* S *then* $\|Bf\| \leqslant b\|f\|$, *and*

(3) *if* α *is in* S^+ *and* f *is in* H_α *then* $N_\alpha(Bf) \leqslant b\, N_\alpha(f)$,

and (ii) *for each* f *in* S, $\mu(f) = \int_{L/F} Bf$ *with respect to the norm* $|\cdot|$.

PROOF. Suppose μ is in E and b is a nonnegative number such that condition (1) holds. Clearly there is a linear transformation B from S such that if f is in S then Bf is a finitely additive function from R to Y and

$$Bf(t) = \mu(P_t f) \text{ for each } t \text{ in } R.$$

If f is in S then, for each member M of the family F,

$$\Sigma_{t \text{ in } M} |Bf(t)| \leqslant b\, \Sigma_{t \text{ in } M} \|P_t f\|$$

so that Bf is in S_0 and $\|Bf\| \leqslant b\|f\|$; if f is in S and each of t and v is in R,

$$P_t(Bf)(v) = \mu(P_t P_v f) = \mu(P_v P_t f) = B(P_t f)(v)$$

so that $B(P_t f) = P_t(Bf)$. By Theorem 16, B maps S into S, if α is in S^+ then B maps H_α into H_α, and conditions (2) and (3) hold.

Suppose, now, that B is a continuous linear transformation in $\{S, \|\cdot\|\}$ with the property that if t is in R and f is in S then $B(P_t f) = P_t(Bf)$. By Theorem 16, if α is in S^+ then B maps H_α into H_α and, for each nonnegative number b, the conditions (2) and (3) are equivalent. Let b be a nonnegative number such that condition (2) holds. Clearly there is a linear transformation μ, from S to Y, such that if f is in S then

$$\mu(f) = \int_{L/F} Bf \text{ with respect to the norm } |\cdot|.$$

If f is in S then, for each member M of the family F,

$$|\Sigma_{t \text{ in } M} Bf(t)| \leqslant \Sigma_{t \text{ in } M} |Bf(t)| \leqslant \|Bf\| \leqslant b\|f\|,$$

so that $|\mu(f)| \leqslant b\|f\|$. Hence, μ is in E and, for each $\{t, f\}$ in $R \times S$,

$$\mu(P_t f) = \int_{L/F} B(P_t f) = \int_{L/F} P_t(Bf) = \int_{t/F} Bf = Bf(t).$$

The foregoing arguments suffice to establish Theorem 17.

THEOREM 18. *The equations* $\omega(\mu)(\alpha)(t)\xi = \mu(P_t \alpha \cdot \xi)$, *for* μ *in* E *and* α *in* S^+ *and* t *in* R *and* ξ *in* Y, *define a linear isomorphism* ω *from* E *onto the subspace of* INV-LIM-$\{H,Q,\pi\}$ *to which the point* G *of* INV-LIM-$\{H,Q,\pi\}$ *belongs only in case there is a nonnegative number* b *such that, for each* α *in* S^+ *and* t *in* R *and* ξ *in* Y, $[\![G(\alpha)(t)\xi]\!] \leqslant b\,\alpha(t)[\![\xi]\!]$, *in which case the norm of the member* $\omega^{-1}(G)$ *of* E *is the least such number* b. *If the ordered pair* $\{\mu,G\}$ *belongs to* ω *and* f *is in* S *then* $\mu(f)$ *is an integral over* L *relatively to* F *in the following sense: for each* α *in* S^+ *such that* f *belongs to* H_α, $\mu(f) = \int_{L/F} G(\alpha)f/\alpha$ *with respect to* $[\![\cdot]\!]$.

Theorem 18 may be proved as a consequence of Theorems 14, 15, 16, and 17, with the help of Theorem 8: regarding the nature of the integral representation, one lets $\frac{G(\alpha)f}{\alpha}(t)$ denote 0 or $G(\alpha)(t)f(t)/\alpha(t)$ accordingly as $\alpha(t)$ is 0 or not.

In the next two Theorems, it is supposed that $\{X,|\cdot|\}$ is a linear normed complete space of functions from a set R_0 into Y such that, if s is in R_0, there is a positive number p such that, for every member g of X, $[\![g(s)]\!] \leqslant p|g|$. The linear space C(S,X), of all continuous linear transformations from $\{S,\|\cdot\|\}$ to the space $\{X,|\cdot|\}$, is normed in the usual manner: the norm of the member B of C(S,X) is the least nonnegative number b such that if f is in S then $|Bf| \leqslant b\|f\|$.

THEOREM 19. *Suppose* α *is in* S^+, $m_\alpha(X)$ *is the set to which* Γ *belongs only in case* Γ *is a function from* $R_0 \times R$ *to* L(Y) *such that* (i) *if* $\{t,\eta\}$ *is in* $R \times Y$ *then* $\Gamma(\cdot,t)\eta$ *is in* X, (ii) *if* u *is in* R_0 *then* $\Gamma(u,\cdot)$ *is finitely additive, and* (iii) *there is a nonnegative number* b *such that if* M *is a member of the family* F *and* x *is a function from* M *to* Y *then*

$$|\Sigma_{t \text{ in } M}\Gamma(\cdot,t)x(t)| \leqslant b\, \Sigma_{t \text{ in } M}\alpha(t)[\![x(t)]\!],$$

and $T_\alpha(X)$ *is the set to which* B *belongs only in case* B *is a linear transformation from* H_α *to* X *and there is a nonnegative number* b *such that, for each member* f *of* H_α, $|Bf| \leqslant b\|f\|$. *Then the equations*

$$Z_\alpha(B)(u,t)\eta = B(P_t\alpha\cdot\eta)(u), \textit{for } B \textit{ in } T_\alpha(X) \textit{ and } \{u,t\} \textit{ in } R_0 \times R \textit{ and } \eta \textit{ in } Y,$$

define a reversible linear transformation Z_α *from* $T_\alpha(X)$ *onto* $m_\alpha(X)$, *such that if the ordered pair* $\{B,\Gamma\}$ *belongs to* Z_α *then*

(1) *in order that the nonnegative number* b *should satisfy the condition (iii) it is necessary and sufficient that, for each member* f *of* H_α, $|Bf| \leqslant b\|f\|$,

(2) *for each member* f *of* H_α, *and each* u *in* R_0 *and* ξ *in* Y, *the function* $\Gamma(u,\cdot)^*\xi$

belongs to H_α *and* $\langle Bf(u),\xi\rangle = Q_\alpha(f,\Gamma(u,\cdot)^*\xi)$, *and*

(3) *for each* f *in* H_α, Bf *is an integral over* L *relatively to* F *in the following sense: if* h *is the function from* R *to* X *such that, for each member* t *of* R, h(t) *is the constant* 0 *or* $\Gamma(\cdot,t)f(t)/\alpha(t)$ *accordingly as* $\alpha(t)$ *is the number* 0 *or not, then* $Bf = \int_{L/F} h$ *with respect to the norm* $|\cdot|$.

PROOF. Suppose B is a member of $T_\alpha(X)$, and let k be the least nonnegative number b such that if f is in H_α then $|Bf| \leqslant b\|f\|$. It is clear that there is a function Γ from $R_0 \times R$ to L(Y) such that

$$\Gamma(u,v)\eta = B(P_t\alpha\cdot\eta)(u) \text{ for each } \{u,t\} \text{ in } R_0 \times R \text{ and } \eta \text{ in } Y,$$

and that, for each such $\{u,t\}$ and η, $\Gamma(u,\cdot)$ is finitely additive and $\Gamma(\cdot,t)\eta$ is in X. If M is in F and x is a function from M to Y and $f = \Sigma_{t \text{ in } M} P_t\alpha\cdot x(t)$ then f is in H_α and $\|f\| = \Sigma_{t \text{ in } M}\alpha(t)\|x(t)\|$: hence the condition (iii) is satisfied with b the number k, and Γ belongs to $m_\alpha(X)$. Moreover, if u is in R_0 then there is a positive number p such that, for every t in R and η in Y,

$$\|\Gamma(u,t)\eta\| \leqslant p|\Gamma(\cdot,t)\eta| \leqslant p\,k\,\alpha(t)\|\eta\|, \text{ so that}$$

$$\|\Gamma(u,t)^*\xi\| \leqslant p\,k\,\alpha(t)\|\xi\| \text{ for every } \xi \text{ in } Y:$$

hence, if $\{u,t\}$ is in $R_0 \times R$ and $\{\xi,\eta\}$ is in $Y \times Y$, $\Gamma(u,\cdot)^*\xi$ is in H_α and

$$\langle B(P_t\alpha\cdot\eta),\xi\rangle = \langle\Gamma(u,t)\eta,\xi\rangle = \langle\eta,\Gamma(u,t)^*\xi\rangle = Q_\alpha(P_t\alpha\cdot\eta,\Gamma(u,\cdot)^*\xi).$$

Assertion (2) follows since, by assertions (3) of Theorem 3 and (2) of Theorem 7, the family U_α is dense in H_α with respect to the norm $\|\cdot\|$.

Suppose, now, that Γ belongs to $m_\alpha(X)$ and that k is the least nonnegative number b such that the condition (iii) holds. It follows that the equations

$$B_0(\Sigma_{t \text{ in } M} P_t\alpha\cdot x(t)) = \Sigma_{t \text{ in } M}\Gamma(\cdot,t)x(t),$$

for members M of F and functions x from M to Y, define a linear transformation B_0 from U_α into X such that $|B_0 f| \leqslant k\|f\|$ for each f in U_α. By the density of U_α in H_α with respect to $\|\cdot\|$, as noted in the preceding paragraph, there is only one member B of $T_\alpha(X)$ of which B_0 is a subset and, if f is in H_α, $|Bf| \leqslant k\|f\|$.

As in the Proof of Theorem 13, the foregoing arguments suffice to establish all but assertion (3) of this Theorem; (3) is again a consequence of Theorem 8.

THEOREM 20. *The equations* $\Omega(B(\alpha)(u,t)\eta = B(P_t\alpha\cdot\eta)(u)$, *for* B *in* C(S,X) *and* α *in* S^+ *and* $\{u,t\}$ *in* $R_0 \times R$ *and* η *in* Y, *define a linear isomorphism* Ω *from* C(S,X) *onto the set to which* Γ *belongs only in case* Γ *is a function from* S^+ *to a set of functions from* $R_0 \times R$ *to* L(Y) *such that* (i) *if* α *is in* S^+ *and* t *is in* R *and* η *is in* Y *then* $\Gamma(\alpha)(\cdot,t)\eta$ *belongs to* X, (ii) *if* u *is in* R_0 *then there is a member* G *of* INV-LIM-$\{H,Q,\pi\}$ *such that, for each* α *in* S^+, $G(\alpha) = \Gamma(\alpha)(u,\cdot)$, *and* (iii) *there is a nonnegative number* b *such that, for each member* M *of* F *and each function* x *from* M *to* Y,

$$|\Sigma_{t \text{ in } M}\Gamma(\alpha)(\cdot,t)x(t)| \leqslant b\Sigma_{t \text{ in } M}\alpha(t)\|x(t)\|,$$

in which case the norm of the member $\Omega^{-1}(\Gamma)$ *of* C(S,X) *is the least such number* b. *If the ordered pair* $\{B,\Gamma\}$ *belongs to* Ω *and* f *is in* S *then* Bf *is an integral over* L *relatively to* F *in the following sense: for each member* α *of* S^+ *such that* f *belongs to* H_α, *if* h *is the function from* R *to* X *such that if* t *is in* R *then* h(t) *is the constant* 0 *in* X *or* $\Gamma(\alpha)(\cdot,t)f(t)/\alpha(t)$ *accordingly as* $\alpha(t)$ *is the number* 0 *or not,* $Bf = \int_{L/F} h$ *with respect to the norm* $|\cdot|$.

Theorem 20 may be proved as a consequence of Theorem 19 - with the help of assertion (2) of Theorem 15, the type of argument given in the first paragraph of the Proof of Theorem 19, and the fact that the H-image of S^+ fills up S.

THEOREM 21. *If the ordered pair* $\{B,\Gamma\}$ *belongs to the isomorphism* Ω, *defined in Theorem 20, then the following two statements are equivalent:*

(1) *if* f *is in* S *then* $|Bf| = \|f\|$, *and*

(2) *if* α *is in* S^+ *and* M *is in* F *and* x *is a function from* M *to* Y *then*

$$|\Sigma_{t \text{ in } M}\Gamma(\alpha)(\cdot,t)x(t)| = \Sigma_{t \text{ in } M}\alpha(t)\|x(t)\|.$$

PROOF. If (1) is true then, for each such α and M and x as indicated,

$$|B(\Sigma_{t \text{ in } M}P_t\alpha\cdot x(t))| = \|\Sigma_{t \text{ in } M}P_t\alpha\cdot x(t)\| = \Sigma_{t \text{ in } M}\alpha(t)\|x(t)\|$$

whence (2) is true. Suppose that (2) is true. For each α in S^+ and each g in the family U_α, $|Bg| = \|g\|$: hence (1) is true, since the H-image of S^+ fills up S and, for each α in S^+, the family U_α is dense in H_α with respect to $\|\cdot\|$.

THEOREM 22. *Suppose that* (1) *if* μ *is in* E *then* $n(\mu)$ *denotes the norm of* μ, (2) $\{X,|\cdot|\}$ *is the normed linear space of all continuous linear transformations from*

$\{E,n\}$ *to* $\{Y,\lVert\cdot\rVert\}$, *and* (3) σ *is the member of* $C(S,X)$ *given by* $\sigma(f)\mu = \mu(f)$, *for* f *in* S *and* μ *in* E. *Then* σ *is an isometry: if* f *is in* S *then* $|\sigma(f)| = \|f\|$.

PROOF. It should be noted that $\{X,|\cdot|\}$ is an example of the type of space indicated in the Central Problem (and in Theorems 19 and 20): for each nonzero member μ of E, $n(\mu)$ is a positive number p such that $\lVert\phi(\mu)\rVert \leqslant p|\phi|$ for every member ϕ of X. Therefore the notation $C(S,X)$ is appropriate in this context.

Moreover, it is clear that the indicated transformation σ belongs to $C(S,X)$ and that if f is in S then $|\sigma(f)| \leqslant \|f\|$, since $\lVert\mu(f)\rVert \leqslant \|f\|n(\mu)$ for μ in E.

Suppose, now, that f is a nonzero member of S. To know that $|\sigma(f)| = \|f\|$, it will suffice to have a member μ of E such that $n(\mu) = 1$ and $\lVert\mu(f)\rVert = \|f\|$. In accordance with the Hahn-Banach extension theorem (or the Bohnenblust-Sobczyk version thereof in the case of complex scalars [3, page 86]), there is a member λ of D such that $\lambda(f) = \|f\|$ and $|\lambda(g)| \leqslant \|g\|$ for every g in S. Let ξ be a member of Y such that $\lVert\xi\rVert = 1$: the equations $\mu(g) = \lambda(g)\xi$, for g in S, define a member μ of E with the indicated property. This completes the Proof.

REMARK 1. By Theorem 12, a second description of the norm of a member λ of the space D is: the least nonnegative number b such that if α is in S^+ and t is in R then $\lVert A(\lambda)(\alpha)(t)\rVert \leqslant b\alpha(t)$. In accordance with Theorems 17 and 18, there are four descriptions of the norm n for the space E: if μ is in E then $n(\mu)$ is the least nonnegative number b such that

(1) if f is in S then $\lVert\mu(f)\rVert \leqslant b\|f\|$.

(2) if f is in S then $\|\zeta(\mu)f\| \leqslant b\|f\|$.

(3) if α is in S^+ and f is in H_α then $N_\alpha(\zeta(\mu)f) \leqslant bN_\alpha(f)$.

(4) if α is in S^+ and t is in R and ξ is in Y then $\lVert\omega(\mu)(\alpha)(t)\xi\rVert \leqslant b\,\alpha(t)\lVert\xi\rVert$.

Variants of these descriptions are available from the observation (*cf.* Theorem 8) that if f is in S then, for each M in F and α in S^+ such that f belongs to H_α,

$$\|\Pi_\alpha(M)f\| = \Sigma_{t \text{ in } M}\lVert f(t)\rVert.$$

REMARK 2. If f is in S and M is in F then $\|\Sigma_{t \text{ in } M}P_t f\| = \Sigma_{t \text{ in } M}\|P_t f\|$ - whereas, for each α in S^+ such that f is in H_α, (*cf.* Theorem 6)

$$N_\alpha(\Sigma_{t \text{ in } M}P_t f)^2 = \Sigma_{t \text{ in } M}N_\alpha(P_t f)^2.$$

This seeming anomaly may be resolved by showing that if f is in S then Vf is a member δ of S^+ such that, for each t in R, $\|P_t f\| = N_\delta(P_t f)^2$: a proof may be based on such a system of inequalities as is indicated in the Proof of Theorem 1.

Modification of the Initial Supposition. Henceforth, instead of supposing that R is a pre-ring of subsets of the set L filling up L, it is supposed only that the following Axiom holds.

SUBDIVISION AXIOM. *The collection* R *of subsets of the set* L *fills up* L *and, if* G *is a finite subcollection of* R, *there is a subcollection* M *of* R *such that*

(i) *if* X *is a finite subcollection of* M *then no member of* R *which is covered by* X *lies in any member of* M *which does not belong to* X, *and*

(ii) *each set in the collection* G *is filled up by a finite subcollection of* M.

It has been shown earlier [15] that this Axiom is a necessary and sufficient condition on the collection R (relatively to its additve extension) for there to exist a function γ from R onto a pre-ring of subsets of some set such that, if u is a member of R and G is a finite subcollection of R, u is covered by G only in case $\gamma(u)$ is covered by the γ-image of G. The following definitions have been introduced [15]: (1) the subcollection M of R is *nonoverlapping relatively to* R provided that condition (i) of the Subdivision Axiom holds, and (2) the function f from R to an additive Abelian semigroup is R-*additive* provided that if M is a finite subcollection of R which is nonoverlapping relatively to R and M fills up the member u of R then $\Sigma_{t \text{ in } M} f(t) = f(u)$. If γ is a function from R, of the type indicated earlier in this paragraph then, inasmuch as [15; Theorem 0] the γ-images of those nondegenerate subcollections of R which are nonoverlapping relatively to R *are* the collections of mutually exclusive members of the γ-image of R, it is clear that γ and γ^{-1} provide for a translation of all the results from the preceding sections of this report to the present context. A more direct transition is available here: let the letter F now stand for the family of all finite subcollections M of R such that M is nonoverlapping relatively to R, and let "R-additive" replace "finitely additive" everywhere the latter has appeared.

Only one more change need be made, this in the definition of the function P, in order to validate the resulting body of propositions: P is now a function from R such that, for each t in R, P_t is a function from S_0 to S_0 such that if f is in S_0 then, for each u in R, $P_t f(u)$ is 0 or $\Sigma_{v \text{ in } M} f(v)$ accordingly as (i) no member of R lies both in u and in t or (ii) M is a member of F such that each set in M lies both in u and in t and each set in R which lies both in u and in t is covered by M. This change is sufficient: by [15, Theorem 10], if u and t are sets in R such that some set in R lies in both of them and W is a member of F of which some subcollection fills up u and some subcollection fills up t, there is a subcollection M of W such that each set in M lies in both u and t and each set in R which lies in both u and t is covered by M. The change is necessary: there may be sets u and t in R such that some set in R lies in both of them but there is no subcollection of R which fills up the common part of u and t [15, Example 4].

REMARK. The primitive instance of the Subdivision Axiom is the case that L is the real line and R is the collection of all (closed and bounded) intervals of real numbers. Another instance, one where the existence of a function γ (of the type indicated) from R onto a pre-ring of subsets of some set is perhaps somewhat less obvious, is the case that L is the ordinary Euclidean plane and R is made up of all subsets t of L such that t consists of a triangle plus its interior.

Continuous and Quasi-Continuous Functions. Suppose, for the purposes of illustration in this section, that L belongs to the collection R and $\{X,|\cdot|\}$ is the usual normed algebra of all continuous linear transformations in $\{Y,\langle\cdot,\cdot\rangle\}$: if the member k of L(Y) belongs to X, $|k|$ is the least nonnegative number b such that if ξ is in Y then $\|k\xi\| \leqslant b\|\xi\|$. If ϕ is a function from L to a bounded subset of $\{X,|\cdot|\}$ and t is in R then $|\phi|_t$ denotes the least upper bound of $|\phi(p)|$ for p in t; there are the implicit multiplication and involution, in the class of such functions ϕ, as determined by the equations

$$(\phi_1\phi_2)(p)\xi = \phi_1(p)\phi_2(p)\xi \text{ and } \phi'(p) = \phi(p)^*, \text{ for p in L and } \xi \text{ in Y},$$

as well as the customary addition among functions from a set L to a linear space.

Let A(R,X) denote the set of all function ϕ from L to X such that if $\epsilon > 0$ then there is a member M of F filling up L such that, if t is in M and both p and q belong to t, $|\phi(p)\text{-}\phi(q)| < \epsilon$. Let B(R,X) denote the closure, with respect to $|\cdot|_L$, of the set of all finite linear combinations (with coefficients from X) of characteristic functions of sets in R. It is clear that A(R,X) is an involution-algebra, that B(R,X) is a linear space, and that each of A(R,X) and B(R,X) is complete with respect to the norm $|\cdot|_L$. It can be shown that if R is a pre-ring of subsets of L then B(R,X) is A(R,X); it can happen, however, that B(R,X) is an algebra of which A(R,X) is a proper subalgebra. Consider the following Example.

EXAMPLE 1. Let L be the unit interval [0,1] and R be the collection of all subintervals of [0,1]: A(R,X) is the set of all continuous functions from L to X, and B(R,X) is the set of all quasi-continuous functions from L to X which are continuous at 0 and at 1, *i.e.*, the set of all functions $\emptyset$ from [0,1] to X such that $\emptyset$ is continuous at 0 and at 1 and such that if p is a number between 0 and 1 then each of the limits $\emptyset(p-)$ and $\emptyset(p+)$ exists (with respect to $|\cdot|$). It may be shown that if [a,b] is an interval lying in (0,1) then the set QC([a,b],X) of all quasi-continuous functions from [a,b] to X is the set of restrictions to [a,b] of members of B(R,X). If R_1 is the collection consisting of [0,1] together with all subsets t of [0,1] such that either t is degenerate or there is a member [p,q] of R such that t is the open interval (p,q), then $A(R_1,X)$ is the set QC([0,1],X). (From investigations by J. A. Reneke [19, pages 106-112], there are other cases of this type of example - with L a rectangular interval in some Euclidean space.)

As implicitly suggested by Reneke [19], if $\emptyset$ is a function from L to X and f is R-additive from R to Y then the Stieltjes integral $\int_L \emptyset f$ (of $\emptyset$ "with respect to f") may be interpreted as a member T of Y such that, if c is a choice function for R (*i.e.*, c is a function from R such that if u is in R then c(u) is in u), then $T = \int_{L/F} \emptyset[c] f$ with respect to $\|\cdot\|$ 0 in the sense previously indicated in the Introduction, with h the function given by $h(u) = \emptyset(c(u))f(u)$, for u in R. Here, now, is an adaptation to the present context of one of T. H. Hildebrandt's results [6] (which might be termed the Hildebrandt-Fichtenholz-Kantorovitch Theorem, see [3, argument pages 258-259 and comment page 373]). The adaptation seems to include some instances of A(R,X) as a linear subspace of QC([0,1],X); for such instances, with Y the complex plane and X identified with Y, *cf.* G. F. Webb [27].

THEOREM 23. *If* R *is a pre-ring, then the Stieltjes integral equations*

$$\Delta(f)(\emptyset) = \int_L \emptyset f, \textit{ for } f \textit{ in } S_0 \textit{ and } \emptyset \textit{ in } A(R,X),$$

define an isometric linear isomorphism Δ *from* $\{S_0, \|\cdot\|\}$ *onto the space consisting of* *all continuous linear functions* λ *from* $\{A(R,X),|\cdot|_L\}$ *to* $\{Y,\|\cdot\|\}$, *normed in the usual manner, such that if* k *is in* X *and* $\emptyset$ *is in* A(R,X) *then* $\lambda(k\emptyset) = k\lambda(\emptyset)$.

INDICATION OF PROOF. For each t in R let 1_t be the function from L to X defined by $1_t(p) = j^2$ or 0, for p in L, accordingly as p is or is not in t. Since R is a pre-ring of subsets of L, it follows that A(R,X) is B(R,X), as noted previously, and that if λ is such a function as indicated then the function f defined by $f(t) = \lambda(1_t)$, for t in R, is finitely additive and is clearly the candidate to be a member of S_0 such that $\lambda = \Delta(f)$. The essence of the Theorem will therefore be established provided that (given a nontrivial λ, and an f which is so defined) if W is in F then there is a member M of F filling up L, such that each set in W is filled up by a subcollection of M, and a member $\emptyset$ of A(R,X) such that $|\emptyset|_L = 1$ and $\|\lambda(\emptyset)\| = \Sigma_{t \text{ in } M}\|f(t)\|$. This may be shown as follows.

Suppose λ is a nontrivial linear function, as indicated, and that f is the function defined by $f(t) = \lambda(1_t)$, for t in R. Since A(R,X) is B(R,X), there is a member u of R such that $f(u) \neq 0$; let W be a member of F. There is a member M of F filling up L, with a subcollection filling up u, such that each set in W is filled up by a subcollection of M: there is at least one t in M such that f(t) is not 0 (in Y). Let ξ be a member of Y such that $\|\xi\| = 1$, let k be a function from M to X such that if t is in M and η is in Y then

$$k(t)\eta = 0 \text{ or } \frac{\langle\eta,f(t)\rangle}{\|f(t)\|}\xi \text{ accordingly as } f(t) \text{ is } 0 \text{ or not,}$$

and let $\emptyset = \Sigma_{t \text{ in } M}k(t)1_t$, so that

$$\lambda(\emptyset) = \Sigma_{t \text{ in } M}\|f(t)\|\xi \text{ and } \|\lambda(\emptyset)\| = \Sigma_{t \text{ in } M}\|f(t)\|.$$

Now, if p is in L, there is only one t in M which contains p: therefore, if η is in Y then $\|\emptyset(p)\eta\| = \|k(t)\eta\| \leqslant \|\eta\|$, whence $|\emptyset(p)| \leqslant 1$. Since there is some t in M such that $f(t) \neq 0$ and, for each p in t, $\|\emptyset(p)f(t)\| = \|f(t)\|$, it follows that $|\emptyset|_L = 1$. This completes the suggested argument.

As an instance of this type of theorem, for a case where the collection R is not a pre-ring and S is not all of S_0, the following Example is basic.

EXAMPLE 2. To establish connection with the Riesz Theorem alluded to in the Introduction, let (i) L be [0,1] and R be the collection of all subintervals of [0,1],

(ii) Y be the complex plane, j be ordinary complex conjugation, and $\langle\cdot,\cdot\rangle$ be the usual complex inner product for Y given by $\langle\xi,\eta\rangle = \xi j(\eta)$, and (iii) S be the set of all functions f in S_0 such that, if $\epsilon > 0$ and $0 < p < 1$, there is a number r in (p,1] such that if q is a number in (p,r] then $|f([p,q])| < \epsilon$. Inasmuch as Y is one-dimensional, there is the usual identification of X with Y: the space D from the Central Problem of this report is the same as E, the space INV-LIM-$\{H,Q,\pi\}$ is identified with the inverse limit space inv-lim$\{H,Q,\pi\}$, and there is a coalescence of Theorems 12 and 18. A statement of the Riesz Theorem is this: the Stieltjes integral equations

$$\Delta(f)(\emptyset) = \int_L \emptyset f, \text{ for f in S and } \emptyset \text{ in } A(R,X),$$

define an isometric linear isomorphism Δ from $\{S,\|\cdot\|\}$ onto the dual of the normed linear space $\{A(R,X),|\cdot|_L\}$ (*cf.* Example 1). Hence, the space $\{E,n\}$ (Theorem 22) is identifiable as the second dual of $\{A(R,X),|\cdot|_L\}$ with the natural embedding δ of A(R,X) in E taking the form $\delta(\emptyset)(f) = \Delta(f)(\emptyset)$, for $\emptyset$ in A(R,X) and f in S. Composites of the isomorphisms ζ and ω (Theorems 17 and 18) with δ have the forms

$$(\zeta(\delta(\emptyset))f)(t) = \int_t \emptyset f \text{ and } \omega(\delta(\emptyset))(\alpha)(t) = \int_t \emptyset\alpha,$$

for $\emptyset$ in A(R,X) and f in S and t in R and α in S^+. It follows from Theorems 15, 17, and 18 that the ζ-image of E is commutative and it may be seen, with the help of these Theorems, that $\zeta[\delta]$ is an involution-preserving algebra-isomorphism.

It can be proved, independently of the special suppositions of this section, that (in the context of Theorem 18) if the ordered pair $\{\mu,G\}$ belongs to ω and α is in S^+ then $\mu(f) = \int_{L/F} G(\alpha)f/\alpha$, with respect to $[\![\cdot]\!]$, for every f in the closure of H_α with respect to the norm $\|\cdot\|$ - one might invoke the obvious extension of each $\Pi_\alpha(M)$ (for M in F) to include f by the formulas from Theorem 8, and then use the consequent inequalities $\|\Pi_\alpha(M)f - \Pi_\alpha(M)g\| \leq \|f-g\|$ (in continuation of the observation at the end of Remark 1 after Theorem 22), with which the identities

$$\mu(\Pi_\alpha(M)f) = \Sigma_{t \text{ in } M} \frac{G(\alpha)f}{\alpha}(t), \text{ for M in F},$$

serve to establish the result. There is, however, a limitation to the procedure.

It can not be proved that if the ordered pair $\{\mu,G\}$ belongs to ω, and α is a positive member of S^+, then (*cf.* Theorem 18) $\mu(f) = \int_{L/F} G(\alpha)f/\alpha$ for every f in S such that the latter integral exists. This could not be proved even in the real version of Example 2. Consider the following Example.

EXAMPLE 3. Let (i) L be [0,1] and R be the collection of all subintervals of [0,1], (ii) Y be the real line, j be the identity function on Y, and $\langle\cdot,\cdot\rangle$ be real multiplication so that $\langle\xi,\eta\rangle = \xi\eta$ for ξ and η in Y, and (iii) S be the set of all functions f in S_0 such that, if $\epsilon > 0$ and $0 < p < 1$, there is a number r in (p,1] such that if q is a number in (p,r] then $|f([p,q])| < \epsilon$. Consider the member α of S^+ defined by $\alpha([p,q]) = q-p$, for [p,q] in R: let c be a number between 0 and 1, $\emptyset$ be the function defined by $\emptyset(p) = |p-c|$, for p in L, and f be the member of S defined by $f([p,q]) = 1$ or 0, for [p,q] in R, accordingly as the number c does or does not belong to (p,q]. Let λ be the function defined by the Stieltjes integral equations $\lambda(g) = \int_L \emptyset g$, for g in H_α: it may be shown that $\|f\| = 1$ and that, if g is in H_α, $\|f-g\| = 1 + \|g\| \geqslant 1 + |\lambda(g)|$ so that

$$\lambda(g) - \|f-g\| \leqslant -1 < 1 \leqslant \lambda(g) + \|f-g\|.$$

According to the Hahn-Banach extension process, E contains extensions μ_1 and μ_2 of λ such that $\mu_1(f) = -1$ and $\mu_2(f) = 1$ and $n(\mu_1) = n(\mu_2) = 1$: identifying L(Y) with Y itself as in the complex case, one may see that if μ is μ_1 or μ_2 and $G = \omega(\mu)$ then $G(\alpha)(t) = \int_t \emptyset\alpha$, for each t in R, and $\int_{L/F} G(\alpha)f/\alpha = 0 \neq \mu(f)$.

As recorded, *e.g.*, by Dunford and Schwartz [3, pages 373-381], there have been extensions of the Riesz Theorem to contexts more general than that in which L is the unit interval. Accordingly, it seems appropriate to record some consequences of present results in a theorem in which the Riesz Theorem (in the form suggested in Example 2) is taken to be part of the hypothesis. The space E is regarded as an involution-algebra with multiplication induced by ζ and involution induced by ω (as indicated in the section Description of Solutions, justified by Theorems 15 through 18). With these conventions, the following is such a Theorem.

THEOREM 24. *Suppose* S *is a linear subspace of* S_0 *such that* (i) S *is closed with respect to the norm* $\|\cdot\|$, (ii) *if* t *is in* R *and* f *is in* S *and* ξ *is in* Y *then the function* $P_t Vf\cdot\xi$ *belongs to* S, *and* (iii) *the Stieltjes integral equations*

$$\Delta(f)(\emptyset) = \int_L \emptyset f, \textit{ for } f \textit{ in } S \textit{ and } \emptyset \textit{ in } A(R,X),$$

define an isometric linear isomorphism Δ *from* $\{S,\|\cdot\|\}$ *onto the space consisting of all* *continuous linear functions* λ *from* $\{A(R,X),|\cdot|_L\}$ *to* $\{Y,\|\cdot\|\}$, *normed in the usual manner, such that if* k *is in* X *and* $\emptyset$ *is in* A(R,X) *then* $\lambda(k\emptyset) = k\lambda(\emptyset)$. *Then* (1) *the equations* $\delta(\emptyset)(f) = \Delta(f)(\emptyset)$, *for* $\emptyset$ *in* A(R,X) *and* f *in* S, *define an isometric involution-preserving algebra-isomorphism* δ *from* $\{A(R,X),|\cdot|_L\}$ *into the normed algebra* $\{E,n\}$ *(taking the multiplicative identity in* A(R,X) *to that in* E*),* (2) *composites of* ζ *and* ω *with* δ *are given by the Stieltjes integral equations*

$$(\zeta(\delta(\emptyset))f)(t) = \int_t \emptyset f \text{ and } \omega(\delta(\emptyset))(\alpha)(t)\xi = \int_t \emptyset\alpha\cdot\xi,$$

for $\emptyset$ *in* A(R,X) *and* f *in* S *and* t *in* R *and* α *in* S^+ *and* ξ *in* Y, *and* (3) *if* μ *is in* E *and* $\emptyset$ *is in* A(R,X) *then, for each* α *in* S^+ *and* t *in* R *and* ξ *in* Y,

$$\omega(\zeta^{-1}(\zeta(\delta(\emptyset))\zeta(\mu)))(\alpha)(t)\xi = \int_t \emptyset\omega(\mu)(\alpha)\cdot\xi.$$

INDICATION OF PROOF. Since $\|\Delta(f)(\emptyset)\| \leq |\emptyset|_L\|f\|$ for $\emptyset$ in A(R,X) and f in S, the indicated equations clearly define a linear transformation δ from A(R,X) into E such that if $\emptyset$ is in A(R,X) then $n(\delta(\emptyset)) \leq |\emptyset|_L$: suppose $\emptyset_0$ is in A(R,X) and $n(\delta(\emptyset_0)) < |\emptyset_0|_L$. There is a member p of L such that $n(\delta(\emptyset_0)) < |\emptyset_0(p)|$ and, therefore, a member ξ of Y such that $\|\xi\| = 1$ and $n(\delta(\emptyset_1)) < \|\emptyset_0(p)\xi\|$. Let λ be the function defined by $\lambda(\emptyset) = \emptyset(p)\xi$, for $\emptyset$ in A(R,X): there exists a member f of S such that $\|f\| \leq 1$ and if $\emptyset$ is in A(R,X) then $\lambda(\emptyset) = \int_L \emptyset f$. Now,

$$n(\delta(\emptyset_0)) < \|\emptyset_0(p)\xi\| = \|\lambda(\emptyset_0)\| = \|\delta(\emptyset_0)(f)\| \leq n(\delta(\emptyset_0))\|f\| \leq n(\delta(\emptyset_0)).$$

This involves a contradiction, so that δ is an isometry. The other assertions of the Theorem may be established with the help of Theorems 15, 17, and 18.

There is another type of problem, involving cases where R is not a pre-ring, which falls within the scope of the present report. In 1962, in connection with a survey [14] of some investigations concerning the notion of an ordinary linear differential equation, I presented a result (*loc.cit.*, pages 321-322) from which it is easy to arrive at the following Example.

EXAMPLE 4. Let L be the unit interval [0,1] and R be the collection of all subintervals of [0,1]: c_0, c_1, and c_2 denote choice functions for R such that if t is the member [p,q] of R then $c_0(t) = p < c_1(t) < q = c_2(t)$. If λ is such a linear function from QC([0,1],X) to a set of R-additive functions from R to Y that, if k is in X and $\emptyset$

is in QC([0,1],X), $\lambda(k\emptyset) = d\lambda(\emptyset)$, then the following two conditions are equivalent:

(i) there is an R-additive function β from R to the nonnegative numbers such that if $\emptyset$ is in QC([0,1],X) and t is in R then $\|\lambda(\emptyset)(t)\| \leqslant |\emptyset|_t\beta(t)$, and

(ii) there is a member $\{f_0,f_1,f_2\}$ of $S_0 \times S_0 \times S_0$ such that if $\emptyset$ belongs to QC([0,1],X) and t is in R then

$$\lambda(\emptyset)(t) = \int_{t/F}\emptyset[c_0]f_0 + \int_{t/F}\emptyset[c_1]f_1 + \int_{t/F}\emptyset[c_2]f_2.$$

Let $\Delta(f_0,f_1,f_2)$ denote λ in (ii), and note that Theorems 1-22 are applicable with the interpretation that S is S_0. For f in S_0 and $\emptyset$ in QC([0,1],X), it may be seen that $\Delta(f,-f,f)(\emptyset)$ is W. H. Young's version [28] of the Lebesgue-Stieltjes integral designed to yield interval-additive functions, *i.e.,* R-additive $\lambda(\emptyset)$. A substitution theorem for the Young integral [8, page 91], readily adaptable to the present context, can be used to produce an algebra-isomorphism δ, from the space $\{QC([0,1],X),|\cdot|_L\}$ into $\{E,n\}$, having the same character as that in Theorem 24 and justified by much the same type of argument as indicated there. Thus, W. H. Young's idea may be regarded as producing a somewhat general notion of integral.

There is another interpretation of the result from [14] cited in Example 4, making explicit use of the possible multi-dimensional character of the space Y in the present report. J. A. Reneke [19] has discovered higher dimensional versions of the result, with Y the complex plane and X identified with Y, exhibiting (for each positive integer r) a set Φ of 3^{r+1} choice functions for the collection R of all rectangular subintervals of $[0,1]^{r+1}$ such that the integral equations

$$\Delta(f)(\emptyset)(t) = \Sigma_{c \text{ in } \Phi}\int_{t/F}\emptyset[c]f_c,$$

for f in S_0^Φ and $\emptyset$ in QC($[0,1]^{r+1}$,X) and t in R, define a linear homomorphism Δ from the set S_0^Φ of all functions from Φ into S_0 onto the set consisting of all linear functions λ from QC($[0,1]^{r+1}$,X) into S_0 such that, for some R-additive β from R to the nonnegative numbers, $\|\lambda(\emptyset)(t)\| \leqslant |\emptyset|_t\beta(t)$ for each function $\emptyset$ in QC($[0,1]^{r+1}$,X) and each set t in R. Reneke's results are readily adaptable to the present situation by adding the condition $\lambda(k\emptyset) = k\lambda(\emptyset)$ for k in X. More generally, however, Reneke has investigated conditions on the ordered pair {L,R} which imply [15, Geometric Perspectives] the Subdivision Axiom (but are not implied by it [15, Example 5]) and which, together with the assumption that the set B(R,X) is an algebra, are consistent with the existence of a finite set Φ of choice functions yielding $\Delta(f)(\emptyset)$ as above, for f in S_0^Φ and $\emptyset$ in B(R,X). Now, supposing only such a finite set Φ, consider the following procedure: replace the space $\{Y,\langle\cdot,\cdot\rangle\}$ by the product space $\{Y^\Phi,\langle\cdot,\cdot\rangle_\Phi\}$, where

$$\langle\xi,\eta\rangle_\Phi = \Sigma_{c \text{ in } \Phi}\langle\xi_c,\eta_c\rangle \text{ for } \xi \text{ and } \eta \text{ in } Y^\Phi,$$

throughout appropriate earlier sections of this report, and regard X^Φ as a subset of $L(Y^\Phi)$ in the usual way - $\langle k\xi,\eta\rangle_\Phi = \Sigma_{c \text{ in } \Phi}\langle k_c\xi_c,\eta_c\rangle$ for k in X^Φ. Because of the finite cardinality of Φ, $S_0(Y^\Phi)$ is easily identified with S_0^Φ, and certain linear embeddings of Y in Y^Φ can be made to yield linear embeddings of B(R,X) in the algebra E corresponding to $S_0(Y^\Phi)$. To avoid notational complications here, details and variations of all this are left to suggest themselves to the reader.

Finally, to see that certain types of linear subspaces of S_0 which sometimes occur in measure theoretic investigations (with bounded nonnegative measures) are instances of the tyep S of the present report, consider the following Example.

EXAMPLE 5. Independently of the special suppositions of this section, let δ be a nontrivial R-additive function from R to the nonnegative numbers such that $\int_{L/F}\delta$ exists, and let S be the closure with respect to the norm $\|\cdot\|$ of the space U_δ as described in Theorem 4. Note that Theorems 1-22 hold with S taken to be S_0 and that δ belongs to S_0^+ and that, for each ξ in Y, $\delta\cdot\xi$ belongs to H_δ. Now, by Theorem 7, U_δ is dense in H_δ with respect to N_δ and therefore, by Theorem 3, with respect to $\|\cdot\|$: hence, S is the closure with respect to $\|\cdot\|$ of H_δ. If g is in U_δ and ξ is in Y then it may be seen that $Vg\cdot\xi$ is in U_δ; by Theorem 5, if t is in R then P_t maps H_δ into H_δ; from this it may be argued that if t is in R and f is in S then $P_tVf\cdot\xi$ belongs to S: thus, S satisfies the condition stipulated in stating the Central Problem of this report. Moreover, since $\delta\cdot\xi$ belongs to H_δ for each ξ in Y, the function δ belongs to S^+. Therefore, Theorems 1-22 hold as stated for this space S and, as has been noted in the first paragraph following Example 2, if μ is in E and $G = \omega(\mu)$ then $\mu(f) = \int_{L/F}G(\delta)f/\delta$ for every function f in S. A similar result holds of course for each α in S^+ such that U_α is dense in $\{S,\|\cdot\|\}$.

REMARK 1. Suppose Y is the complex plane, X is identified with Y, and R is a pre-ring. Let α be a nontrivial member of S_0^+, M be a member of F, x be a function from M to Y, and g be the member $\Sigma_{u \text{ in } M} P_u \alpha \cdot x(u)$ of the family U_α: it may be shown in this case that if G is the member $\Sigma_{u \text{ in } M} x(u) 1_u$ of B(R,X) then

$$g(t) = \int_t G\alpha, \text{ for each set t in R, and } \|g\| = \int_L |G|\alpha.$$

Thus, some of the present results have obvious measure theoretic interpretations.

REMARK 2. One effect of the introduction of the Subdivision Axiom has been provision of a framework within which the Hildebrandt-Fichtenholz-Kantorovitch Theorem (in the form of Theorem 23) is seen as the instance of the Riesz Theorem in which R is a pre-ring and S is all of S_0. This suggests an inquiry, then, as to the general existence of such a subspace S of S_0 as postulated in Theorem 24.

REMARK 3. The questions of cardinality alluded to in the Introduction, in connection with R. D. Mauldin's investigations [16, 17], may be viewed (in the general context of Theorems 1-22) as suggesting an inquiry as to conditions on the ordered pair {R,S} which might insure that, if λ is in D and $g = A(\lambda)$ as in the statement of Theorem 12, there exists a member α of S^+ such that

$$\lambda(f) = \int_{L/F} \langle g(\alpha), Jf\rangle / \alpha \text{ for each f in S.}$$

The effect of Examples 3 and 5 is not represented as obviating any such inquiry.

THEOREM 25. *In the algebra* A_0 *of all continuous linear transformations in the product space* $\{\times\{H,Q\}, Q^\wedge\}$, *the Z-image of* E *is closed in the weak operator topology - the representation* Z *being given by*

$$Q^\wedge(Z(\mu)f,g) = \Sigma_\alpha Q_\alpha(\zeta(\mu)f_\alpha, g_\alpha) \text{ for } \mu \text{ in E, f and g in } \times\{H,Q\}.$$

REMARK. The indicated weak operator topology is that introduced by von Neumann (*cf.* [9, page 53]), and the indicated product space is the direct sum over S^+ of the spaces $\{H_\alpha, Q_\alpha\}$ which was denoted by $\{X,(\cdot,\cdot)\}$ in the section entitled Description of Solutions. As indicated in that section, and as now justified by Theorems 14-18, the Z-image of E is the B*-algebra A_3: A_1 denotes the algebra of all members B of A_0 iwith a representation Ψ such that

$$Q^\wedge(Bf,g) = \Sigma_\alpha Q_\alpha(\Psi(B)_\alpha f_\alpha, g_\alpha) \text{ for f and g in } \times\{H,Q\}$$

where, for each α in S^+, $\Psi(B)_\alpha$ is a continuous linear transformation in $\{H_\alpha, Q_\alpha\}$, A_2 is the algebra of all members B of A_1 such that if α is in S^+ and t is in R and h is in H_α

then $\Psi(B)_\alpha P_t h = P_t \Psi(B)_\alpha h$, and A_3 is the algebra of all members B of A_2 such that if α and β are members of S^+ such that H_α is a subset of H_β then $\Psi(B)_\alpha$ is the restriction to H_α of $\Psi(B)_\beta$. Hence, it is asserted here that the Z-image of E is what is called a (W*)-algebra in the case of complex scalars (*cf.* [9, page 161]). It should become clear that this assertion is independent of the special supposition of this section that L belongs to the collection R.

PROOF. Suppose B is in the weak closure of the Z-image A_3 of E. For each $Q^\wedge$-orthogonal projection Φ in the commutant (in A_0) of A_3 and each μ in E,

$$Q^\wedge(B\Phi f - \Phi Bf, g) = Q^\wedge((B - Z(\mu))\Phi f, g) + Q^\wedge((Z(\mu) - B)f, \Phi g)$$

for all f and g in $X\{H,Q\}$: since each weak neighborhood of B contains $Z(\mu)$ for some μ in E, it follows that $B\Phi = \Phi B$. It is immediate that B belongs to A_1; by considering $\Phi_t = Z(\zeta^{-1}(P_t))$ for each t in R, one may see that B belongs to A_2. If α and β are in S^+ and H_α is a subset of H_β then, for each h in H_α and t in R and η in Y and μ in E,

$$\langle \Psi(B)_\beta h(t) - \Psi(B)_\alpha h(t), \eta\rangle = Q_\beta((\Psi(B)_\beta - \zeta(\mu))h, P_t\beta\cdot\eta) + Q_\alpha((\zeta(\mu) - \Psi(B)_\alpha)h, P_t\alpha\cdot\eta).$$

It follows, as above, that B belongs to A_3. This completes the Proof.

Hierarchy of Dual Spaces in the Scalar Case. Suppose, throughout this section, that the space Y is one-dimensional, and that the algebra $\{X, |\cdot|\}$ of the preceding section is identified with the scalars in the usual manner, L(Y) being all of X. There are three Observations which are useful in this special situation.

OBSERVATION 1. In every instance of a triple {L,R,S} as postulated in the Introduction, it follows from assertion (3) of Theorem 15 that the multiplication induced in E by the representation ζ (*cf.* Theorem 17) is commutative. In the case of real scalars, each member of the Z-image A_3 of E (*cf.* Theorem 25) is seen to be Hermitian with respect to the inner product $Q^\wedge$; in the alternative case of complex scalars, each member of A_3 is normal with respect to $Q^\wedge$. Since Theorem 25 implies that A_3 is closed in the strong operator topology for A_0, it follows that in either case the spectral resolution of each member of A_3 has all of its values in A_3 (this may be seen from the argument due to Riesz [22, pages 272-288] for Hilbert's spectral theorem). Therefore A_3 is the closure, with respect to the uniform operator norm for A_0, of the set of all finite linear combinations (with coefficients from X) of nonzero

$Q^\wedge$-orthogonal projections belonging to A_3. Of course, if B is such a linear combination $\Sigma_{p \text{ in } M} k(p)p$ and $pq = 0$ for each two members p and q of M then $n(Z^{-1}(B)) = \sup_{p \text{ in } M} |k(p)|$.

One may note in passing that is a consequence, due to Stone (*cf.* [9, pages 162-163]), of Theorem 25 that the set of all $Q^\wedge$-orthogonal projections belonging to A_3 is a complete Boolean lattice relatively to the usual partial ordering that is induced by the inner product $Q^\wedge$.

OBSERVATION 2. In amplification of the pattern of ideas from Observation 1, let R' be the Z^{-1}-image of the set of all nonzero $Q^\wedge$-orthogonal projections in A_3 and $\leqslant$ denote the partial ordering determined for R' by the $Q^\wedge$-induced partial ordering of the Z-image of R': if each of μ_0 and μ_1 is in R' then the statement that $\mu_0 \leqslant \mu_1$ means that $Q^\wedge(f,Z(\mu_0)f) \leqslant Q^\wedge(f,Z(\mu_1)f)$ for every f in $X\{H,Q\}$, clearly equivalent to saying that if α is in S^+ then $Q_\alpha(f,\zeta(\mu_0)f) \leqslant Q_\alpha(f,\zeta(\mu_1)f)$ for every f in H_α, this latter in turn being equivalent to saying that if α is in S^+ and t is in R then $\omega(\mu_0)(\alpha)(t) \leqslant \omega(\mu_1)(\alpha)(t)$ (*cf.* Theorems 17 and 18). If μ is in E and $\epsilon > 0$, there exists a function k from a finite subset M of R' into X such that $Z(p)Z(q) = 0$ for each two members p and q of M, $\Sigma_{p \text{ in } M} Z(p)$ is the identity transformation on $X\{H,Q\}$, and $n(\mu - \Sigma_{p \text{ in } M} k(p)p) < \epsilon$. It may be noted that, in order that the nonzero member μ of E should belong to R', it is necessary and sufficient that if α is in S^+ and t is in R then $\omega(\mu)(\alpha)(t)$ be real and $\zeta(\mu)^2 = \zeta(\mu)$, *i.e.*, $\int_{t/F} \omega(\mu)(\alpha)^2/\alpha = \omega(\mu)(\alpha)(t)$ (*cf.* Theorem 15).

OBSERVATION 3. With $\{R', \leqslant\}$ the upper semi-lattice from Observation 2, let R'' be the collection of all subsets x of R' such that x has, and is maximal with respect to having, the property that if g is a finite subset of x then there is a member p of R' such that $p \leqslant q$ for each q in g. Let γ be a function from R' such that if p is in R' then $\gamma(p)$ is the subset of R'' to which the member x of R'' belongs only in case p belongs to x. In consequence of the properties of the algebra A_3 from Observation 1, especially the commutativity of multiplication, it follows from [15] that the collection $\gamma^\rightarrow(R')$, the γ-image of R', is a pre-ring of subsets of R'' to which R'' belongs and that, if p is an element of R' and M is a finite subset of R', $p \leqslant \sup_{\{R',\leqslant\}} M$ only in case the set $\gamma(p)$ is covered by the collection $\gamma^\rightarrow(M)$. Moreover, from Observation 2, there exists an isometric involution-preserving algebra-isomorphism, here denoted simply by the suffix ^, from the ζ-image of E onto the space $B(\gamma^{\to}(R'),X)$, determined by the formulas

$$\zeta(\Sigma_{p \text{ in } M} k(p)p)^\wedge = \Sigma_{p \text{ in } M} k(p) 1_{\gamma(p)}$$

for functions k from finite subsets M of R' to X, and taking the identity on the space S to the constant 1 on the set R''. Hence, the Hildebrandt Theorem (scalar version of Theorem 23, Hildebrandt-Fichtenholz-Kantorovitch Theorem) may be seen to yield a representation of the dual of $B(\gamma^{\to}(R'),X)$ as the space $S_0(\gamma^{\to}(R'))$, of all finitely additive functions of bounded variation from $\gamma^{\to}(R')$ to the space Y.

The suggested hierarchy now emerges. Explicitly, one may imagine starting with a triple $\{L_1,R_1,S_1\}$ such as postulated in the Introduction, or as amended in the section Modification of the Initial Supposition, and such that if t is in R_1 then $f(t) \neq 0$ for some f in S_1: use Theorems 17 and 18 to arrive at ζ and ω for the dual space E_1 so that Theorem 25 is applicable; use Observations 1, 2, and 3 to determine the set L_2, the pre-ring R_2 of subsets of L_2 to which L_2 belongs, and the space S_2 as $S_0(R_2)$; the way is now open to repeat this procedure starting with the triple $\{L_2,R_2,S_2\}$. Integrals of the Hellinger and Stieltjes type arise, alternately, as this process is continued indefinitely. Integral formulas, for the transition from $\{L_k,R_k,S_k,E_k\}$ to the next stage $\{L_{k+1},R_{k+1},S_{k+1},E_{k+1}\}$, may be written as follows:

$$(\zeta(\mu)f)(t) = \int_{t/F_k} \omega(\mu)(\alpha)f/\alpha \text{ and } \lambda(\zeta(\mu)^\wedge) = \int_{L_{k+1}} \zeta(\mu)^\wedge \sigma(\lambda),$$

where F_k is the family of all finite subcollections of R_k nonoverlapping relatively to R_k, ζ and ω are the functions from E_k as determined by Theorems 17 and 18, the notation $\zeta(\mu)^\wedge$ (for μ in E_k) describes the representation of $\zeta(\mu)$ as a member of the space $B(R_{k+1},X)$ as in Observation 3, and σ is the Hildebrandt mapping (inverse to the Δ of Theorem 23) from the dual of $B(R_{k+1},X)$ onto S_{k+1}.

Implicit in the foregoing hierarchy, also, are the following functions:

(1) with $P(R_k)$ the function from R_k into the ζ-image of E_k as amended from the Introduction, there is the inclusion-preserving mapping, from R_k into R_{k+1}, to which the member $\{t,T\}$ of $R_k \times R_{k+1}$ belongs only in case t is in R_k and T is the subset of L_{k+1} to which a point x of L_{k+1} belongs only in case $P(R_k)_t^\wedge(x) = 1$;

(2) with the assurance of isometry provided, *e.g.*, by Theorem 22, there is the linear isomorphism from S_k into S_{k+1} which consists of all ordered pairs of the form $\{f,\sigma(\lambda_f)\}$ such that f is in S_k and λ_f is the member of the dual of $B(R_{k+1},X)$ determined by the equations $\lambda_f(\zeta(\mu)^\wedge) = \mu(f)$ for μ in E_k [with the mapping (1) from R_k into R_{k+1}, in view of the integral formulas $f(t) = \int_{L_{k+1}} P(R_k)_t^\wedge \sigma(\lambda_f)$ for t in R_k, one might regard $\sigma(\lambda_f)$ as an extension to R_{k+1} of an f in S_k] ; and

(3) there is the isometric linear isomorphism from E_k into E_{k+1} which consists of all ordered pairs of the form $\{\mu,\delta(\zeta(\mu)^\wedge)\}$ such that μ is in E_k and δ is the linear function from $B(R_{k+1},X)$ into E_{k+1} determined by the formulas

$$\delta(\emptyset)(\sigma(\lambda)) = \lambda(\emptyset) = \int_{L_{k+1}} \emptyset\sigma(\lambda) \text{ for } \emptyset \text{ in } B(R_{k+1},X) \text{ and } \lambda \text{ in its dual.}$$

REMARK. Independently of the dimensionality of the space Y, it may be seen from Theorem 15 that the center of the Z-image A_3 of E (*cf.* Observation 1) is the Z-image of the set of all μ in E such that if α is in S^+ and t is in R then the transformation $\omega(\mu)(\alpha)(t)$ is in the center of the algebra X, *i.e.*, is a scalar. Hence, the heuristic evidence of the scalar case suggests that in general R' be the Z^{-1}-image of the set of all nonzero $Q^\wedge$-orthogonal projections in the center of A_3, with $\ll$ the partial ordering indicated in Observation 2. There is the natural multiplication of members of E by members of X, leading one to note that if k is a function from a finite subset M of R' into X such that $Z(p)Z(q) = 0$ for each two members p and q of M then $n(\Sigma_{p \text{ in } M} k(p)p) = \sup_{p \text{ in } M} |k(p)|$ as in the scalar case. Therefore, the ζ-image of the closure in $\{E,n\}$ of the set of all such finite linear combinations (with coefficients from X) of members of R' has a representation as $B(\gamma^{\rightarrow}(R'),X)$, just as in the scalar case (Observation 3). Thus the question naturally arises as to whether or not that closure is all of E.

Recapitulation and Extension of Results. The pattern of ideas in Theorems 1 through 10, and the arguments given in support of those theorems, have been presented in such a way as to allow for an extension, with minor modifications, to a somewhat more general situation. This section includes a re-examination of that pattern from such a viewpoint. The Subdivision Axiom, as previously enunciated, continues to supplant the pre-ring hypothesis, and the propositions arising from the aforementioned Theorems by taking S to be all of S_0 are designated as Theorems 1-0 through 10-0. Let S_2 be the set of all R-additive functions f from R to Y such that, for some nonnegative R-additive function α, $\|f(t)\| \leqslant \alpha(t)$ for each t in R; let the functions P_t (for t in R) and V and J be extended to S_2 in the natural way; let S_2^+ be the V-image of S_2, *i.e.* , the set of all nonnegative R-additive functions. It may be noted that if t is in R then P_t maps S_2 into S_0 and $\|P_t f\| = \int_{t/F} \|f\|$ for each f in S_2.

Attention is now directed to the intermediate set S_1 to which f belongs only in case f is an R-additive function from R to Y and, for some member h of S_0^+ and some member α of S_2^+, if t is in R then $\|f(t)\|^2 \leqslant h(t)\alpha(t)$: let H now denote a function from S_2^+ such that, for each α in S_2^+, H_α is the set to which f belongs only in case f is in S_1 and, for some h in S_0^+, the preceding inequalities hold for each t in R. By definition, now, S_1 is filled up by the H-image of S_2^+. It can happen that S_0 is a proper subset of S_1 and S_1 is a proper subset of S_2: the case in which L is an uncountable set, and R is the collection of all degenerate subsets of L, may easily be seen to present such a siutation (with R a pre-ring).

There now arise Theorems 1-1 through 10-1, from Theorems 1-0 through 10-0, upon (i) replacing S_0^+ by S_2^+ throughout, (ii) changing assertions (1) and (3) in Theorem 3-0 to read, respectively, that H_α is a linear subspace of S_1 and that if t is in R and f is in H_α then $\|P_t f\|^2 \leqslant N_\alpha(f)^2\alpha(t)$, and (iii) deleting assertion (3) from Theorem 7-0. A survey of the indicated arguments reveals the necessity of two explicit modifications, called for since S_0^+ may not be all of S_2^+:

(1) In the first paragraph of the Proof for Theorem 4-0, $P_t\alpha\cdot\xi$ may be shown to belong to H_α by computation (from Theorem 2-0) yielding $N_\alpha(P_t\alpha\cdot\xi)^2 = \alpha(t)\|\xi\|^2$.

(2) The second display in the second paragraph of the Proof of Theorem 10-1 is

$$P_u\gamma\cdot\xi = \pi(\alpha,\alpha+\beta)(P_u\beta\cdot\xi) = \lambda(\beta,\alpha+\beta)(P_u\alpha\cdot\xi),$$

asserted as holding for each u in R and ξ in Y, and showing U_γ to lie in $H_\alpha H_\beta$. Only minor modifications, consistent with (1) and (2), now serve to establish the suggested sequence of theorems; and the H-image of S_2^+ is a distributive lattice.

What is intended, now, is to provide an extension ω_1 of $\omega[\zeta^{-1}]$ as described in Theorems 17 and 18 to a collection of linear transformations in the space S_1. The function π having been defined (in Theorem 9-1) on the subset of $S_2^+ \times S_2^+$ to which

$\{\alpha,\beta\}$ belongs only in case H_α is a subset of H_β, INV-LIM-$\{H,Q,\pi\}$ is now taken to denote the linear space to which G belongs only in case G is a function from S_2^+ such that if α is in S_2^+ and ξ is in Y and t is in R then (i) $G(\alpha)$ is an R-additive function from R to L(Y), $G(\alpha)\cdot\xi$ belongs to S_2, and $P_tG(\alpha)\cdot\xi$ belongs to H_α, and (ii) if β is a member of S_2^+ such that H_α is a subset of H_β then

$$P_tG(\alpha)\cdot\xi = \pi(\alpha,\beta)(P_tG(\beta)\cdot\xi).$$

On the foregoing basis, the following Theorem may be interpreted as arising from the circle of ideas indicated in Theorems 16, 17, and 18 (with S taken to be S_0), and as being a part thereof in case the collection R is such that S_1 is S_0.

THEOREM 26. *Suppose* $\{X,|\cdot|\}$ *is a normed linear space such that* B *is in* X *only in case* B *is a linear transformation in* S_1 *such that* (i) *if* t *is in* R *and* f *is in* S_1 *then* $B(P_tf) = P_t(Bf)$, (ii) *if* α *is in* S_2^+ *then* B *maps* H_α *into* H_α, *and* (iii) *there is a nonnegative number* b *such that* $N_\alpha(Bf) \leqslant b\, N_\alpha(f)$ *for each* α *in* S_2^+ *and each* f *in* H_α, *in which case* $|B|$ *is the least such number* b. *Then, the equations* $\omega_1(B)(\alpha)(t)\xi = \int_{L/F} B(P_t\alpha\cdot\xi)$, *for* B *in* X *and* α *in* S_2^+ *and* t *in* R *and* ξ *in* Y, *define a linear isomorphism* ω_1 *from* X *onto the subspace of* INV-LIM-$\{H,Q,\pi\}$ *to which the point* G *of* INV-LIM-$\{H,Q,\pi\}$ *belongs only in case there is a nonnegative number* b *such that if* α *is in* S_2^+ *and* t *is in* R *and* ξ *is in* Y *then* $\|G(\alpha)(t)\xi\| \leqslant b\alpha(t)\|\xi\|$, *in which case* $|\omega_1^{-1}(G)|$ *is the least such number* b. *If the ordered pair* $\{B,G\}$ *belongs to* ω_1 *and* f *is in* S_1 *then* Bf *is an integral over* L *relatively to* F *in the following sense: for each member* α *of* S_2^+ *such that* f *is in* H_α, *if* h *is the function from* R *to* H_α *such that if* t *is in* R *then* $h(t) = 0$ *or* $\frac{1}{\alpha(t)}P_tG(\alpha)\cdot f(t)$ *accordingly as* $\alpha(t)$ *is the number* 0 *or not*, $Bf = \int_{L/F} h$ *with respect to the norm* N_α - *in particular, for each member* α *of* S_2^+ *such that* f *is in* H_α *and each set* u *in* R, $Bf(u) = \int_{u/F} G(\alpha)f/\alpha$ *with respect to the norm* $\|\cdot\|$.

INDICATION OF PROOF. It may first be noted that Theorem 13, and its Proof, hold as stated with S_2^+ replacing S^+: let Theorem 13-1 denote the result when so amended. Similarly, a result which may be designated as Theorem 14-1 arises from Theorem 14 with this same replacement and the following amendments:

(1) the displayed description of the transformation Ψ should be made to read

"$\Psi(B)(t)\xi = \int_{L/F} B(P_t\beta\cdot\xi)$, for B in $T_{\alpha\beta}(P)$ and t in R and ξ in Y,"

(2) the assertion (2) should be made to read, in part, "$P_t G^* \eta$ belongs to H_β and $\langle Bf(t), \eta \rangle = Q_\beta(f, P_t G^* \eta)$," with similar amendments in the Proof, and

(3) the assertion (3), and the last paragraph of the Proof, should be deleted.

Finally, a suitable Theorem 15-1 is available from Theorem 15, and its Proof, as amended consistently with (2) of Theorem 14-1. Theorem 26 may now be established directly on the basis of the emerging pattern of ideas.

REMARK 1. There is a context in which the present results may be given an interpretation analogous to that suggested in Remark 1 following Example 5. One may, *e.g.*, start with a sequential-ring (or σ-ring [8, page 147]) Σ of subsets of L filling up L and a (nontrivial and finite-valued) nonnegative measure δ on Σ, and let R be the collection of all members of Σ having positive δ-measure. It is not difficult to see that the Subdivision Axiom is satisfied, and that members of S_2 (suitably extended to Σ) are absolutely continuous with respect to the measure δ.

REMARK 2. With the help of the Theorem 15-1, the algebra X in Theorem 26 is equipped with a natural norm-preserving involution and is amendable to the type of representation indicated, in Description of Solutions, for the ζ-image of E.

REMARK 3. The algebra X in Theorem 26 contains the identity transformation on S_1 as well as, for each t in R, the restriction of P_t to S_1. It also contains certain transformations arising from Stieltjes integral equations of differential type [14]. Suppose, *e.g.*, that W is a function from L to a bounded collection of continuous linear transformations in the space $\{Y, \langle \cdot, \cdot \rangle\}$ and that c is a choice function for R such that $\int_{t/F} W[c]\alpha \cdot \xi$ exists with respect to $[\![\cdot]\!]$ for each t in R and α in S_2^+ and ξ in Y: it may be shown that the equations $B_1 f(t) = \int_{t/F} w[c] f$, for f in S_1 and t in R, define a member B_1 of X and that if $\{B_2, G_2\}$ belongs to ω_1 and $G = \omega_1(B_1 B_2)$ then $G(\alpha)(t)\xi = \int_{t/F} W[c]\, G_2(\alpha) \cdot \xi$ for all α in S_2^+ and ξ in Y.

It can not be proved that if S_0 is a proper subset of S_2 then S_1 is a proper subset of S_2. Consider the following Example:

EXAMPLE 6. Let L be the real line and R be the collection of all intervals of real numbers, and suppose f is a member of S_2 which does not belong to S_1: let β be a member of S_2^+ such that $[\![f(t)]\!] \leqslant \beta(t)$ for each t in R. There exists a continuous function p, from L to the positive numbers, such that the Stieltjes integral equations

$h(t) = \int_t \frac{1}{p}\beta$, for t in R, define a member h of S_0^+: let α be the member of S_2^+ defined by the Stieltjes integral equations $\alpha(t) = \int_t p\,\beta$, for t in R. It may be seen that if t is in R then $\|f(t)\|^2 \leqslant h(t)\alpha(t)$: the function f thus belongs to H_α, and so to S_1. This involves a contradiction. Apropos of the foregoing Remark 3, there are special interpretations in this instance of the algebra $\{X,|\cdot|\}$ from Theorem 26, which are available by analogy with Theorem 16.

REMARK 4. There is a simple device for contemplating additive extensions of members f of such spaces as S_0, S_1, or S_2, which is somewhat in the spirit of the Riesz and Daniell approaches [22] to measurable sets in the sense of postponement until after identification of certain dual spaces. Typically, one might replace each f (from the appropriate space S) by a function $\hat{f}$ defined on the Boolean ring which is generated by the restrictions to S of members of the P-image of R - by setting $\hat{f}(P_tP_u) = P_tf(u)$ and $\hat{f}(P_t + P_u - P_tP_u) = f(t) + f(u) - P_tf(u)$, for $\{t,u\}$ in $R \times R$. There are then extrapolations of the functions $\hat{f}$ to some of the idempotent elements of an algebra of operators, closed with respect to one of the usual linear operator topologies, which includes the aforementioned Boolean ring.

Apropos of lattice-theoretic questions, it can not be proved that if α and β are in S_0^+ and H_α is a proper subset of H_β then there is a nontrivial member γ of S_2^+ such that the intersection $H_\alpha H_\gamma$ is H_0. Consider the following final Example.

EXAMPLE 7. Let L be the set of all nonnegative integers, and R the pre-ring of all degenerate subsets of L; let Y be the real line, with $\langle\cdot,\cdot\rangle$ ordinary real multiplication (as in Example 3). There is a natural identification of Y-valued R-additive functions with infinite real number sequences, *i.e.*, with functions on L to Y: with this identification, let α and β be members of S_0^+ such that if m is in L then $\alpha(m) = (m+1)^{-4}$ and $\beta(m) = (m+1)^{-2}$. It follows from Theorem 9-0 that H_α is a proper subset of H_β, and from Theorem 10-1 that if γ is in S_2^+ and $H_\alpha H_\gamma = H_0$ then $\alpha(m)\gamma(m) = 0$ for each m in L - so that γ is the constant 0.

Prospectus. Consider the family S_4^+, consisting of all functions α from R to L(Y) such that α is R-additive and $\langle\alpha(t)\xi,\eta\rangle = \langle\xi,\alpha(t)\eta\rangle$ and $\langle\xi,\alpha(t)\xi\rangle \geqslant 0$, for each t in R and $\{\xi,\eta\}$ in $Y \times Y$. Special members α of S_4^+, such that if ξ is in Y then $\int_{L/F} \alpha\cdot\xi = \xi$ with respect to $\|\cdot\|$, are important in the general theory of Hilbert spaces; the entire family S_4^+ is important in the theory of Stieltjes integral equations of differential type (as indicated in [14, page 328]). In extrapolation of an idea introduced in [13], the (R-additive) function f from R to Y is said to be of *bounded variation with respect to the inner product* $\langle\cdot,\cdot\rangle$ provided there is a member h of S_0^+ and a member α of S_4^+ such that $|\langle f(t),\xi\rangle|^2 \leqslant h(t)\langle\xi,\alpha(t)\xi\rangle$, for each t in R and ξ in Y: let S_3 denote the set of all such functions f. In a third report it will be shown that the pattern of ideas from Theorems 1-1 through 10-1 can be extended to the family S_3, with loss only of the distributivity which is indicated in Theorem 10-1, but with a corresponding extension of Theorem 26. This proposed extension involves, in part, integrals of the Hellinger-type which were indicated in [13, pages 76-77], and includes the ideas introduced in [12] when there seemed to be a technical convenience in requiring that, for each t in some such collection as R, the $\alpha(t)$-image of Y be closed with respect to the norm $[\![\cdot]\!]$. It may be noted that Hellinger's ideas originally [4,5] involved a study of the scalar functions $g(t;\xi) = \langle f(t),\xi\rangle$ derived from the aforementioned functions f, a study incorporated in [24, Chapters V-VII] by use of Radon-Stieltjes integrals. Suppose, however, that R is the collection of all right-closed intervals lying in (0,1] and S_0^+ is identified as a subset of S_4^+ in the natural way and, for each β in S_4^+, H_β is the subset of S_3 clearly suggested above - it will be seen that if Y is infinite dimensional then there is a projection-valued member β of S_4^+ such that (1) $\beta((0,1])$ is the identity j^2 on Y, (2) if α is in S_0^+ and H_α is a subset of H_β then $\alpha = 0$, and (3) there is no member γ of S_0^+ such that H_β lies in H_γ: in this case, H_β consists of all R-additive f from R to Y such that if t is in R then f(t) is in the $\beta(t)$-image of Y, and $Q_\beta(f,g) = \int_{L/F}\langle f,g\rangle$ for f and g in H_β.

REFERENCES

1. W. M. Bogdanowicz, *Lebesgue-Bochner type spaces of group-valued summable functions and abstract integrals,* Queen's Papers Pure Appl. Math., 10(1967), 205-208.
2. W. M. Bogdanowicz and J. N. Welch, *Integration generated by projective volumes over Hilbert spaces,* Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 19(1971), 49-57.
3. N. Dunford and J. T. Schwartz, *Linear Operators Part I,* Interscience, New York, 1958.
4. E. Hellinger, *Die Orthogonalinvarianten quadratischen Formen von unendlichvielen Variablen,* Dissertation, Gottingen, 1907.
5. ———, *Neue Begrundung der Theorie quadratischer Formen von unendlichvielen Veranderlichen,* J. reine angew. Math., 136(1909), 210-271.

6. T. H. Hildebrandt, *On bounded linear functional operations,* Trans. Amer. Math. Soc., 36(1934), 868-875.
7. ———, *Linear operations on functions of bounded variation,* Bull. Amer. Math. Soc., 44(1938), 75.
8. ———, *Introduction to the Theory of Integration,* Academic Press, New York, 1963.
9. E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups,* Amer. Math. Soc. Colloq. Publ., (revised), Providence, 31(1957).
10. J. L. Kelley, Isaac Namioka, *et al., Linear Topological Spaces,* Van Nostrand, Princeton, 1963.
11. A. Kolmogoroff, *Untersuchungen uber den Integralbegriff,* Math. Ann., 103(1930), 654-696.
12. J. S. Mac Nerney, *Hellinger integrals in inner product spaces,* J. Elisha Mitchell Sci. Soc., 76(1960), 252-273.
13. ———, *Hermitian moment sequences,* Trans. Amer. Math. Soc., 103(1962), 45-81.
14. ———, *A linear initial value problem,* Bull. Amer. Math. Soc., 69(1963), 314-329.
15. ———, *Order-characterization of a pre-ring of subsets of a set,* Finitely Additive Set Functions I, Houston J. Math.
16. R. D. Mauldin, *A representation theorem of the second dual of C[0,1],* Studia Math., 46(1973), 197-200.
17. ———, *The continuum hypothesis, integration and duals of measure spaces,* Illinois J. Math.
18. J. Radon, *Theorie und Anwendungen der absolut additiven Mengenfunktionen,* Sitzber. Akad. Wiss. Wien, 122(1913), Abt. II a, 1295-1438.
19. J. A. Reneke, *On the existence and representation of integrals,* Illinois J. Math., 14(1970), 99-112.
20. C. E. Rickart, *General Theory of Banach Algebras,* Van Nostrand, Princeton, 1960.
21. F. Riesz, *Sur les operations fonctionelles lineaires,* Comptes Rendus Acad. Sci. Paris, (1909), 974-977.
22. F. Riesz and B. Sz.-Nagy, *Functional Analysis,* Ungar, New York, 1955, (translation of second French edition, 1952).
23. W. Rudin, *Functional Analysis,* McGraw-Hill, New York, 1973.
24. M. H. Stone, *Linear Transformations in Hilbert Space,* Amer. Math. Soc. Colloq. Publ., New York, 15(1932).
25. J. von Neumann, *Allgemeine Eigenwerttheroie Hermitescher Funktionaloperatoren,* Math. Ann., 102(1930), 49-131.
26. ———, *Functional Operators Vol. II,* Annals of Math. Studies, 22(1950), Princeton University Press, Princeton.
27. G. F. Webb, *Dual spaces of spaces of quasi-continuous functions,* Math. Nachr., 55(1973), 309-323.
28. W. H. Young, *On integration with respect to a function of bounded variation,* Proc. London Math. Soc., 13(1914), 109-150.

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FINITELY ADDITIVE SET FUNCTIONS
III. THE LINEAR SPAN OF A FAMILY OF FUNCTIONAL HILBERT SPACES

J. S. Mac Nerney*

ABSTRACT. Let R be a pre-ring of subsets of the set L filling up L, F be the family of all finite subcollections M of R such that no element of L belongs to two sets in M, $\{Y,\langle\cdot,\cdot\rangle\}$ be a complete inner product space with $\|\cdot\|$ the norm corresponding to the inner product $\langle\cdot,\cdot\rangle$, and $L(Y)^+$ be the set of all nonnegative Hermitian linear transformations in $\{Y,\langle\cdot,\cdot\rangle\}$. Suppose H is a linear space of finitely additive functions from R to Y, and for each $\{t,f\}$ in R × H, $P_tf(s)$ is 0 (in the space Y) or $\Sigma_{v \text{ in } M}f(v)$ accordingly as s does not intersect t or M is a member of F which fills up the common part of s and t. THEOREM. *In order that* Q *should be an inner product for* H *such that* (i) $\{H,Q\}$ *is complete,* (ii) *if* s *is in* R *then evaluation at* s *is continuous from* $\{H,Q\}$ *to* $\{Y,\langle\cdot,\cdot\rangle\}$, *and* (iii) *for each* t *in* R, P_t *maps* H *into* H *and is Hermitian with respect to* Q, *it is necessary and sufficient that there be a finitely additive function* α *from* R *to* $L(Y)^+$ *such that* (1) H *consists of all finitely additive functions* f *from* R *to* Y *such that if* u *is in* R *then* f(u) *is in* $\alpha(u)^{1/2}(Y)$ *and, for some* $b \geq 0$, $\Sigma_{v \text{ in } M}\|\alpha(v)^{-1/2}f(v)\|^2 \leq b$ *for each* M *in* F, *and* (2) *for each* $\{f,g\}$ *in* H × H, Q(f,g) *is the integral* $\int\langle\alpha^{-1/2}f,\alpha^{-1/2}g\rangle$ *over* L *relatively to the subdivision-refinement process* F. The class of all such spaces H_α, with an inner product Q as in the Theorem, is shown to be closed with respect to intersection and vector addition: this fact leads to integral representations for certain linear operations on the linear span S of a family of such spaces H_α.

Introduction. Throughout the body of this report, except where explicit relaxation of the condition is indicated, it is assumed (as in [12]) that R is a pre-ring of subsets of a set L filling up L, *i.e.*, that the collection R of subsets of the set L fills up L and has the property that, if G is a finite collection of members of R, there is a collection M of mutually exclusive members of R such that each set belonging to G is filled up by a finite subcollection of M. The letter F again stands for the family of all

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finite subcollections M of R such that no element of L belongs to two sets in M. If K is a member of R or is L itself, and h is a function from R to a normed linear space, the notion of the integral over K relatively to F of the function h is that introduced in [12].

However, $\{Y,\langle\cdot,\cdot\rangle\}$ is now taken to be a complete *complex* inner product space, *i.e.*, either a complex Euclidean space or a Hilbert space or a hyper-Hilbert space (in the terminology followed by J. von Neumann [25, especially 96-104] shortly after his introduction [23] of the phrase "a Hilbert space"): most of what is used here, concerning continuous (or bounded) linear transformations in and between spaces of this type, is adequately described in von Neumann's lectures [25] and in the book [20] by M. H. Stone, with some augmentation from the book [14] by F. Riesz and B. Szokefalvi-Nagy. The norm $[\![\cdot]\!]$ for Y is that arising from the inner product $\langle\cdot,\cdot\rangle$, so that $[\![\xi]\!] = \langle\xi,\xi\rangle^{1/2}$ for ξ in Y, L(Y) is the set of all linear transformations from Y to Y, and $L(Y)^c$ is the set of all members of L(Y) which are continuous with respect to $[\![\cdot]\!]$. If B is in $L(Y)^c$ then B* denotes the adjoint of B with respect to the inner product $\langle\cdot,\cdot\rangle$, and B^{-1} denotes the inverse of the restriction of B to the $[\![\cdot]\!]$-closure of the B*-image of Y; $L(Y)^+$ denotes the set of all nonnegative and Hermitian members of $L(Y)^c$, *i.e.*, all B in L(Y) such that if ξ is in Y then $\langle\xi,B\xi\rangle$ is a nonnegative real number; if B is in $L(Y)^+$ then $B^{1/2}$ denotes that square root of B which belongs to $L(Y)^+$, and $B^{-1/2}$ denotes $(B^{1/2})^{-1}$. Customary identification of complex scalars with members of $L(Y)^c$ leads to this notational convention: if B is a complex number then B* denotes the complex conjugate of B.

Now, for each set t in R, P_t is a transformation such that if k is a finitely additive function from R to Y or to L(Y) then P_tk is a function from R determined as follows: is s is in R, $P_tk(s) = 0$ or $\Sigma_{v \text{ in } M}k(v)$ accordingly as s does not intersect t or M is a member of F which fills up the intersection st of s with t. In [12] it was established, among other things, that if S_0 is the linear space of all finitely additive functions from R to Y which are of bounded variation with respect to $[\![\cdot]\!]$ and S_0 is coupled with the total variation norm $\|\cdot\|$ then members of the space E (of all continuous linear functions from $\{S_0,\|\cdot\|\}$ to $\{Y,[\![\cdot]\!]\}$) may be described in terms of integrals over L relatively to F: these representations were seen to arise from the existence of a family of complete inner product spaces (of finitely additive functions from R to Y) $\{H_\alpha, Q_\alpha\}$, indexed by the finitely additive functions α from R to the nonnegative real numbers such that $\int_{L/F} \alpha$ exists, having certain special properties including the following. (1) In each $\{H_\alpha, Q_\alpha\}$, if t is in R, evaluation at t is continuous from $\{H_\alpha, Q_\alpha\}$ to $\{Y, \langle\cdot,\cdot\rangle\}$ had the restriction to H_α of P_t is an orthogonal projection relatively to the inner product Q_α; (2) as α ranges over the indicated class, the spaces H_α constitute a lattice with respect to intersection and vector addition, and the linear span of the H_α is all of S_0.

The following Central Problem of the present report is thus seen to involve a pattern of ideas which includes the development in the preceding report [12] as an instance in which attention was focussed on bounded variation with respect to $\|\cdot\|$.

CENTRAL PROBLEM. Characterize those complete inner product spaces $\{H,Q\}$, of finitely additive functions from R to Y, such that if t is in R then evaluation at t is continuous from $\{H,Q\}$ to $\{Y, \langle\cdot,\cdot\rangle\}$ and the restriction to H of P_t maps H into H and is Hermitian with respect to Q; investigate intersections and vector sums of pairs of such spaces H (each with such an inner product Q); seek integral formulas for continuous linear transformations in and between such spaces $\{H,Q\}$, and for certain linear operations on the linear span S of a family of such spaces H; and investigate the possible existence of a significant norm for such a linear span S.

After the (essentially self-contained) presentation in Theorems 1-7 of some elementary arithmetic about complete inner product spaces $\{H,Q\}$ of functions from a (nonstructured) set R to Y such that evaluation at each element of the set R is a continuous linear transformation from $\{H,Q\}$ to $\{Y, \langle\cdot,\cdot\rangle\}$, all such spaces $\{H,Q\}$ as specified in the Central Problem are characterized. It is shown (*cf.* Abstract and Theorems 8-13) that the only such spaces are those Hellinger integral spaces generated by finitely additive functions from R to $L(Y)^+$. Namely, in order that $\{H,Q\}$ should be such a complete inner product space as indicated in the Problem, it is necessary and sufficient that there be a finitely additive function α from R to $L(Y)^+$ such that (i) H is the set H_α consisting of all finitely additive f from R to Y such that if u is in R then f(u) is in the $\alpha(u)^{1/2}$-image of Y and, for some nonnegative number b, if M is in F then $\Sigma_{v \text{ in } M}\|\alpha(v)^{-1/2}f(v)\|^2 \leq b$ and (ii) Q is the function Q_α from $H \times H$ such that if $\{f,g\}$ is in $H_\alpha \times H_\alpha$ then $Q_\alpha(f,g)$ is the integral $\int\langle\alpha^{-1/2}f,\alpha^{-1/2}g\rangle$ over L relatively to the family F, *i.e.*,

$$Q_\alpha(f,g) = \int_{L/F} h, \text{ with } h(u) = \langle\alpha(u)^{-1/2}f(u),\alpha(u)^{-1/2}g(u)\rangle \text{ for each u in R;}$$

moreover, for each finitely additive function α from R to $L(Y)^+$, the indicated set H_α is a linear family of functions and the foregoing formulas do define an inner product Q_α for H_α such that the space $\{H_\alpha,Q_\alpha\}$ is complete.

As a special instance of the results described in the preceding paragraph, it may be noted that, in case L is the real line and R is the pre-ring of all bounded right-closed intervals of real numbers and α is a finitely additive function from R to the (orthogonal) projections in $L(Y)^+$, the space $\{H_\alpha,Q_\alpha\}$ is as follows: H_α is the space of all finitely additive functions f from R to Y such that if u is in R then f(u) is in the $\alpha(u)$-image of Y and, for some nonnegative number b, if M is in F then $\Sigma_{v \text{ in } M}\|f(v)\|^2 \leq b$; and $Q_\alpha(f,g) = \int_{L/F}\langle f,g\rangle$ for each $\{f,g\}$ in $H_\alpha \times H_\alpha$.

In Theorems 14-20, there are developed integral representations for all the continuous linear transformations in and between (the pairs of) Hellinger integral spaces $\{H_\alpha,Q_\alpha\}$ generated by the members α of an unrestricte family Ω of finitely additive functions from R to $L(Y)^+$. Peculiar to the context of finitely additive set functions are the representations (Theorems 17 and 20) for the transformations which, for each set t in R, commute with the restrictions of P_t to the appropriate Hellinger integral spaces. In particular, for α and β in Ω such that H_α lies in (*i.e.*, is a subset of) H_β, the special transformation $\pi(\alpha,\beta)$ which is determined by $Q_\alpha(f,\pi(\alpha,\beta)g) = Q_\beta(f,g)$ for $\{f,g\}$ in $H_\alpha \times H_\beta$ is also given (Theorem 18) by

$$\pi(\alpha,\beta)g(s) = \int_{s/F}[\beta^{-1/2}\alpha]^*\beta^{-1/2}g \text{ with respect to } \|\cdot\|, \text{ for each s in R.}$$

In this same Theorem 18, it is shown that if α and β are in Ω then $H_{\alpha+\beta}$ is the vector sum $H_\alpha \dotplus H_\beta$ of H_α and H_β and that the formulas, for $\{t,\eta\}$ in R $\times$ Y,

$$(\alpha:\beta)(t)\eta = \int_{t/F}[(\alpha+\beta)^{-1/2}\alpha]^*[(\alpha+\beta)^{-1/2}\beta]\eta \text{ with respect to } \|\cdot\|,$$

define a finitely additive function $\alpha:\beta$ (the *parallel sum* of α and β) from R to $L(Y)^+$ such that $H_{\alpha:\beta}$ is the intersection of H_α with H_β and $Q_{\alpha:\beta} = Q_\alpha + Q_\beta$. For any β in Ω, the space of all continuous linear functions from $\{H_\beta,Q_\beta\}$ to $\{Y,\langle\cdot,\cdot\rangle\}$ is characterized (Theorem 15) in terms of the Hellinger operator integrals studied by Yu. L. Shmulyan

[17] (see Remark 2 following Theorem 15; the papers by Shmulyan [18] and by Habib Salehi [16] are concerned with the case of finite dimensional Y) and this new connection between those integrals and the space $\{H_\beta,Q_\beta\}$ is shown to arise as one case of the somewhat more general Theorem 5 reinforced by Theorem 14.

After Theorem 20 it is assumed that if α and β are such members of the set Ω, of finitely additive functions from R to $L(Y)^+$, that neither of H_α and H_β lies in the other then both of $\alpha+\beta$ and $\alpha:\beta$ belong to Ω. The linear span of the spaces H_α, for α in Ω, is denoted by $S(\Omega)$: there are integral representations for certain normed spaces of linear functions from $S(\Omega)$ to Y, and for certain normed algebras of linear transformations from $S(\Omega)$ to $S(\Omega)$. These algebras have (Theorem 25) isometric algebra-isomorphisms onto weakly closed algebras of continuous linear transformations in the direct sum $\{\Sigma_\Omega\{H,Q\},\hat{\Omega_\Omega}\}$ of the spaces $\{H_\alpha,Q_\alpha\}$ (for α in Ω). It is the space $S(\Omega)$ to which reference is made in the title of this report, as a linear span of a family of functional Hilbert spaces, although it may happen that some of the spaces $\{H_\alpha,Q_\alpha\}$ are finite dimensional and some fail to be separable.

Finally, in Theorem 26, it is further assumed that if α is in Ω and ξ is in Y then $\alpha\cdot\xi$ belongs to H_α (*i.e.*, the integral $\int_{L/F}\alpha\cdot\xi$ exists with respect to $[\![\cdot]\!]$): a norm $\|\cdot\|$ for $S(\Omega)$ is then described by such a procedure that, in the case [12] of each member of Ω being (real) scalar valued, $\|\cdot\|$ is the total variation norm. The miscellany of examples illustrates some effects of this and of other procedures.

Elementary Arithmetic of Kernel Systems. A *kernel system* (relatively to the space $\{Y,\langle\cdot,\cdot\rangle\}$) is a sequence $\{K,R,H,Q\}$ such that $\{H,Q\}$ is a complete inner product space of functions from the set R to the linear space Y and K is a function from $R \times R$ to $L(Y)$ such that if $\{t,\eta\}$ is in $R \times Y$ then $K(\cdot,t)\eta$ is in H and, for each f in H, $Q(f,K(\cdot,t)\eta) = \langle f(t),\eta\rangle$. The Examples 5 and 6, at the end of the present report, may be viewed as models of the general notion of a kernel system, described here but taken from [8, page 259].

In this section, no special structure need be assumed for the set R. All the results accumulated here in Theorems 1, 2, and 3, together with the special cases of Theorems 1^{sp}, 2^{sp}, and 3^{sp}, have been customary classroom exercises in my usual introductory Hilbert Spaces course since the paper [8] was written in 1955 (see Abstracts

728t-733t, Bull. Amer. Math. Soc., 61(1955), 537-539). Proofs may be based on the Hellinger-Toeplitz Theorem, as extended by von Neumann [24] (Stone's footnote [20, page iv], and Rudin's contemporary version [15, page 110], may be noted) and incorporated in the following Theorem which is stated here for easy reference.

THEOREM 0. *If the inner product space* $\{H_1,Q_1\}$ *is complete,* D *is a dense set in the inner product space* $\{H_2,Q_2\}$, A *is a function from* H_1 *to* H_2, *and* B *is a function from* D *to* H_1 *such that* $Q_1(x,By) = Q_2(Ax,y)$ *for each* $\{x,y\}$ *in* $H_1 \times D$, *then* (i) A *is linear and continuous from* $\{H_1,Q_1\}$ *to* $\{H_2,Q_2\}$, (ii) *there is only one linear extension* C *of* B *mapping* H_2 *to* H_1 *such that* $Q_1(x,Cy) = Q_2(Ax,y)$ *for each* $\{x,y\}$ *in* $H_1 \times H_2$, *and* (iii) *the norm of* A *is the norm of* C, *in the sense that if* $b \geqslant 0$ *then, in order that* $Q_2(Ax,Ax) \leqslant b^2 Q_1(x,x)$ *for each* x *in* H_1, *it is necessary and sufficient that* $Q_1(Cy,Cy) \leqslant b^2 Q_2(y,y)$ *for each* y *in* H_2.

INDICATION OF ELEMENTARY ARGUMENT FOR CONTINUITY. Let N_2 be the norm for H_2 corresponding to the inner product Q_2. Suppose k is a nonnegative number and x is a point in H_1 such that $k < N_2(Ax)$. Let $u = Ax/N_2(Ax)$, so that $N_2(u) = 1$ and $N_2(Ax) = Q_2(Ax,u)$: there is a number ϵ such that $0 < 2\epsilon < 1$ and such that if v is a point in H_2 and $N_2(v-u) < 2\epsilon$ then $k < |Q_2(Ax,v)|$. Let y be a point in D such that $N_2(y-(1-\epsilon)u) < \epsilon$:

$$N_2(y) \leqslant N_2(y-(1-\epsilon)u) + (1-\epsilon) < 1 \text{ and } N_2(y-u) \leqslant N_2(y-(1-\epsilon)u) + \epsilon < 2\epsilon,$$

so that $k < |Q_2(Ax,y)| = |Q_1(x,By)|$. There is a positive number r such that if N_1 is the norm for H_1 corresponding to Q_1 and z is in H_1 and $N_1(z-x) < r$ then

$$k < |Q_1(z,By)| = |Q_2(Az,y)| \leqslant N_2(Az)N_2(y) < N_2(Az).$$

Now, this lower semi-continuity with respect to N_1 of the composite of N_2 with A, coupled with the completeness of H_1 with respect to N_1, leads directly to the N_2-boundedness of the A-image of some N_1-open set in H_1; continuity then follows from the easily established linearity of the transformation A.

THEOREM 1. *If* $\{H,Q\}$ *is a complete inner product space of functions from the set* R *to the space* Y *then the following are equivalent:*

(1) *there is a dense linear subspace* D *of* $\{Y,\langle\cdot,\cdot\rangle\}$ *and a function* g *from* $R \times D$ *to* H *such that, for each* $\{f,t,\eta\}$ *in* $H \times R \times D$, $Q(f,g(t,\eta)) = \langle f(t),\eta\rangle$,

(2) *for each* t *in* R, *evaluation at* t *is a continuous linear transformation from the space* $\{H,Q\}$ *to the space* $\{Y,\langle\cdot,\cdot\rangle\}$, *and*

(3) *there is a function* K *from* R × R *to* L(Y) *such that if* $\{t,\eta\}$ *is in* R × Y *then* $K(\cdot,t)\eta$ *is in* H *and, for each* f *in* H, $Q(f,K(\cdot,t)\eta) = \langle f(t),\eta\rangle$.

THEOREM 1^{sp}. *If* Q *is an inner product for the linear subspace* H *of* Y *such that* $\{H,Q\}$ *is complete then the following are equivalent:*

(1) *there is a dense linear subspace* D *of* $\{Y,\langle\cdot,\cdot\rangle\}$ *and a function* g *from* D *to* H *such that, for each* $\{x,\eta\}$ *in* H × D, $Q(x,g(\eta)) = \langle x,\eta\rangle$,

(2) *the identity function is continuous from* $\{H,Q\}$ *into* $\{Y,\langle\cdot,\cdot\rangle\}$, *and*

(3) *there is a member* A *of* L(Y) *such that if* η *is in* Y *then* $A\eta$ *is in* H *and, for each* x *in* H, $Q(x,A\eta) = \langle x,\eta\rangle$.

Theorem 1 is a direct consequence of Theorem 0 as applied, for each t in R, to the pair $\{H,Q\}$ and $\{Y,\langle\cdot,\cdot\rangle\}$ of spaces, with A the function consisting of all ordered pairs $\{f,f(t)\}$ for f in H. Theorem 1^{sp} arises from Theorem 1 in the case that the set R is degenerate, with the identification of functions from R to Y as points in Y, and of functions from R × R to L(Y) as elements of L(Y).

TERMINOLOGY. In a kernel system $\{K,R,H,Q\}$, the function K is the *evaluation kernel* (or reproducing kernel, or kernel function, or kernel) in the space $\{H,Q\}$, and functions of the form $\Sigma_{t \text{ in } M}K(\cdot,t)x(t)$ (for functions x from finite subsets M of R to Y) are called *K-polygons*. The latter terminology seems appropriate from the observation that, for $K(s,t) = 1 + \inf\{s,t\}$ on $[0,1] \times [0,1]$, each function f on $[0,1]$ which is polygonal in the usual sense may be represented in this form: one has $f = \Sigma_0^n K(\cdot,u_p)x_p$ for some increasing number sequence $\{u_p\}_0^n$ with $u_0 = 0$ and $u_n = 1$, $f(u_0) = \Sigma_0^n x_p$, and $f(u_j) - f(u_{j-1}) = (u_j - u_{j-1})\Sigma_j^n x_p$ $(j = 1,\ldots,n)$.

OBSERVATIONS. If $\{K,R,H,Q\}$ is a kernel system then (*cf.* [8, page 256 ff.])

(1) for each $\{s,t\}$ in R × R and $\{\xi,\eta\}$ in Y × Y,

$$\langle K(t,s)\xi,\eta\rangle = Q(K(\cdot,s)\xi,K(\cdot,t)\eta) = \langle\xi,K(s,t)\eta\rangle,$$

so that, by Theorem 0, K(s,t) is in $L(Y)^c$ and $K(s,t)^* = K(t,s)$; it may be shown by this type of computation that $\{H,Q\}$ uniquely determines the function K.

(2) for each finite subset M of R and each function x from M to Y,

$$\Sigma_{\{s,t\} \text{ in } M \times M}\langle x(s),K(s,t)x(t)\rangle \geqslant 0$$

since this sum is $Q(\Sigma_{s \text{ in } M}K(\cdot,s)x(s),\Sigma_{t \text{ in } M}K(\cdot,t)x(t))$; the fact that each such double sum is real implies Observation 1, since complex scalars are assumed.

(3) The set of all K-polygons is a dense linear subspace of $\{H,Q\}$: if this were not dense then, since $\{H,Q\}$ is complete, there would be a nontrivial member f of H in the Q-orthogonal complement of that set - an immediate contradiction. Thus one sees that if f is in H then Q(f,f) is the least nonnegative number b with the property that if g is a K-polygon then $|Q(f,g)|^2 \leqslant b\, Q(g,g)$.

That the foregoing Observations serve to identify the space $\{H,Q\}$ in terms of the function K on R × R, is the essence of Theorem 2 (*cf.* [8, pages 257-258]).

THEOREM 2. *If* R *is a set and* K *is a function from* R × R *to* L(Y) *then, in order that there should exist a complete inner product space* $\{H,Q\}$ *of functions from* R *to* Y *such that if* $\{t,\eta\}$ *is in* R × Y *then* $K(\cdot,t)\eta$ *is in* H *and, for each* f *in* H, $Q(f,K(\cdot,t)\eta) = \langle f(t),\eta\rangle$, *it is necessary and sufficient that, for each function* x *from a finite subset* M *of* R *to* Y,

$$\Sigma_{\{s,t\} \text{ in } M \times M}\langle x(s),K(s,t)x(t)\rangle \geqslant 0;$$

in case this latter condition is satisfied, there is only one such complete space $\{H,Q\}$: *a function* f *from* R *to* Y *belongs to* H *only in case there is a nonnegative number* b *such that, for each function* x *from a finite subset* M *of* R *to* Y,

$$|\Sigma_{t \text{ in } M}\langle f(t),x(t)\rangle|^2 \leqslant b\Sigma_{\{s,t\} \text{ in } M \times M}\langle x(s),K(s,t)x(t)\rangle,$$

in which case Q(f,f) *is the least such number* b.

THEOREM 2^{sp}. *If* A *is a member of* L(Y) *then, in order that there should be a complete inner product space* $\{H,Q\}$ *such that* H *is a linear subspace of* Y *which includes* A(Y) *and such that if* $\{x,\eta\}$ *is in* H × Y *then* $Q(x,A\eta) = \langle x,\eta\rangle$, *it is necessary and sufficient that* A *belong to* $L(Y)^+$; *in case* A *does belong to* $L(Y)^+$, *then there is only one such complete space* $\{H,Q\}$: *for each member* B *of* $L(Y)^c$ *such that* $BB^* = A$, $H = B(Y)$ *and* $Q(x,y) = \langle B^{-1}x,B^{-1}y\rangle$ *for each* $\{x,y\}$ *in* H × H - *a point* z *of* Y *is in* H *only in case there is a nonnegative number* β *such that, if* η *is in* Y, $|\langle z,\eta\rangle| \leqslant \beta \|B^*\eta\|$, *in which case* $\|B^{-1}z\|$ *is the least such number* β.

A proof of those assertions in Theorem 2 not covered by the preceding set of Observations is given in [8, Theorem 2.5 and 2.7] and is based on identification of the sequential completion of the space of K-polygons with a space of functions; as with

Theorem 1^{sp}, Theorem 2^{sp} arises from Theorem 2 in the case that the set R is degenerate, with the identification of functions from R to Y as points in Y, and of functions from $R \times R$ to L(Y) as elements of L(Y). Alternatively, there is an argument [7, Theorem 1, page 665] for Theorem 2^{sp} which is readily accessible independently of the present somewhat more general considerations, and from which (as many students have seen in classroom exercises) the remaining assertions from Theorem 2 emerge in consequence. Some detail is included here because the pattern of argument seems essential to Theorems 10, 11, and 12 of the present report. It is, therefore, supposed that K is a function from $R \times R$ to L(Y) satisfying the nonnegativeness condition indicated in the statement of Theorem 2, that H is the set of all functions f from R to Y as there indicated, and that if f is in H then N(f) is the square root of the least nonnegative number b such that the indicated inequalities hold for each function x from a finite subset M of R to Y.

INDICATION OF LINEARITY AND NORMABILITY. As indicated in Observation 2, the K-image of $R \times R$ lies in $L(Y)^c$ and $K(s,t)^* = K(t,s)$ for each $\{s,t\}$ in $R \times R$. Since it is true that if f is in H then $|\langle f(t),\eta\rangle|^2 \leqslant N(f)^2\langle\eta,K(t,t)\eta\rangle$ for each $\{t,\eta\}$ in $R \times Y$, it follows that if $N(f) = 0$ then f is the constant 0 from R to the space Y. It follows directly from the definition of H and N that, if $\{f,g\}$ is in $H \times H$ and c is a scalar, then (i) f+g is in H and $N(f+g)^2 \leqslant [N(f)+N(g)]^2$ and (ii) cf is in H and $N(cf)^2 \leqslant |c|^2N(f)^2$: hence, H is linear and N is a norm on H.

INDICATION OF COMPLETENESS. Suppose that f is an infinite H-sequence which is (Cauchy) convergent with respect to the norm N. Letting ϵ be a positive number and j a positive integer such that $N(f_{j+m} - f_{j+n}) < \epsilon$ for m,n = 1,2,3,..., the first inequality in the preceding paragraph (with the completeness of $\{Y,\langle\cdot,\cdot\rangle\}$) yields a function g from R to Y which is the pointwise limit of the sequence f on R with respect to the norm $\|\cdot\|$. Now, for ϵ and j as above, it follows from the defining inequalities for N that (for n = 1,2,...) $g\text{-}f_{j+n}$ belongs to the space H and $N(g\text{-}f_{j+n}) \leqslant \epsilon$. Hence, with the help of the linearity of the space H, one sees that g itself belongs to H and is the N-limit of the infinite sequence f.

INDICATION OF THE INNER PRODUCT Q. Attention is now drawn to the collection R^{fi} of all finite subsets of R and also, for each set M in R^{fi}, to the complete inner product space $\{Y^M, \langle\cdot,\cdot\rangle_M\}$ of all functions from M to Y with the customary

$$\langle x,y\rangle_M = \Sigma_{t \text{ in } M}\langle x(t),y(t)\rangle \text{ for } \{x,y\} \text{ in } Y^M \times Y^M.$$

For each M in R^{fi} and f in the space H, let N(f;M) be the square root of the least nonnegative number b such that if x is in Y^M then

$$|\Sigma_{t \text{ in } M}\langle f(t),x(t)\rangle|^2 \leqslant b\Sigma_{\{s,t\} \text{ in } M \times M}\langle x(s),K(s,t)x(t)\rangle.$$

If f is in H and M_1 and M_2 are members of R^{fi} such that M_1 is a subset of M_2 then, since each function x from M_1 to Y has an extension z to M_2 such that z(t) = 0 in Y for each t in $M_2 - M_1$, it is clear that $N(f;M_1) \leqslant N(f;M_2)$: hence, for each f in H and $\epsilon > 0$, there is a member M_0 of R^{fi} such that

$$N(f) - \epsilon < N(f;M) \leqslant N(f) \text{ for every M in } R^{fi} \text{ which includes } M_0.$$

Now, with a view to applying Theorem 2^{sp} in each of the spaces $\{Y^M, \langle\cdot,\cdot\rangle_M\}$, it may be noted that if M is in R^{fi} then the restriction to M × M of K determines, by

$$A_M x(s) = \Sigma_{t \text{ in } M}K(s,t)x(t) \text{ for each x in } Y^M \text{ and s in M},$$

a member A_M of $L(Y^M)^+$: with the obvious notational conventions, if f is in H and M is in R^{fi}, it follows from the suggested application of Theorem 2^{sp} that $f|_M$ (the restriction to M of f) belongs to $A_M^{1/2}(Y^M)$ and that $\|A_M^{-1/2}(f|_M)\|_M = N(f;M)$. For f and g in H and M in R^{fi}, one has $Q_M(f,g) = \langle A_M^{-1/2}(f|_M), A_M^{-1/2}(g|_M)\rangle_M$ and

$$N(f+g;M)^2 - N(f-g;M)^2 = 4 \text{ Re } Q_M(f,g);$$

hence, one has an inner product Q for H such that $Q(f,f) = N(f)^2$ for each f in H and such that, if $\{f,g\}$ is in H × H and $\epsilon > 0$, there is an M_0 in R^{fi} such that

$$|Q(f,g) - Q_M(f,g)| < \epsilon \text{ for every M in } R^{fi} \text{ which includes } M_0.$$

INDICATION OF ACTION OF K IN {H,Q}. Suppose f is in H, t is in R, and η is in Y. Let M be any member of R^{fi} which contains t, and x be a member of Y^M such that if s is in M then $x(s) = \eta$ of 0 accordingly as s is or is not t. It may be seen that, in the notation of the preceding paragraph, $A_M x = K(\cdot,t)\eta|_M$ so that it follows from Theorem 2^{sp} (still as applied in $\{Y^M, \langle\cdot,\cdot\rangle_M\}$) that

$$Q_M(f,K(\cdot,t)\eta) = \langle A_M^{-1/2}(f|_M), A_M^{-1/2}(A_M x)\rangle_M = \langle f|_M, x\rangle_M = \langle f(t),\eta\rangle.$$

It follows from the definition of the function Q that $Q(f,K(\cdot,t)\eta) = \langle f(t),\eta\rangle$.

INDICATION OF UNIQUENESS OF $\{H,Q\}$. Supposing that $\{H_1,Q_1\}$ is also a complete inner product space of functions from R to Y in which K is the evaluation kernel, it follows (see Observation 3 preceding Theorem 2) that the set H_0 consisting of all K-polygons is dense both in $\{H,Q\}$ and in $\{H_1,Q_1\}$; moreover Q_1 agrees with Q on $H_0 \times H_0$. It follows that if f is an infinite H_0-sequence with limit g in $\{H,Q\}$ then f is (Cauchy) convergent in $\{H_1,Q_1\}$, and so has a limit h in $\{H_1,Q_1\}$: since

$$Q(f_n,K(\cdot,t)\eta) = \langle f_n(t),\eta\rangle = Q_1(f_n,K(\cdot,t)\eta) \text{ for } \{t,\eta\} \text{ in } R \times Y \ (n = 0,1,\ldots),$$

it follows that $h = g$. By symmetry, H_1 is a subset of H, and so is H. Now, by Observation 3 again, $Q_1(f,f) = Q(f,f)$ for each f in H. From the familiar

$$4 \operatorname{Re} Q_1(f,g) = Q(f+g,f+g) - Q(f-q,f-q) = 4 \operatorname{Re} Q(f,g)$$

for $\{f,g\}$ in $H \times H$, it follows that Q_1 is Q. This completes the argument.

THEOREM 3. *If* $\{K_1,R,H_1,Q_1\}$ *and* $\{K_2,R,H_2,Q_2\}$ *are kernel systems then*

(1) H_1 *is a subset of* H_2 *only in case there is a nonnegative number* b *such that, for each function* x *from a finite subset* M *of* R *to* Y,

$$\Sigma_{\{s,t\} \text{ in } M \times M}\langle x(s),K_1(s,t)x(t)\rangle \leqslant b\Sigma_{\{s,t\} \text{ in } M \times M}\langle x(s),K_2(s,t)x(t)\rangle,$$

and in this case the least such b *is the least nonnegative number* c *such that*

$$Q_2(f,f) \leqslant c\, Q_1(f,f) \textit{ for each } f \textit{ in } H_1,$$

(2) *there is a kernel system* $\{K_3,R,H_3,Q_3\}$ *such that* $K_3 = K_1+K_2$, H_3 *is the vector sum* $H_1 \dot{+} H_2$ *of* H_1 *and* H_2, *and if* h *is in* H_3 *then* $Q_3(h,h)$ *is the minimum value of* $Q_1(f,f)+Q_2(g,g)$ *for all* f *in* H_1 *and* g *in* H_2 *such that* $f+g = h$ *and*

(3) *there is a kernel system* $\{K_4,R,H_4,Q_4\}$ *such that* H_4 *is the common part* H_1H_2 *of* H_1 *and* H_2, $Q_4 = Q_1+Q_2$, *and* K_4 *is given by the formulas (with* Q_3 *as in (2))*

$$\begin{aligned}\langle\xi,K_4(s,t)\eta\rangle &= Q_3(K_1(\cdot,s)\xi,K_2(\cdot,t)\eta)\\ &= \tfrac{1}{4}\{\langle\xi,(K_1+K_2)(s,t)\eta\rangle\\ &\quad - Q_3((K_1-K_2)(\cdot,s)\xi,(K_1-K_2)(\cdot,t)\eta)\}\end{aligned}$$

for each $\{\xi,\eta\}$ *in* $Y \times Y$ *and* $\{s,t\}$ *in* $R \times R$.

THEOREM 3^{sp}. *If* A *and* B *are members of* $L(Y)^+$ *then*

(1) $A^{1/2}(Y)$ *is a subset of* $B^{1/2}(Y)$ *only in case there is a nonnegative number* b *such that, for each* ξ *in* Y, $\langle\xi,A\xi\rangle \leqslant b\langle\xi,B\xi\rangle$, *and in this case the least such* b *is the least*

$c \geqslant 0$ *such that* $\|B^{-1/2}x\|^2 \leqslant c\|A^{-1/2}x\|^2$ *for each* x *in* $A^{1/2}(Y)$,

(2) A+B *is a member* C *of* $L(Y)^+$ *such that* $C^{1/2}(Y)$ *is the vector sum of* $A^{1/2}(Y)$ *and* $B^{1/2}(Y)$, *and if* z *is in* $C^{1/2}(Y)$ *then* $\|C^{-1/2}z\|^2$ *is the minimum value of* $\|A^{-1/2}x\|^2+\|B^{-1/2}y\|^2$ *for all* $\{x,y\}$ *in* $A^{1/2}(Y) \times B^{1/2}(Y)$ *such that* $x+y = z$, *and*

(3) *there is a member* D *of* $L(Y)^+$ *such that* $D^{1/2}(Y)$ *is the common part of* $A^{1/2}(Y)$ *and* $B^{1/2}(Y)$, $\langle D^{-1/2}x, D^{-1/2}y\rangle = \langle A^{-1/2}x, A^{-1/2}y\rangle + \langle B^{-1/2}x, B^{-1/2}y\rangle$ *for each* x *and* y *in* $D^{1/2}(Y)$, *and* D *is given by the formulas (with* $C = A+B$*)*

$$D = [C^{-1/2}A]^*[C^{-1/2}B] = \tfrac{1}{4}\{C - [C^{-1/2}(A-B)]^*[C^{-1/2}(A-B)]\}.$$

As with Theorems 1^{sp} and 2^{sp}, Theorem 3^{sp} arises from Theorem 3 in the case that the set R is degenerate, with the identification of fucntions from R to Y as points in Y, and of functions from R $\times$ R to L(Y) as elements of L(Y).

INDICATION OF PROOF OF 3(1). The sufficiency is a consequence of Theorem 2. If H_1 is a subset of H_2 then $Q_1(f,K_1(\cdot,t)\eta) = \langle f(t),\eta\rangle = Q_2(f,K_2(\cdot,t)\eta)$ for all f in H_1 and $\{t,\eta\}$ in R $\times$ Y; density in $\{H_2,Q_2\}$ of the set of all K_2-polygons then makes Theorem 0 directly applicable (A being the identity function from H_1 to H_2).

INDICATION OF PROOF OF 3(2). By Theorem 0, 2, and 3(1), there exists a linear transformation A from H_3 to H_1 such that $Q_1(f,Ah) = Q_3(f,h)$ for all f in H_1 and h in H_3: in particular, for each $\{t,\eta\}$ in R $\times$ Y, $K_1(\cdot,t)\eta = A(K_3(\cdot,t)\eta)$ so that $(1-A)(K_3(\cdot,t)\eta) = K_2(\cdot,t)\eta$. Therefore, 1-A maps H_3 into H_2 and, for each $\{g,h\}$ in $H_2 \times H_3$, $Q_2(g,(1-A)h) = Q_3(g,h)$. Now, for each h in H_3,

$$Q_1(Ah,Ah) + Q_2((1-A)h,(1-A)h) = Q_3(Ah+(1-A)h,h) = Q_3(h,h).$$

If $\{f,g\}$ is in $H_1 \times H_2$ and x is a function from a finite subset M of R to Y then (with Σ''_M denoting $\Sigma_{\{s,t\} \text{ in } M \times M}$)

$$\begin{aligned} &|\Sigma_{t \text{ in } M}\langle f(t)+g(t),x(t)\rangle| \\ &\leqslant [Q_1(f,f)\Sigma''_M\langle x(s),K_1(s,t)x(t)\rangle]^{1/2} + [Q_2(g,g)\Sigma''_M\langle x(s),K_2(s,t)x(t)\rangle]^{1/2} \\ &\leqslant [Q_1(f,f) + Q_2(g,g)]^{1/2}[\Sigma''_M\langle x(s),(K_1+K_2)(s,t)x(t)\rangle]^{1/2}, \end{aligned}$$

so that (by Theorem 2) f+g is in H_3 and $Q_3(f+g,f+g) \leqslant Q_1(f,f) + Q_2(g,g)$.

INDICATION OF PROOF OF 3(3). Note that the linear transformation A from the preceding argument is Hermitian with respect to Q_3; moreover, the first formula defines $K_4(\cdot,t)\eta = (A-A^2)(K_3(\cdot,t)\eta)$, belonging to H_1 and to H_2 for each $\{t,\eta\}$ in $R \times Y$ and is equivalent to the second since $A-A^2 = \frac{1}{4}\{1-(2A-1)^2\}$. Since it is clear that Q_1+Q_2 is an inner product Q_4 for the common part H_4 of H_1 and H_2, such that the space $\{H_4,Q_4\}$ is complete, there remains only the computation

$$Q_4(f,K_4(\cdot,t)\eta) = Q_1(f,A(K_2(\cdot,t)\eta))+Q_2(f,(1-A)(K_1(\cdot,t)\eta)) = Q_3(f,K_3(\cdot,t)\eta),$$

which is $\langle f(t),\eta\rangle$, for f in H_4 and $\{t,\eta\}$ in $R \times Y$, to establish the result.

REMARK 1. An argument for Theorem 3 has been recorded by P. H. Jessner (his 1962 Dissertation [6, Chapter II], Theorems 2.1 and 2.2, and Corollaries), whose primary interest was the "approximate inclusion" relation between kernel systems: $\{H_1,Q_1\}$ is said to be approximately included in $\{H_2,Q_2\}$ provided that H_1 contains a subset of H_2 which is dense in $\{H_1,Q_1\}$. Among other things [6, Theorem 3.1], Jessner showed that this provision is equivalent to each of the following:

(1) there is a function Γ from $R \times R$ to $L(Y)^c$ such that if $\{s,t\}$ is in $R \times R$ then $\Gamma(s,t)^* = \Gamma(t,s)$ and, for each $\{\xi,\eta\}$ in $Y \times Y$, $\Gamma(\cdot,t)\eta$ is in H_2 and

$$\langle\xi,K_1(s,t)\eta\rangle = Q_2(\Gamma(\cdot,s)\xi,\Gamma(\cdot,t)\eta),$$

(2) H_2 is dense in the space $\{H_3,Q_3\}$ of the present Theorem 3(2), and

(3) if A is the transformation given in the Indication of Proof for Theorem 3(2) and C is the square root of $A-A^2$ which is Hermitian and nonnegative with respect to the inner product Q_3 and M is the function from $R \times R$ to $L(Y)^c$ given by the equations $M(s,t)\eta = C((K_1+K_2)(\cdot,t)\eta)(s)$ for $\{s,t\}$ in $R \times R$ and η in Y, then

$$\langle\xi,K_1(s,t)\eta\rangle = Q_2(M(\cdot,s)\xi,M(\cdot,t)\eta).$$

As I have shown elsewhere [10], if the space Y is not finite dimensional then this approximate inclusion relation is not transitive. Neither Jessner's results nor my example constitutes an integral part of the present development, but there is a Comment following Theorem 18 with an illustrative application of his main results.

TERMINOLOGY. If each of $\{K_1,R,H_1,Q_1\}$ and $\{K_2,R,H_2,Q_2\}$ is a kernel system, the *parallel sum of* K_1 *and* K_2, denoted by $K_1:K_2$, is the function K_4 described in Theorem 3(3), so that $\{K_4,R,H_4,Q_4\} = \{K_1:K_2,R,H_1H_2,Q_1+Q_2\}$. From the Observation 1 preceding Theorem 2, to the effect that if $\{K,R,H,Q\}$ is a kernel system then the function K is uniquely determined by $\{H,Q\}$, it is clear that parallel summation is both commutative and associative: *viz*., if each of $\{K_j,R,H_j,Q_j\}$ $(j = 1,2,3)$ is given then $K_1:K_2 = K_2:K_1$ since $H_1H_2 = H_2H_1$ and $Q_1+Q_2 = Q_2+Q_1$, and also $K_1:(K_2:K_3) = (K_1:K_2):K_3$ since $H_1(H_2H_3) = (H_1H_2)H_3$ and $Q_1+(Q_2+Q_3) = (Q_1+Q_2)+Q_3$.

REMARK 2. The foregoing terminology and notation are consistent with those used by W. N. Anderson and R. J. Duffin [1], who have investigated this idea in considerable depth for the matrix algebra case (R a finite set and Y the complex plane; or, alternatively, R a degenerate set and Y finite dimensional) using the generalized inverses of nonnegative definite Hermitian matrices. Attention should be drawn, in passing, to the penetrating study by M. R. Hestenes [5] of the idea of the generalized inverse (generalized reciprocal, pseudo inverse). An extension of the parallel sum idea to the intermediate case of Theorem 3^{sp} (Y unrestricted) has been recorded in 1971 by P. A. Fillmore and J. P. Williams [4, page 276 ff.]: there A:B was taken to mean $A^{1/2}[(A+B)^{-1/2}A^{1/2}]*[(A+B)^{-1/2}B^{1/2}]B^{1/2}$, and this is easily seen to be the D indicated in the former of the last two formulas in $3^{sp}(3)$ (another notation, $A:B = A(A+B)^{-1}B$, has already been justified [7, Theorem 4 and the Defintition, pages 666-667]; see, also , discussion on page 277 of [4]). It seems, from the comments and query bridging pages 279-280 of the latter paper [4], that the affirmative answer to the general question about the associativity of the parallel summation was not noticed in that intermediate context at that time.

REMARK 3. One of the remarkable discoveries by Anderson and Duffin [1] was that, in the notation of Theorem 3^{sp} with Y finite dimensional, if each of A, B, C, and D belongs to $L(Y)^+$ then so does $(A+C):(B+D) - (A:B + C:D)$. A digression seems in order here for the purpose of exhibiting *in retrospect* a general case of that Anderson-Duffin result in the form of a Corollary to the present Theorem 3.

COROLLARY TO THEOREM 3. (Anderson-Duffin). *If each of* K_1, K_2, K_3, *and* K_4 *is the evaluation kernel in a complete inner product space of functions from the set* R *to* $\{Y,\langle\cdot,\cdot\rangle\}$ *then so is the function* $(K_1+K_3):(K_2+K_4) - (K_1:K_2 + K_3:K_4)$.

PROOF. Suppose that (for $j = 1,2,3,4$) $\{K_j,R,H_j,Q_j\}$ is a kernel system, and that $\{K_0,R,H_1H_2\dot{+}H_3H_4,Q_0\}$ and $\{K_5,R,(H_1\dot{+}H_3)(H_2\dot{+}H_4),Q_5\}$ are the kernel systems with $K_0 = K_1:K_2 + K_3:K_4$ and $K_5 = (K_1+K_3):(K_2+K_4)$ provided by Theorem 3, and with $\dot{+}$ denoting vector sum. By Theorem 3, one has the systems $\{K_1{:}K_2,R,H_1H_2,Q_1+Q_2\}$ and $\{K_3{:}K_4,R,H_3H_4,Q_3+Q_4\}$; also, the assertion to be established is equivalent to the assertion that $H_1H_2\dot{+}H_3H_4$ is a subset of $(H_1\dot{+}H_3)(H_2\dot{+}H_4)$ and that $Q_5(h,h) \leqslant Q_0(h,h)$ for each h in $H_1H_2\dot{+}H_3H_4$. Inasmuch as the indicated inclusion is clear, let h be a member of $H_1H_2\dot{+}H_3H_4$. By Theorem 3(2), there exists a member f_0 of H_1H_2 and a member g_0 of H_3H_4 such that $f_0+g_0 = h$ and

$$Q_0(h,h) = (Q_1+Q_2)(f_0,f_0) + (Q_3+Q_4)(g_0,g_0).$$

Also by Theorem 3(2), there exists a member $\{f_1,g_1\}$ of $H_1 \times H_3$ and a member $\{f_2,g_2\}$ of $H_2 \times H_4$ such that $f_1+g_1 = h = f_2+g_2$ and

$$Q_5(h,h) = \{Q_1(f_1,f_1)+Q_3(g_1,g_1)\} + \{Q_2(f_2,f_2)+Q_4(g_2,g_2)\}$$

and such that both the following statements are true:

(i) $Q_1(f_1,f_1)+Q_3(g_1,g_1) \leqslant Q_1(f,f)+Q_3(g,g)$ for $\{f,g\}$ in $H_1 \times H_3$, $f+g = h$.

(ii) $Q_2(f_2,f_2)+Q_4(g_2,g_2) \leqslant Q_2(f,f)+Q_4(g,g)$ for $\{f,g\}$ in $H_2 \times H_4$, $f+g = h$.

As a particular consequence, there is the inequality:

$$Q_5(h,h) \leqslant \{Q_1(f_0,f_0)+Q_3(g_0,g_0)\}+\{Q_2(f_0,f_0)+Q_4(g_0,g_0)\} = Q_0(h,h).$$

Thus the equivalent assertion is established, and the Proof is complete.

REMARK 4. As a prelude to the next Theorem, one might consider the following special situation: let R be the set of all nonnegative integers and $C_0,C_1,\ldots$ be an infinite sequence of Hermitian members of $L(Y)^c$. The condition that, for each finite subset M of R, there should exist an interval $[a_M,b_M]$ of real numbers such that if x is a function from M to Y then (with Σ''_M denoting $\Sigma_{\{s,t\} \text{ in } M \times M}$)

$$a_M\Sigma''_M\langle s(x),C_{s+t}x(t)\rangle \leqslant \Sigma''_M\langle x(s),C_{s+t+1}x(t)\rangle \leqslant b_M\Sigma''_M\langle x(s),C_{s+t}x(t)\rangle$$

is equivalent [9] to the condition that C be a special type moment sequence. Only for the case that some interval $[\alpha,\beta]$ includes all the $[a_M,b_M]$ does such a moment sequence fall within the scope of Sz.-Nagy's Principal Theorem [22] concerning the representation of a *-semi-group by a family of *continuous* linear transformations in an extension space of $\{Y,\langle\cdot,\cdot\rangle\}$. In any case, however, with $K(s,t) = C_{s+t}$ for $\{s,t\}$ in $R \times R$, the kernel system $\{K,R,H,Q\}$ plays a central role, and $\{H,Q\}$ provided a realization of the extension space, with $C_0 = 1$ as added hypothesis. It may be noted that if $\{K,R,H,Q\}$ is any kernel system such that $K(\epsilon,\epsilon) = 1$ for some element ϵ of R then the equations $\lambda(\eta)(s) = K(s,\epsilon)\eta$, for $\{s,\eta\}$ in $R \times Y$, define a linear isometry λ from $\{Y,\langle\cdot,\cdot\rangle\}$ into $\{H,Q\}$, the Q-orthogonal projection π from H onto $\lambda(Y)$ being given simply by $\pi f = K(\cdot,\epsilon)f(\epsilon)$ for f in H.

THEOREM 4. *If* $\{K,R,H,Q\}$ *is a kernel system and* ϵ *is an element of the set* R *and* Z *is the* $\|\cdot\|$*-closure of* $K(\epsilon,\epsilon)^{1/2}(Y)$, *then*

(1) *the equations* $\lambda(\eta)(s) = [K(\epsilon,\epsilon)^{-1/2}K(\epsilon,s)]^*\eta$, *for* $\{\eta,s\}$ *in* $Z \times R$, *define a linear isometry* λ *from the space* $\{Z,\langle\cdot,\cdot\rangle\}$ *into* $\{H,Q\}$,

(2) *the equations* $\pi f(s) = [K(\epsilon,\epsilon)^{-1/2}K(\epsilon,s)]^*K(\epsilon,\epsilon)^{-1/2}f(\epsilon)$, *for* $\{f,s\}$ *in* $H \times R$, *define the* Q*-orthogonal projection* π *from* H *onto* $\lambda(Z)$, *and*

(3) *if* $\{f,\eta\}$ *is in* $H \times Z$ *then* $Q(f,\lambda(\eta)) = \langle K(\epsilon,\epsilon)^{-1/2}f(\epsilon),\eta\rangle$, *from which it follows that the* Q*-orthogonal complement of* $\lambda(Z)$ *in* H *is the subset of* H *to which the member* f *of* H *belongs only in case* $f(\epsilon) = 0$.

The result enunciated here as Theorem 4, is the first step established in the inductive proof of Lemma 11 in [9, page 58]: there, ϵ was the zero in the set R of all nonnegative integers, but the argument was independent of the character of R. That argument, it may be recalled, commences with two applications of the present Theorem 2^{sp} [9, Lemma 1] to show that if f is in H then $f(\epsilon)$ is in $K(\epsilon,\epsilon)^{1/2}(Y)$ and $\|K(\epsilon,\epsilon)^{-1/2}f(\epsilon)\|^2 \leqslant Q(f,f)$ so that, in particular, if s is in R and η is in Y then $\|K(\epsilon,\epsilon)^{-1/2}K(\epsilon,s)\eta\| \leqslant \|K(s,s)^{1/2}\eta\|$, whence $K(\epsilon,\epsilon)^{-1/2}K(\epsilon,s)$ is in $L(Y)^c$; the argument is essentially completed with identification of the function Γ,

$$\Gamma(s,t) = K(s,t) - [K(\epsilon,\epsilon)^{-1/2}K(\epsilon,s)]^*[K(\epsilon,\epsilon)^{-1/2}K(\epsilon,t)],$$

as a matrix representation of a Q-orthogonal projection in the sense of Theorem 6 to be enunciated presently (an alternate argument may be based on the Theorems 1 and 2). The following interim result serves to characterize, for a kernel system $\{K,R,H,Q\}$, those functions G from R to $L(Y)^c$ such that $G\cdot^*\xi$ is in H for ξ in Y; it will be the basis for Theorem 15, even as Theorem 4 will be for Theorem 14, and a proof is given for Theorem 5 which established a pattern adaptable to Theorem 6.

THEOREM 5. *If* $\{K,R,H,Q\}$ *is a kernel system then the equations,*

$$\sigma(\mu)(t)\eta = \mu(K(\cdot,t)\eta) \textit{ for } \{t,\eta\} \textit{ in } R \times Y,$$

define a reversible linear transformation σ *from the collection of all continuous linear* *functions* μ *from* $\{H,Q\}$ *to* $\{Y,\langle\cdot,\cdot\rangle\}$ *onto the collection of all functions* G *from* R *to* $L(Y)^c$ *such that* $G\cdot{}^*\xi$ *is in* H *for each* ξ *in* Y, *and if* $\{\mu,G\}$ *is in* σ *then*

(1) *for each* f *in* H *and* ξ *in* Y, $\langle\mu(f),\xi\rangle = Q(f,G\cdot{}^*\xi)$, *and*

(2) σ *is an isometry in the sense, that, if* $b \geqslant 0$, *these are equivalent:*

(i) $\|\mu(f)\|^2 \leqslant b^2Q(f,f)$ *for each* f *in* H, *and*

(ii) $Q(G\cdot{}^*\xi,G\cdot{}^*\xi) \leqslant b^2\|\xi\|^2$ *for each* ξ *in* Y.

PROOF. Suppose μ is a continuous linear function from $\{H,Q\}$ to $\{Y,\langle\cdot,\cdot\rangle\}$ and $b \geqslant 0$ such that $\|\mu(f)\|^2 \leqslant b^2Q(f,f)$ for each f in H: let G be a function from R to L(Y) such that $G(t)\eta = \mu(K(\cdot,t)\eta)$ for $\{t,\eta\}$ in $R \times Y$. Inasmuch as, for each such $\{t,\eta\}$, $\|G(t)\eta\|^2 \leqslant b^2\langle\eta,K(t,t)\eta\rangle$, G maps R into $L(Y)^c$. If x is a function from a finite subset M of R to Y then, for each ξ in Y,

$$|\Sigma_{t \text{ in } M}\langle G(t)^*\xi,x(t)\rangle|^2 = |\langle\xi,\mu(\Sigma_{t \text{ in } M}K(\cdot,t)x(t))\rangle|^2$$
$$\leqslant \|\xi\|^2b^2\Sigma_{\{s,t\} \text{ in } M\times M}\langle x(s),K(s,t)x(t)\rangle,$$

so that, by Theorem 2, $G\cdot{}^*\xi$ is in H and $Q(G\cdot{}^*\xi,G\cdot{}^*\xi) \leqslant b^2\|\xi\|^2$; moreover,

$$\langle\mu(\Sigma_{t \text{ in } M}K(\cdot,t)x(t)),\xi\rangle = \Sigma_{t \text{ in } M}\langle x(t),G(t)^*\xi\rangle = Q(\Sigma_{t \text{ in } M}K(\cdot,t)x(t),G\cdot{}^*\xi),$$

so that assertion (1) follows from the density in $\{H,Q\}$ of the set of K-polygons.

Suppose, now, that G is such a function from R to $L(Y)^c$ that $G\cdot{}^*\xi$ is in H for each ξ in Y: for functions x from finite subsets M of R to Y, the formulas

$$Q(\Sigma_{t \text{ in } M}K(\cdot,t)x(t),G\cdot{}^*\xi) = \Sigma_{t \text{ in } M}\langle x(t),G(t)^*\xi\rangle = \langle\lambda(\Sigma_{t \text{ in } M}K(\cdot,t)x(t)),\xi\rangle,$$

define a linear function λ from a dense subset of $\{H,Q\}$ to $\{Y,\langle\cdot,\cdot\rangle\}$ so that, by Theorem 0, there is only one linear extension μ of λ mapping H into Y such that $Q(f,G\cdot{}^*\xi) = \langle\mu(f),\xi\rangle$ for each $\{f,\xi\}$ in $H \times Y$, μ is continuous from $\{H,Q\}$ to the space $\{Y,\langle\cdot,\cdot\rangle\}$, and if $b \geqslant 0$ then conditions (i) and (ii) are equivalent.

REMARK 1. From Theorem 5 there is available some sharpening of Theorem 1, in the following sense. Suppose that, for some r in R, $\mu(f) = f(r)$ for f in H and let $G = \sigma(\mu)$: for each t in R, $G(t) = K(r,t)$. If ξ is in Y, $G\cdot{}^*\xi = K(\cdot,r)\xi$ and $Q(G\cdot{}^*\xi,G\cdot{}^*\xi) = \langle\xi,K(r,r)\xi\rangle$; hence, if $b \geqslant 0$, the following are equivalent:

(i) $\|f(r)\|^2 \leqslant b^2Q(f,f)$ for each f in H, and

(ii) $\langle\xi,K(r,r)\xi\rangle \leqslant b^2\|\xi\|^2$ for each ξ in Y.

REMARK 2. If, in the context of Theorem 5, L(H,Y) denotes the collection of all continuous linear functions from $\{H,Q\}$ to $\{Y,\langle\cdot,\cdot\rangle\}$ then the σ-image of $L(H,Y)$ is *complete* with respect to the norm indicated implicitly in Theorem 5(2); one may introduce an $L(Y)^c$-valued inner product for the σ-image of $L(H,Y)$ by the equations

$$Q(G_1\cdot{}^*\xi, G_2\cdot{}^*\eta) = \langle\xi,(G_1,G_2)\eta\rangle \text{ for } \{\xi,\eta\} \text{ in } Y \times Y,$$

whereupon $(\cdot,\cdot)$ is readily seen to have the following properties:

(i) $(G_1+G_2,G_3) = (G_1,G_3) + (G_2,G_3)$ and $(kG_1,G_2) = k(G_1,G_2)$ for k in $L(Y)^c$,

(ii) $(G_1,G_2)^* = (G_2,G_1)$, (G,G) is in $L(Y)^+$ and is 0 only in case G is 0,

(iii) $(G,K(t,\cdot)) = G(t)$ for each t in R, and

(iv) $|\langle\xi,(G_1,G_2)\eta\rangle|^2 \leqslant \langle\xi,(G_1,G_1)\xi\rangle\,\langle\eta,(G_2,G_2)\eta\rangle$ for each $\{\xi,\eta\}$ in $Y \times Y$,

and if μ is in $L(H,Y)$ and $G = \sigma(\mu)$, the aforementioned norm of G is the least nonnegative number b such that $\|(G,G)^{1/2}\xi\| \leqslant b\|\xi\|$ for each ξ in Y. In case Y is finite dimensional, this norm may be shown to be equivalent to that corresponding to the complex-valued inner product q: $q(G_1,G_2) =$ trace of (G_1,G_2).

REMARK 3. With $K(s,t) = 1$ or 0 accordingly as s is or is not t, $\{H,Q\}$ is seen to be the familiar "direct sum of R copies of the space $\{Y,\langle\cdot,\cdot\rangle\}$" described as follows: H is the set of all functions f from R to Y such that there is a nonnegative number b such that $\Sigma_{t \text{ in } M}\|f(t)\|^2 \leqslant b$ for each finite subset M of R, and $Q(f,g) = \Sigma_{t \text{ in } R}\langle f(t),g(t)\rangle$ for $\{f,g\}$ in $H \times H$ - in the sense that if ϵ is a positive number and $\{f,g\}$ is in $H \times H$ then there is a finite subset M_0 of R such that $|Q(f,g) - \Sigma_{t \text{ in } M}\langle f(t),g(t)\rangle| < \epsilon$ for each finite subset M of R which includes M_0. The final set (or range) of the transformation σ from Theorem 5 is the set of all functions G from R to $L(Y)^c$ such that if ξ is in Y then there is a nonnegative number β such that $\Sigma_{t \text{ in } M}\langle\xi,G(t)G(t)^*\xi\rangle \leqslant \beta$ for each finite subset M of R: assertion 5(1) may be read as $\langle\mu(f),\xi\rangle = \Sigma_{t \text{ in } R}\langle G(t)f(t),\xi\rangle$. It will follow from Theorem 15(1) that $\mu(f) = \Sigma_{t \text{ in } R}G(t)f(t)$ with respect to $\|\cdot\|$, but this may also be proved directly from Theorem 5 because of membership in $L(Y)^c$ of the weak (hence, strong) limit $\Sigma_{t \text{ in } R}G(t)G(t)^*$. Finally, in connection with the preceding Remark 2, the formulas $(G_1,G_2) = \Sigma_{t \text{ in } R}G_1(t)G_2(t)^*$ may be noted.

THEOREM 6. *If each of* $\{K_\alpha,R,H_\alpha,Q_\alpha\}$ *and* $\{K_\beta,R,H_\beta,Q_\beta\}$ *is a kernel system,* $m_{\alpha\beta}$ *is the set to which* Γ *belongs only in case* (i) Γ *is a function from* $R \times R$ *to the set* $L(Y)$ *and* (ii) *there is a nonnegative number* b *such that, if each of* x *and* y *is a function from a finite subset* M *of* R *to* Y,

$$|\Sigma_M''\langle x(s),\Gamma(s,t)y(t)\rangle|^2 \leqslant b^2\Sigma_M''\langle x(s),K_\alpha(s,t)x(t)\rangle\Sigma_M''\langle y(s),K_\beta(s,t)y(t)\rangle$$

(with Σ_M'' *denoting* $\Sigma_{\{s,t\}}$ in $M \times M$*), and* $T_{\alpha\beta}$ *is the space of all continuous linear transformations from* $\{H_\beta,Q_\beta\}$ *to* $\{H_\alpha,Q_\alpha\}$*, then the equations*

$$\Phi(C)(s,t)\eta = C(K_\beta(\cdot,t)\eta)(s), \textit{for } C \textit{ in } T_{\alpha\beta}, \{s,t\} \textit{ in } R \times R, \textit{and } \eta \textit{ in } Y,$$

define a reversible linear transformation Φ *from* $T_{\alpha\beta}$ *onto* $m_{\alpha\beta}$ *such that, if the ordered pair* $\{C,\Gamma\}$ *belongs to* Φ *and* A *is the (adjoint) transformation from* H_α *to* H_β *determined by* $Q_\alpha(f,Cg) = Q_\beta(Af,g)$ *for* $\{f,g\}$ *in* $H_\alpha \times H_\beta$*, then*

(1) *in order that the nonnegative number* b *should satisfy condition (ii), it is necessary and sufficinet that, for each* g *in* H_β, $Q_\alpha(Cg,Cg) \leqslant b^2Q_\beta(g,g)$,

(2) $\Gamma(s,t)^*\xi = A(K_\alpha(\cdot,s)\xi)(t)$ *for each* $\{s,t\}$ *in* $R \times R$ *and* ξ *in* Y,

(3) $\langle Af(t),\eta\rangle = Q_\alpha(f,\Gamma(\cdot,t)\eta)$ *for each* f *in* H_α *and* $\{t,\eta\}$ *in* $R \times Y$,

(4) $\langle\xi,Cg(s)\rangle = Q_\beta(\Gamma(s,\cdot)^*\xi,g)$ *for each* g *in* H_β *and* $\{s,\xi\}$ *in* $R \times Y$*, and*

(5) *in case* α *is* β *and each of* $\{C_1,\Gamma_1\}$ *and* $\{C_2,\Gamma_2\}$ *belongs to* Φ *and* $C = C_1,C_2$,

$$\langle\xi,\Gamma(s,t)\eta\rangle = Q_\beta(\Gamma_1(s,\cdot)^*\xi,\Gamma_2(\cdot,t)\eta) \textit{ for } \{s,t\} \textit{ in } R \times R,\ \{\xi,\eta\} \textit{ in } Y \times Y;$$

if, moreover, H_α *is a subset of* H_β *and* π *is the member of* $T_{\alpha\beta}$ *determined by the condition that* $Q_\alpha(f,\pi g) = Q_\beta(f,g)$ *for all* $\{f,g\}$ *in* $H_\alpha \times H_\beta$*, then* $\Phi(\pi) = K_\alpha$.

Theorem 6 arises as in instance of Theorem 3.1 of [8, pages 259-260], with the three Corollaries there indicated, but a proof may be constructed with the help of the present Theorems 0, 1, and 2 along the lines of the Proof which has been given for Theorem 5. The somewhat more general Theorem 3.1 of [8] allows a possibility of different underlying sets R_α and R_β for the two kernel systems: the present Theorem 5 is seen to arise therefrom by taking the first set R_α to be degenerate.

COROLLARY 6.1. *In the context of Theorem 6, if* Γ *is a function from* $R \times R$ *to* $L(Y)^c$ *then, in order that* Γ *should be such a member of* $m_{\alpha\beta}$ *that* $\Phi^{-1}(\Gamma)$ *is a linear isometry from* $\{H_\beta,Q_\beta\}$ *onto* $\{H_\alpha,Q_\alpha\}$*, it is necessary and sufficient that, for each* $\{s,t\}$ *in* $R \times R$ *and* $\{\xi,\eta\}$ *in* $Y \times Y$, $\Gamma(\cdot,s)\xi$ *be in* H_α, $\Gamma(t,\cdot)^*\eta$ *be in* H_β,

$$Q_\alpha(\Gamma(\cdot,s)\xi,\Gamma(\cdot,t)\eta) = \langle\xi,K_\beta(s,t)\eta\rangle,$$

and

$$Q_\beta(\Gamma(s,\cdot)^*\xi,\Gamma(t,\cdot)^*\eta) = \langle\xi,K_\alpha(s,t)\eta\rangle.$$

INDICATION OF PROOF. As to the necessity, with A the adjoint (as in

Theorem 6) of $C = \Phi^{-1}(\Gamma)$, the conditions are consequent to: $CA = 1$ on H_α, and $AC = 1$ on H_β. Regarding the sufficiency, supposing Γ as indicated, it must be shown that Γ belongs to $m_{\alpha\beta}$: if each of x and y is a function from the finite subset M of R to Y then (by Schwarz's inequality for Q_α, with Σ''_M as before)

$$|\Sigma''_M\langle x(s),\Gamma(s,t)y(t)\rangle|^2 = |Q_\alpha(\Sigma_{s \text{ in } M}K_\alpha(\cdot,s)x(s),\Sigma_{t \text{ in } M}\Gamma(\cdot,t)y(t))|^2$$
$$\leqslant \Sigma''_M\langle x(s),K_\alpha(s,t)s(t)\rangle\ \Sigma''_M\langle y(s),K_\beta(s,t)y(t)\rangle,$$

whence Γ is in $m_{\alpha\beta}$. Now, with C and A as before, if $\{s,t\}$ is in $R \times R$ and $\{\xi,\eta\}$ is in $Y \times Y$ then

$$\langle \xi,CA(K_\alpha(\cdot,t)\eta)(s)\rangle = Q_\beta(\Gamma(s,\cdot)^*\xi,AK_\alpha(\cdot,t)\eta) = Q_\beta(\Gamma(s,\cdot)^*\xi,\Gamma(t,\cdot)^*\eta),$$

and

$$\langle \xi,AC(K_\beta(\cdot,t)\eta)(s)\rangle = Q_\alpha(\Gamma(\cdot,s)\xi,CK_\beta(\cdot,t)\eta) = Q_\alpha(\Gamma(\cdot,s)\xi,\Gamma(\cdot,t)\eta),$$

so that $CA = 1$ on the K_α-polygons and $AC = 1$ on the K_β-polygons. Continuity and density, on and in the respective spaces, insure the asserted conclusion.

COROLLARY 6.2. *If, in the context of Theorem 6 with* $\alpha = \beta$, Γ *is a function from* $R \times R$ *to* $L(Y)$ *then, in order that* Γ *should be such a member of* $m_{\beta\beta}$ *that* $\Phi^{-1}(\Gamma)$ *is a* Q_β*-orthogonal projection, it is necessary and sufficient that, for each* $\{s,t\}$ *in* $R \times R$ *and* $\{\xi,\eta\}$ *in* $Y \times Y$, $\Gamma(\cdot,s)\xi$ *be in* H_β *and*

$$Q_\beta(\Gamma(\cdot,s)\xi,\Gamma(\cdot,t)\eta) = \langle \xi,\Gamma(s,t)\eta\rangle.$$

INDICATION OF PROOF. As to the necessity, with $C = \Phi^{-1}(\Gamma)$, the condition is a consequence of: $C = C^2$ and is Hermitian with respect to Q_β. Regarding the sufficiency, supposing Γ as indicated, if each of x and y is a function from the finite subset M of R to Y then (with Σ''_M as before), *seriatim*,

$$\Sigma''_M\langle y(s),\Gamma(s,t)y(t)\rangle = Q_\beta(\Sigma_{s \text{ in } M}\Gamma(\cdot,s)y(s),\Sigma_{t \text{ in } M}\Gamma(\cdot,t)y(t)) \geqslant 0$$

(so that Γ maps $R \times R$ into $L(Y)^c$ and $\Gamma(s,t)^* = \Gamma(t,s)$ for $\{s,t\}$ in $R \times R$),

$$\{\Sigma''_M\langle y(s),\Gamma(s,t)y(t)\rangle\}^2 = |Q_\beta(\Sigma_{s \text{ in } M}K_\beta(\cdot,s)y(s),\Sigma_{t \text{ in } M}\Gamma(\cdot,t)y(t))|^2$$
$$\leqslant \Sigma''_M\langle y(s),K_\beta(s,t)y(t)\rangle\ \Sigma''_M\langle y(s),\Gamma(s,t)y(t)\rangle,$$

$$\Sigma''_M\langle y(s),\Gamma(s,t)y(t)\rangle \leqslant \Sigma''_M\langle y(s),K_\beta(s,t)y(t)\rangle,$$

and

$$|\Sigma''_M\langle x(s),\Gamma(s,t)y(t)\rangle|^2 = |Q_\beta(\Sigma_{s \text{ in } M}K_\beta(\cdot,s)x(s),\Sigma_{t \text{ in } M}\Gamma(\cdot,t)y(t))|^2$$

$$\leqslant \Sigma_M''\langle x(s),K_\beta(s,t)x(t)\rangle\ \Sigma_M''\langle y(s),\Gamma(s,t)y(t)\rangle$$

$$\leqslant \Sigma_M''\langle x(s),K_\beta(s,t)x(t)\rangle\ \Sigma_M''\langle y(s),K_\beta(s,t)y(t)\rangle,$$

whence Γ is in $m_{\beta\beta}$. Now, with $C = \Phi^{-1}(\Gamma)$, C is Hermitian with respect to Q_β as a consequence of the fact noted above that $\Gamma(s,t)^* = \Gamma(t,s)$ for $\{s,t\}$ in $R \times R$. The assumed condition on Γ now implies that $C = C^2$ on the K_β-polygons and, as in the indicated argument for Corollary 6.1, this insures that $C = C^2$ on all of H_β.

THEOREM 6^{sp}. *Suppose each of* A *and* B *is in* $L(Y)^c$, m *is the set to which* G *belongs only in case* (i) G *is in* L(Y) *and* (ii) *there is a nonnegative number* b *such that, if* $\{\xi,\eta\}$ *is in* $Y \times Y$, $|\langle\xi,G\eta\rangle| \leqslant b\|A^*\xi\|\|B^*\eta\|$, Π_1 *is the orthogonal projection from* Y *onto the* $\|\cdot\|$*-closure of* $A^*(Y)$, Π_2 *is the orthogonal projection from* Y *onto the* $\|\cdot\|$*-closure of* $B^*(Y)$, *and* T *is the set of all members* D *of* $L(Y)^c$ *such that* $\Pi_1 D \Pi_2 = D$. *Then there is a reversible linear transformation* Ψ *from* T *onto* m *such that* $\Psi(D) = ADB^*$ *for each* D *in* T; *if, moreover,* $\{D,G\}$ *is in* Ψ,

(1) *in order that the nonnegative number* b *should satisfy condition (ii), it is necessary and sufficient that, for each* z *in* Y, $\|Dz\| \leqslant b\|z\|$, *and*

(2) *each of* $A^{-1}G$ *and* $B^{-1}G^*$ *is in* $L(Y)^c$ *and* $D = A^{-1}(B^{-1}G^*)^* = (B^{-1}(A^{-1}G)^*)^*$.

The foregoing proposition has been established as Lemma 3 in [9, page 49]; it may be seen to arise from Theorem 6 as follows. Direct translation of Theorem 6 yields $\Phi(C) = CBB^*$ for C in $T_{\alpha\beta}$, where (*cf.* Theorem 2^{sp}) A is a linear isometry from $\{\Pi_1(Y),\langle\cdot,\cdot\rangle\}$ onto $\{A(Y),Q_\alpha\}$ and B is a linear isometry from $\{\Pi_2(Y),\langle\cdot,\cdot\rangle\}$ onto $\{B(Y),Q_\beta\}$, and Φ maps $T_{\alpha\beta}$ onto the set m (corresponding to $m_{\alpha\beta}$ in Theorem 6) in the manner indicated. Now the function ζ, $\zeta(C) = A^{-1}CB$ for C in $T_{\alpha\beta}$, is a linear isometry from $T_{\alpha\beta}$ onto the set T: Ψ is the composite $\Phi[\zeta^{-1}]$ from T onto m. It may be noted that $\Pi_1 = A^{-1}A$ and $\Pi_2 = B^{-1}B$ (*cf.* Lemma 2 in [9, page 48]).

REMARK. For the case that $G = G^*$ in $L(Y)^c$ and A belongs to $L(Y)^+$ and there is a $b \geqslant 0$ such that $|\langle G\xi,\xi\rangle| \leqslant b\langle A^2\xi,\xi\rangle$ for each ξ in Y, the transformation D in T such that $D = D^*$ and $G = ADA$ ($B = A$ in Theorem 6^{sp}) was discovered as a consequence of other considerations in 1952 by B. Sz.-Nagy [21, pages 290-291] in an investigation of Hermitian moment sequences on a (bounded) number interval: a 1955 footnote by Sz.-Nagy [22, page 11] calls attention to those other considerations. A general application of the idea, with $B = A$ not necessarily in $L(Y)^+$, appears in [9, Lemma 6, page 53]; see [9, page 79 f.] for acknowledgement of relevant priorities.

THEOREM 7. *Suppose* $\{H_1,Q_1\}$ *and* $\{H_2,Q_2\}$ *are complete inner product spaces,* H_1 *is a linear subspace of* H_2, *and* π *is a function from* H_2 *to* H_1 *such that, for each* $\{f,g\}$ *in* $H_1 \times H_2$, $Q_1(f,\pi g) = Q_2(f,g)$. *With the notational conventions* (i) N_1 *and* N_2 *are the respective norms corresponding to the inner products* Q_1 *and* Q_2, (ii) *for* $j = 1$ *or* 2, T_j *is the algebra of continuous linear transformations in the space* $\{H_j,Q_j\}$, (iii) *if* $\{A,B\}$ *is in* $T_1 \times T_2$ *then* A' *is the adjoint of* A *with respect to* Q_1 *and* B'' *is the adjoint of* B *with respect to* Q_2, (iv) $\pi^{1/2}$ *is the square root of* π *which is Hermitian and nonnegative with respect to* Q_2, *and* (v) π^{-1} *is the inverse of the restriction of* π, *and* $\pi^{-1/2}$ *the inverse of the restriction of* $\pi^{1/2}$, *to the* N_2*-closure of* H_1, *the following statements are true:*

(1) H_1 *is* $\pi^{1/2}(H_2)$ *and* $Q_1(f,g) = Q_2(\pi^{-1/2}f,\pi^{-1/2}g)$ *for all* $\{f,g\}$ *in* $H_1 \times H_1$ *and if* $\{A,B\}$ *is in* $T_1 \times T_2$ *then* A *is a subset of* B *only in case* $A'\pi = \pi B''$;

(2) *if* B *is in* T_2 *and* $B(H_1)$ *lies in* H_1 *then the restriction to* H_1 *of* B *is in* T_1 *and has norm, with respect to* N_1, *the norm of* $\pi^{-1/2}B\pi^{1/2}$ *with respect to* N_2;

(3) *if* A *is in* T_1 *then* $\pi^{-1/2}A\pi^{1/2}$ *is in* T_2, $(\pi^{-1/2}A\pi^{1/2})'' = \pi^{-1/2}A'\pi^{1/2}$, *and* A *is the restriction to* H_1 *of a member of* T_2 *only in case* $A'\pi(H_2)$ *lies in* $\pi(H_2)$, *in which case* $\pi^{-1}A'\pi$ *belongs to* T_2 *and* A *is a subset of* $(\pi^{-1}A'\pi)''$;

(4) *if* B *is in* T_2 *then* $B\pi = \pi B$ *only in case there is a member* A *of* T_1 *such that* A *is a subset of* B *and* A' *is a subset of* B'';

(5) *if* A *is in* T_1 *then* $A\pi\pi^{1/2} = \pi A\pi^{1/2}$ *only in case there is a member* B *of* T_2 *such that* A *is a subset of* B *and* A' *is a subset of* B''.

A PROOF FOR 7(1). The formulas for H_1 and Q_1 are a consequence of Theorem 2^{sp} applied in the space $\{H_2,Q_2\}$ instead of $\{Y,\langle\cdot,\cdot\rangle\}$. If $\{A,B\}$ is in $T_1 \times T_2$,

$$Q_1(f,A'\pi g - \pi B''g) = Q_1(Af,\pi g) - Q_2(f,B''g) = Q_2(Af - Bf,g)$$

for all $\{f,g\}$ in $H_1 \times H_2$, so that $A = B$ on H_1 only in case $A'\pi = \pi B''$.

A PROOF FOR 7(2). In the presence of Theorem $3^{sp}(1)$, this is an application of Theorems 3 and 5 of [7, pages 666-667]. It may be argued directly, however, as follows. Assuming that B is in T_2 and $B(H_1)$ lies in H_1, Theorem $3^{sp}(1)$ applied in the space $\{H_2,Q_2\}$ assures the existence of a nonnegative number β such that

$$\text{if f is in } H_2 \text{ then } Q_2(f,[B\pi^{1/2}]\,[B\pi^{1/2}]''f) \leqslant \beta^2 Q_2(f,\pi f);$$

by the definition of the norm N_2, this is equivalent to

$$\text{if f is in } H_2 \text{ then } N_2([B\pi^{1/2}]''f) \leqslant \beta\, N_2(\pi^{1/2}f);$$

by Schwarz's inequality and the definition of $[B\pi^{1/2}]''$, this is equivalent to

$$\text{if } \{f,g\} \text{ is in } H_2 \times H_2 \text{ then } |Q_2(f,B\pi^{1/2}g)| \leqslant \beta\, N_2(\pi^{1/2}f)N_2(g);$$

by Theorem 2^{sp} applied in the space $\{H_2,Q_2\}$, this is equivalent to

$$\text{if g is in } H_2 \text{ then } N_2(\pi^{-1/2}B\pi^{1/2}g) \leqslant \beta\, N_2(g);$$

since $B\pi^{1/2}$ is 0 on the Q_2-orthogonal complement of H_1, this is equivalent to

$$\text{if g is in } H_2 \text{ then } N_2(\pi^{-1/2}B\pi^{1/2}g) \leqslant \beta\, N_2(\pi^{-1/2}\pi^{1/2}g);$$

finally, this says that if f is in H_1 then $N_1(Bf) \leqslant \beta\, N_1(f)$, and all is proved.

A PROOF FOR 7(3). Assuming that A is in T_1, if $\{f,g\}$ is in $H_2 \times H_2$ then

$$Q_1(\pi^{1/2}f,A\pi^{1/2}g) = Q_2(\pi^{-1/2}\pi^{1/2}f,\pi^{-1/2}A\pi^{1/2}g) = Q_2(f,\pi^{-1/2}A\pi^{1/2}g),$$

so that $\pi^{-1/2}A\pi^{1/2}$ is in T_2 and $(\pi^{-1/2}A\pi^{1/2})'' = \pi^{-1/2}A'\pi^{1/2}$. According to the Statement 7(1), if A is a subset of the member B of T_2 then $A'\pi = \pi B''$ so that $A'\pi(H_2)$ lies in $\pi(H_2)$. Suppose, now, that $A'\pi(H_2)$ lies in $\pi(H_2)$: by (7(2), the restriction to $H_0 = \pi(H_2)$ of A' is in the algebra T_0 determined by the inner product $Q_0(f,g) = Q_2(\pi^{-1}f,\pi^{-1}g)$ for $\{f,g\}$ in $H_0 \times H_0$. By applying the first part of this argument to the pair $\{T_0,T_2\}$, one sees that $\pi^{-1}A'\pi$ belongs to T_2 and one may consider the member $B = (\pi^{-1}A'\pi)''$ of T_2:

$$B\pi^{1/2} = [\pi^{1/2}(\pi^{-1}A'\pi)]'' = [\pi^{-1/2}A'\pi]'' = \pi^{1/2}(\pi^{-1/2}A'\pi^{1/2})'' = \pi^{1/2}\pi^{-1/2}A\pi^{1/2},$$

so that $B\pi^{1/2} = A\pi^{1/2}$, and this is what remained to be proved.

A PROOF FOR 7(4). Suppose that B is in T_2. By 7(1), if there is a member A of T_1 such that A is a subset of B and A' is a subset of B'', then $\pi B'' = B''\pi$ so that $B\pi = \pi B$. Suppose, now, that $B\pi = \pi B$: it follows that $\pi B'' = B''\pi$, and it is a property of nonnegative Hermitian square roots that $\pi^{1/2}$ commutes with B and with B''. By 7(2), the restriction to H_1 of B is in T_1, as is the restriction to H_1 of B''. Finally, if each of f and g is in the N_2-closure of H_1 then

$$Q_1(\pi^{1/2}f,B\pi^{1/2}g) = Q_1(\pi^{1/2}f,\pi^{1/2}Bg) = Q_2(f,Bg)$$

$$= Q_2(B''f,g) = Q_1(\pi^{1/2}B''f,\pi^{1/2}g)$$
$$= Q_1(B''\pi^{1/2}f,\pi^{1/2}g).$$

A PROOF FOR 7(5). Suppose A is in T_1. If there is a member B of T_2 such that A is a subset of B and A' is a subset of B'' then, by 7(4), $B\pi = \pi B$ so that $A\pi = \pi A$ on H_1, *i.e.*, $A\pi\pi^{1/2} = \pi A\pi^{1/2}$. Suppose, now, that $A\pi = \pi A$ on H_1; it is easily checked that the restriction to H_1 of $\pi^{1/2}$ is Hermitian and nonnegative with respect to Q_1. Therefore $A\pi^{1/2} = \pi^{1/2}A$ and $A'\pi^{1/2} = \pi^{1/2}A'$ on H_1: hence $\pi^{-1/2}A\pi^{1/2} = A$ and $\pi^{-1/2}A'\pi^{1/2} = A'$ on H_1. By 7(2), $\pi^{-1/2}A\pi^{1/2}$ belongs to T_2 and $(\pi^{-1/2}A\pi^{1/2})'' = \pi^{-1/2}A'\pi^{1/2}$, so that the proof is complete.

COROLLARY TO THEOREM 7. *If, with the suppositions of Theorem 7,* $\{H_1,Q_1\}$ *and* $\{H_2,Q_2\}$ *are complete inner product spaces of functions from a set* R *to the space* Y *with evaluation kernels* K_1 *and* K_2*, respectively, and* $\{A,B\}$ *is in* $T_1 \times T_2$*, then*

(1) *in order that* A *should be a subset of* B*, it is necessary and sufficient that if* $\{t,\eta\}$ *is in* $R \times Y$ *then* $A'(K_1(\cdot,t)\eta) = \pi B''(K_2(\cdot,t)\eta)$*, and*

(2) *if* A *is a subset of* B *then, in order that* A' *should be a subset of* B''*, it is necessary and sufficient that, for* $\{t,\eta\}$ *in* $R \times Y$, $B(K_1(\cdot,t)\eta) = \pi B(K_2(\cdot,t)\eta)$.

INDICATION OF PROOF. It is clear that $K_1(\cdot,t)\eta = \pi(K_2(\cdot,t)\eta)$ for each t in R and η in Y since, for each f in H_1,

$$Q_1(f,K_1(\cdot,t)\eta) = \langle f(t),\eta\rangle = Q_2(f,K_2(\cdot,t)\eta);$$

from the density (for j = 1 or 2) of the K_j-polygons in $\{H_j,Q_j\}$, assertions (1) and (2) are consequences, respectively, of (1) and (4) of Theorem 7.

REMARK 1. The supposition in Theorem 7 that $\{H_1,Q_1\}$ is *continuously included* (or continuously situated) in $\{H_2,Q_2\}$ is, according to Theorem 3(1), automatically satisfied with H_1 a subset of H_2 for the spaces of primary concern in the present context, spaces of Y-valued functions having evaluation kernels relatively to the inner product $\langle\cdot,\cdot\rangle$. It may be noted, as I have shown elsewhere [11], that if the complete inner product space $\{H_2,Q_2\}$ is continuously included in a complete inner product space $\{H,Q\}$ such that H_2 is a proper dense linear subspace of $\{H,Q\}$ then there exist two complete inner product spaces $\{H_1,Q_1\}$ and $\{H_3,Q_3\}$ such that

(i) $\{H_1,Q_1\}$ is continuously included in $\{H_2,Q_2\}$, $\{H_2,Q_2\}$ is continuously included in $\{H_3,Q_3\}$, and $\{H_3,Q_3\}$ is continuously included in $\{H,Q\}$, (ii) H_1 is dense in $\{H,Q\}$ and H_3 is not all of H, and (iii) H_1 is not dense in $\{H_2,Q_2\}$ and H_2 is not dense in $\{H_3,Q_3\}$. Hence, the formulation of Theorem 7 may not be tautologically amiss.

REMARK 2. The construction recalled in the preceding Remark was carried out [11] with the help of those portions of the 1959 results [7, Theorems 1-5] (and the 1962 results [9, Lemmas 1-3]) which are included in the present Theorems 2^{sp}, $3^{sp}(1)$, and 6^{sp}. Meanwhile, those earlier results have been noted separately by others (*e.g.*, by R. G. Douglas [2] and by Yu. L. Shmulyan [19]) and have been used effectively by Fillmore and Williams [4, Theorem 2.1 *et seq.*] in exposition about the lattice of operator ranges (or Julia varieties, or semiclosed-subspaces) in Y. As has been observed by Shmulyan [19, page 400], such results can be effective in an investigation of linear fractional transformations with $L(Y)^c$-coefficients [7].

Characterization of Hellinger Integral Spaces. From here onward, as indicated in the Introduction, it is assumed that R is a pre-ring of subsets of the set L filling up L, F is the family consisting of all finite subcollections M of R such that no element of L belongs to two sets in M, and (for each t in R) P_t is a transformation such that if k is a finitely additive function from R to Y or to L(Y) then P_tk is a function from R (to Y or to L(Y), respectively) determined as follows: if s is in R, $P_tk(s) = 0$ or $\Sigma_{v \text{ in } M}k(v)$ accordingly as s does not intersect t or M is a member of F which fills up st.

THEOREM 8. *If* $\{K,R,H,Q\}$ *is a kernel system such that each member of* H *is a finitely additive function from* R *to* Y *and such that, for each* t *in* R, P_t *maps* H *into* H *and the restriction of* P_t *to* H *is Hermitian with respect to* Q, *then the function* α, $\alpha(t) = K(t,t)$ *for each* t *in* R, *is finitely additive from* R *to* $L(Y)^+$ *and, for each* $\{s,t\}$ *in* $R \times R$, $K(s,t) = (P_t\alpha)(s)$.

PROOF. With α defined as indicated, if f is in H and $\{s,t\}$ is in $R \times R$ and η is in Y then $\alpha(t) = K(t,t)$, a member of $L(Y)^+$ by Theorem 2, and

$$Q(f,P_sK(\cdot,t)\eta) = \langle P_sf(t),\eta\rangle = \langle P_tf(s),\eta\rangle = Q(f,P_tK(\cdot,s)\eta)$$

so that $P_sK(\cdot,t)\eta = P_tK(\cdot,s)\eta$ and, for each member M of F filling up t,

$$\alpha(t) = \Sigma_{v \text{ in } M}P_vK(\cdot,t)(v) = \Sigma_{v \text{ in } M}P_tK(\cdot,v)(v) = \Sigma_{v \text{ in } M}K(v,v);$$

hence α is finitely additive from R to $L(Y)^+$. Now, if s and t are members of R, one of these cases arises: either s does not intersect t in which case

(i) $P_t\alpha(s) = 0 = P_tK(\cdot,s)(s) = P_sK(\cdot,t)(s) = K(s,t)$,

or s is a subset of t in which case

(ii) $P_t\alpha(s) = K(s,s) = P_tK(\cdot,s)(s) = P_sK(\cdot,t)(s) = K(s,t)$,

or s intersects t but is not a subset of t in which case there is a member M of F filling up s, with a subcollection W filling up the intersection st, so that

(iii) $P_t\alpha(s) = \Sigma_{v \text{ in } W}K(v,v) = \Sigma_{v \text{ in } W}P_tK(\cdot,v)(v)$
$= \Sigma_{v \text{ in } W}P_vK(\cdot,t)(v) = \Sigma_{v \text{ in } W}K(v,t) = \Sigma_{v \text{ in } M}K(v,t) = K(s,t)$.

THEOREM 9. *Suppose* α *is a finitely additive function from* R *to* $L(Y)^+$, *and* f *is a finitely additive function from* R *to* Y *such that if* v *is in* R *then* f(v) *is in* $\alpha(v)^{1/2}(Y)$. *Then, if* M *and* W *are members of* F *such that each set in* M *is filled up by a subcollection of* W,

$$\Sigma_{s \text{ in } M}\|\alpha(s)^{-1/2}f(s)\|^2 \leqslant \Sigma_{t \text{ in } W}\|\alpha(t)^{-1/2}f(t)\|^2.$$

Theorem 9 is a consequence of Theorem $3^{sp}(2)$, on the basis of which one sees that if U is a subcollection of W filling up the member s of M then

$$\|\alpha(s)^{-1/2}f(s)\|^2 \leqslant \Sigma_{v \text{ in } U}\|\alpha(v)^{-1/2}f(v)\|^2;$$

for scalar valued α this has been seen [12, Theorem 1] from Schwarz's inequality. The inequality from Theorem $3^{sp}(2)$ invoked here is: if each of A and B is in $L(Y)^+$ and x is in $A^{1/2}(Y)$ and y is in $B^{1/2}(Y)$, then x+y is in $(A+B)^{1/2}(Y)$ and

$$\|(A+B)^{-1/2}(x+y)\|^2 \leqslant \|A^{-1/2}x\|^2 + \|B^{-1/2}y\|^2.$$

The original form of this proposition was: if each of A and B is in $L(Y)^+$ and x is a member of A(Y) and y is a member of B(Y), such that x+y is in (A+B)(Y), then

$$\langle x+y,(A+B)^{-1}(x+y)\rangle \leqslant \langle x,A^{-1}x\rangle + \langle y,B^{-1}y\rangle$$

[8, Lemma 1.1 on page 254] (Abstract 728t, Bull. Amer. Math. Soc., 61(1955), 537).

THEOREM 10. *If* α *is a finitely additive function from* R *to* $L(Y)^+$ *and* f *is a finitely additive function from* R *to* Y *and* $b \geqslant 0$, *the following are equivalent:*

(1) *there is a real nonnegative finitely additive function* h *defined on* R *such that if* $\{t,\eta\}$ *is in* R × Y *then* $|\langle f(t),\eta\rangle|^2 \leqslant h(t)\langle\eta,\alpha(t)\eta\rangle$ *and* $\int_{L/F} h \leqslant b$,

(2) *if* M *is a member of* F *then, for each function* x *from* M *to* Y,

$$|\Sigma_{t \text{ in } M}\langle f(t),x(t)\rangle|^2 \leqslant b\ \Sigma_{t \text{ in } M}\langle x(t),\alpha(t)x(t)\rangle, \textit{ and}$$

(3) *if* v *is in* R *then* f(v) *is in* $\alpha(v)^{1/2}(Y)$ *and, for each member* M *of* F,

$$\Sigma_{t \text{ in } M}\|\alpha(t)^{-1/2}f(t)\|^2 \leqslant b.$$

PROOF. As in the Proof given for [12, Theorem 2], if condition (3) holds then it is a consequence of Theorem 9 that the equations $h(t) = \int_{t/F}\|\alpha^{-1/2}f\|^2$, for t in R, define a real nonnegative finitely additive function h on R and, by Theorems 2^{sp} and 9, if $\{t,\eta\}$ is in R × Y then

$$|\langle f(t),\eta\rangle|^2 \leqslant \|\alpha(t)^{-1/2}f(t)\|^2\langle\eta,\alpha(t)\eta\rangle \leqslant h(t)\langle\eta,\alpha(t)\eta\rangle.$$

The implication from (1) to (2) is a consequence of Schwarz's inequality coupled with the finitely additive character of h; if (2) holds then (3) is a consequence of Theorem 2^{sp} applied in the product space Y^M for each member M of the family F.

THEOREM 11. *If* α *is a finitely additive function from* R *to* $L(Y)^+$ *then the collection* H_α, *of all finitely additive functions* f *from* R *to* Y *such that (for some nonnegative number* b*) one of the three conditions in Theorem 10 holds, is a linear space of functions from* R *to* Y, *there is a norm* N_α *for* H_α *such that if* f *is in* H_α *then* $N_\alpha(f)^2 = \int_{L/F}\|\alpha^{-1/2}f\|^2$, *and* H_α *is complete with respect to* N_α.

Theorem 11 may be proved as a consequence of Theorems 9 and 10 with the help of the observation that if f is in H_α then $N_\alpha(f)$ is the square root of the least nonnegative number b such that one of the three conditions in Theorem 10 holds. The result may be viewed as a translation of relevant portions of [12, Theorem 3] from the context in which α was scalar valued and existence of $\int_{L/F}\alpha$ was given.

THEOREM 12. *If* α *is a finitely additive function from* R *to* $L(Y)^+$ *then there is an inner product* Q_α *for the space* H_α *(described in Theorem 11) such that*

$$Q_\alpha(f,g) = \int_{L/F}\langle\alpha^{-1/2}f,\alpha^{-1/2}g\rangle \textit{ for each } \{f,g\} \textit{ in } H_\alpha \times H_\alpha,$$

and if t *is in* R *then* P_t *maps* H_α *into* H_α *and, for each* {f,g} *in* $H_\alpha \times H_\alpha$,

$$Q_\alpha(P_tf,g) = \int_{t/F}\langle\alpha^{-1/2}f,\alpha^{-1/2}g\rangle = Q_\alpha(f,P_tg);$$

with $K_\alpha(s,t) = (P_t\alpha)(s)$ *for* {s,t} *in* R × R, $\{K_\alpha,R,H_\alpha,Q_\alpha\}$ *is a kernel system.*

PROOF. This argument is patterned after that given for [12, Theorem 6], and as in that argument the existence of the indicated function Q_α follows from

$$\Sigma_{u \text{ in } M}\|\alpha(u)^{-1/2}[f(u)+g(u)]\|^2 - \Sigma_{u \text{ in } M}\|\alpha(u)^{-1/2}[f(u)-g(u)]\|^2$$
$$= 4 \text{ Re } \Sigma_{u \text{ in } M}\langle\alpha(u)^{-1/2}f(u),\alpha(u)^{-1/2}g(u)\rangle$$

as a set of identities, $\{f,g\}$ in $H_\alpha \times H_\alpha$ and M in F. It is clear from this that if f is in H_α then $Q_\alpha(f,f) = N_\alpha(f)^2$, so that Q_α is an inner product for H_α with corresponding norm N_α: the space $\{H_\alpha,Q_\alpha\}$ is complete. If $\{f,g\}$ is in $H_\alpha \times H_\alpha$, the indicated integral formulas for $Q_\alpha(P_tf,g)$ and $Q_\alpha(f,P_tg)$ (for t in R) may be verified by considering members of F having subcollections filling up t. There remains only the verification that the indicated function K_α is the evaluation kernel in the space $\{H_\alpha,Q_\alpha\}$; if f is in H_α and $\{t,\eta\}$ is in R $\times$ Y then, for each member W of F having a subcollection M filling up t,

$$\Sigma_{u \text{ in } W}\langle\alpha(u)^{-1/2}f(u),\alpha(u)^{-1/2}\mathbf{K}_\alpha(u,t)\eta\rangle = \Sigma_{v \text{ in } M}\langle\alpha(v)^{-1/2}f(v),\alpha(v)^{1/2}\eta\rangle$$
$$= \Sigma_{v \text{ in } M}\langle f(v),\eta\rangle = \langle f(t),\eta\rangle,$$

so that $Q_\alpha(f,K_\alpha(\cdot,t)\eta) = \langle f(t),\eta\rangle$.

THEOREM 13. *If* H *is a linear space of finitely additive functions from* R *to* Y *then, in order that* Q *should be an inner product for* H *such that*

(i) *the space* {H,Q} *is complete,*

(ii) *if* s *is in* R, *evaluation at* s *is continuous from* {H,Q} *to* $\{Y,\langle\cdot,\cdot\rangle\}$, *and*

(iii) *for each* t *in* R, *the restriction to* H *of* P_t *is a Q-orthogonal projection,*

it is necessary and sufficient that there exist a finitely additive function α *from* R *to* $L(Y)^+$ *such that* {H,Q} *is the space* $\{H_\alpha,Q_\alpha\}$ *described in Theorems 11-12.*

Theorem 13 is a consequence of Theorem 1 and Theorems 8 through 12.

TERMINOLOGY. If α is a finitely additive function from R to $L(Y)^+$ then the *Hellinger integral space generated by* α is the space $\{H_\alpha,Q_\alpha\}$ from Theorems 11-12.

REMARK. Suppose that f is a finitely additive function from R to Y, W is a finite subcollection of R, and x is a function from W to Y. Let M be a member of F such that each set in W is filled up by a subcollection of M and each set in M is a subset of some set in W; if t is in W then M(t) denotes the set to which v belongs only in case v is a member of M lying in t, and if u is in M then W(u) is the set to which s belongs only in case s is a member of W which includes u. Let z be the function from

M to Y such that $z(u) = \Sigma_{x \text{ in } W(u)} x(s)$ for each u in M:

$$\Sigma_{t \text{ in } W}\langle f(t),x(t)\rangle = \Sigma_{t \text{ in } W}\Sigma_{v \text{ in } M(t)}\langle f(v),x(t)\rangle$$

$$= \Sigma_{u \text{ in } M}\Sigma_{s \text{ in } W(u)}\langle f(u),x(s)\rangle$$

$$= \Sigma_{u \text{ in } M}\langle f(u),z(u)\rangle.$$

If, moreover, α is a finitely additive function from R to $L(Y)^+$ and K is defined on $R \times R$ by $K(s,t) = P_t\alpha(s)$ for $\{s,t\}$ in $R \times R$, then

$$\Sigma_{\{s,t\} \text{ in } W\times W}\langle x(s),K(s,t)x(t)\rangle = \Sigma_{\{u,t\} \text{ in } M\times W}\langle z(u),K(u,t)x(t)\rangle$$

$$= \Sigma_{\{u,v\} \text{ in } M\times M}\langle z(u),K(u,v)z(v)\rangle,$$

and this latter sum is $\Sigma_{u \text{ in } M}\langle z(u),\alpha(u)z(u)\rangle$. This type of computation may be seen to establish a connection between the arguments given in support of Theorems 10 through 12 and the indication given for a Proof of Theorem 2: indeed, Theorem 2 may be regarded, in this way, as implicitly including Theorems 10, 11, and 12. It will be seen in Theorem 14, however, that the approximation process indicated for the inner product Q in connection with Theorem 2 has a special significance in the Hellinger integral spaces relatively to the subdivision refinement process F, one which allows a sharpening of assorted results recorded in Theorems 3, 5, and 6.

Special Continuous Linear Transformations. Let Ω now denote a collection of finitely additive functions from R to $L(Y)^+$. For each finitely additive function α from R to $L(Y)^+$, $\{H_\alpha,Q_\alpha\}$ is the Hellinger integral space generated by α, K_α is the evaluation kernel in the space $\{H_\alpha,Q_\alpha\}$ so that $K_\alpha(s,t) = P_t\alpha(s)$ for $\{s,t\}$ in $R \times R$, and N_α is the norm corresponding to the inner product Q_α so that $N_\alpha(f) = Q_\alpha(f,f)^{1/2} = \{\int_{L/F}\|\alpha^{-1/2}f\|^2\}^{1/2}$ for f in H_α.

THEOREM 14. *Suppose* β *is in* Ω *and, for each member* M *of* F, $\Pi_\beta(M)$ *is the function from* H_β *to* Y^R *such that if* f *is in* H_β *and* s *is in* R *then*

$$\Pi_\beta(M)f(s) = \Sigma_{t \text{ in } M}[\beta(t)^{-1/2}P_t\beta(s)]^*\beta(t)^{-1/2}f(t).$$

For each member M *of the family* F,

(1) $\Pi_\beta(M)$ *is the* Q_β*-orthogonal projection from* H_β *onto the subspace of* H_β *to which* g *belongs only in case there is a function* x *from* M *to* Y *such that, if* t *is in* M, x(t) *is in the* $\|\cdot\|$*-closure of the* $\beta(t)$*-image of* Y *and*

$$g(s) = \Sigma_{t \text{ in } M}[\beta(t)^{-1/2}P_t\beta(s)]^*x(t) \textit{ for each } s \textit{ in } R,$$

(2) *the* Q_β*-orthogonal complement of the* $\Pi_\beta(M)$*-image of* H_β *is the subset of* H_β *to which the member* f *of* H_β *belongs only in case* $f(t) = 0$ *for each* t *in* M, *and*

(3) *if* $\{f,g\}$ *belongs to* $H_\beta \times H_\beta$ *then*

$$Q_\beta(f - \Pi_\beta(M)f, g - \Pi_\beta(M)g) = Q_\beta(f,g) - \Sigma_{t \text{ in } M}\langle\beta(t)^{-1/2}f(t), \beta(t)^{-1/2}g(t)\rangle.$$

PROOF. Suppose β is in Ω and M is in F. With reference to Theorem 4, for each t in M, let Z_t be the $\|\cdot\|$-closure of the $\beta(t)$-image of Y, λ_t be the linear isometry from $\{Z_t, \langle\cdot,\cdot\rangle\}$ into $\{H_\beta, Q_\beta\}$ given by

$$\lambda_t(\eta)(s) = [\beta(t)^{-1/2}P_t\beta(s)]^*\eta \text{ for } \{\eta,s\} \text{ in } Z_t \times R,$$

and π_t be the Q_β-orthogonal projection from H_β onto $\lambda_t(Z_t)$ given by

$$\pi_t f(s) = [\beta(t)^{-1/2}P_t\beta(s)]^*\beta(t)^{-1/2}f(t) \text{ for } \{f,s\} \text{ in } H_\beta \times R.$$

If t is in M then it is known from Theorem 4(3) that

(i) $Q_\beta(f,\lambda_t(\eta)) = \langle\beta(t)^{-1/2}f(t),\eta\rangle$ for $\{f,\eta\}$ in $H_\beta \times Z_t$, and

(ii) the Q_β-orthogonal complement of $\lambda_t(Z_t)$ is the subspace of H_β to which the member f of H_β belongs only in case $f(t) = 0$.

Hence, if t and u are in M then $\pi_u\pi_t$ is the zero projection in the space $\{H_\beta, Q_\beta\}$. Inasmuch as $\Pi_\beta(M) = \Sigma_{t \text{ in } M}\pi_t$, all the assertions of the present Theorem may be seen as consequences of the foregoing facts - with the help of the formulas

$$Q_\beta(f - \Pi_\beta(M)f, g - \Pi_\beta(M)g) = Q_\beta(f,g) - \Sigma_{t \text{ in } M}Q_\beta(f, \lambda_t(\beta(t)^{-1/2}g(t)))$$

for $\{f,g\}$ in $H_\beta \times H_\beta$.

REMARK 1. The following approximation process is implicit in Theorem 14. If β is in Ω and f is in H_β then the equations,

$$h(t)(s) = [\beta(t)^{-1/2}P_t\beta(s)]^*\beta(t)^{-1/2}f(t) \text{ for } \{s,t\}, \text{ in } R \times R,$$

define a function h from R to H_β such that $f = \int_{L/F} h$ with respect to the norm N_β. Moreover, in case β is scalar valued (as in [12, Theorem 8]) and M is in F and f is in H_β, $\Pi_\beta(M)f = \Sigma_{t \text{ in } M}P_t\beta\cdot\xi(t)$ where, for each t in M, $\xi(t)$ is 0 or $f(t)/\beta(t)$ accordingly as $\beta(t)$ is the scalar zero or not.

REMARK 2. With reference to the displayed formula in Theorem 14(1), for β in Ω and M in F, it may be shown that $Q_\beta(g,g) = \Sigma_{t \text{ in } M}\|x(t)\|^2$; if, in particular, ξ is a function from M to Y such that $x(t) = \beta(t)^{1/2}\xi(t)$ for t in M then it may be seen from the following type of computation,

$$\begin{aligned}\langle[\beta(t)^{-1/2}P_t\beta(s)]^*\beta(t)^{1/2}\xi(t),\eta\rangle &= \langle\beta(t)^{1/2}\xi(t),\beta(t)^{-1/2}P_t\beta(s)\eta\rangle \\ &= \langle\xi(t),\beta(t)^{1/2}\beta(t)^{-1/2}P_t\beta(s)\eta\rangle \\ &= \langle\xi(t),P_t\beta(s)\eta\rangle,\end{aligned}$$

that $g = \Sigma_{t \text{ in } M} P_t\beta\cdot\xi(t)$ and that $Q_\beta(g,g) = \Sigma_{t \text{ in } M}\langle\xi(t),\beta(t)\xi(t)\rangle$. Now, it is known (*cf.* [7, Theorem 2], [8, Lemma 3.1], or [4, page 259]) that if A belongs to $L(Y)^+$ then, in order that the A-image of Y should be the $A^{1/2}$-image of Y, it is necessary and sufficient that the A-image of Y should be $\|\cdot\|$-closed: hence, unless Y is finite dimensional, it can not be proved that each such function g is of the latter form for some function ξ from M to Y.

REMARK 3. An R-*simple function* (determined by the member M of F) from L to Y is a function ξ from L to Y such that ξ is constant on each set in M and, if p is in L but not in any set in M, $\xi(p) = 0$: it may be noted that if ξ is determined by the member M_0 of F then ξ is also determined by each member M of F such that each set in M_0 is filled up by a subcollection of M. Suppose β is in Ω and c is a choice function from R, *i.e.*, if t is in R then c_t is an element of t: there is a function q_β such that if each of ξ_1 and ξ_2 is an R-simple function determined by the member M of F then $q_\beta(\xi_1,\xi_2) = \Sigma_{t \text{ in } M}\langle\xi_1(c_t),\beta(t)\xi_2(c_t)\rangle$. With the usual identification of a space h_β of equivalence classes of R-simple functions, one has an inner product space which may be denoted by $\{h_\beta,q_\beta\}$: the computations from the preceding Remark serve to identify a linear isometry δ from $\{h_\beta,q_\beta\}$ onto a dense linear subspace of $\{H_\beta,Q_\beta\}$, $\delta(\xi) = \Sigma_{t \text{ in } M} P_t\beta\cdot\xi(c_t)$ with the usual slurring of identification of functions with equivalence classes, so that $\{H_\beta,Q_\beta\}$ is seen as a completion of $\{h_\beta,q_\beta\}$. Indeed it may be seen that each continuous linear function λ from $\{h_\beta,q_\beta\}$ to the scalars has the form $\lambda(\xi) = \Sigma_{t \text{ in } M}\langle\xi(c_t),f(t)\rangle$ for some f in H_β, with ξ (representing an element of h_β) determined by the member M of F. An interpretation of the facts indicated in the latter part of Remark 2 is that, for *fixed* M in F, the set of equivalence classes having representatives determined by M need not be closed in the completion of $\{h_\beta,q_\beta\}$ if Y is infinite dimensional.

THEOREM 15. *If* β *is in* Ω *then the equations* $\sigma(\mu)(t)\eta = \mu(P_t\beta\cdot\eta)$, *for* t *in* R *and* η *in* Y, *define a reversible linear transformation* σ *from the collection of all continuous linear functions* μ *from* $\{H_\beta,Q_\beta\}$ *to* $\{Y,\langle\cdot,\cdot\rangle\}$ *onto the collection of all finitely additive functions* G *from* R *to* $L(Y)^c$ *such that if* ξ *is in* Y *then* $G\cdot{}^*\xi$ *belongs to* H_β, *and if* $\{\mu,G\}$ *is in* σ *then*

(1) *for* f *in* H_β *and* M *in* F, $\mu(\Pi_\beta(M)f) = \Sigma_{t \text{ in } M}[\beta(t)^{-1/2}g(t)^*]^*\beta(t)^{-1/2}f(t)$ *so that, for each* f *in* H_β, $\mu(f) = \int_{L/F}[\beta^{-1/2}G\cdot{}^*]^*\beta^{-1/2}f$ *with respect to* $\|\cdot\|$, *and*

(2) σ *is an isometry in the sense that, if* $b \geqslant 0$, *these are equivalent:*

(i) $\|\mu(f)\| \leqslant b\, N_\beta(f)$ *for each* f *in* H_β, *and*

(ii) $\|\int_{L/F}[\beta^{-1/2}G\cdot{}^*]^*[\beta^{-1/2}G\cdot{}^*]\xi\| \leqslant b^2\|\xi\|$ *for each* ξ *in* Y.

One may view Theorem 15 as an application of Theorem 5 (in the context of the present section), as reinforced by Theorem 14 together with the computations:

$$\langle\mu(\Pi_\beta(M)f),\xi\rangle = Q_\beta(\Pi_\beta(M)f,G\cdot{}^*\xi) = \Sigma_{t \text{ in } M}\langle\beta(t)^{-1/2}f(t),\beta(t)^{-1/2}G(t)^*\xi\rangle$$

for appropriate M, f, and ξ. The integral indicated in 15(2)(ii) is identified as the member B of $L(Y)^+$ such that $\mu(G\cdot{}^*\xi) = B\xi$ for each ξ in Y, as in 15(1); hence this integral exists as a strong limit in $L(Y)^c$. Further proof seems unnecessary.

REMARK 1. Suppose, as in Remark 1 following Theorem 5, that r is in R and $\mu(f) = f(r)$ for f in H_β, and let $G = \sigma(\mu)$: for each t in R, $G(t) = P_t\beta(r)$. If ξ is in Y then $Q_\beta(G\cdot{}^*\xi,G\cdot{}^*\xi) = \langle\xi,\beta(r)\xi\rangle$; if $b \geqslant 0$, these are equivalent:

(i) $\|f(r)\| \leqslant b\, N_\beta(f)$ for each f in H_β, and

(ii) $\|\beta(r)^{1/2}\xi\| \leqslant b\|\xi\|$ for each ξ in Y;

this provides some sharpening of Theorem 13, even as Theorem 5 does of Theorem 1.

REMARK 2. The $L(Y)^c$-valued inner product as suggested in Remark 2 following Theorem 5, adapted to the context of Theorem 15, takes the form

$$(G_1,G_2) = \int_{L/F}[\beta^{-1/2}G_1\cdot{}^*]^*[\beta^{-1/2}G_2\cdot{}^*],$$

the latter integrals existing as strong limits in $L(Y)^c$. It may be shown, with the help of [7, Theorem 5, page 667], that this is the Hellinger operator integral investigated by Yu. L. Shmulyan [17] - but not heretofore connected with the space $\{H_\beta,Q_\beta\}$. If k is a function from a member M of F to $L(Y)^c$ then there is a member $\{\mu,G\}$ of the transformation σ in Theorem 15 which arises from the formulas

$$\mu(f) = \Sigma_{r \text{ in } M} k(r)f(r) \text{ and } G = \Sigma_{r \text{ in } M} k(r)P_r\beta,$$

and it may be shown that $(G,G) = \Sigma_{r \text{ in } M} k(r)\beta(r)k(r)^*$. An analysis, analogous to that indicated in Remarks 2 and 3 following Theorem 14, would be available for R-simple functions from L to $L(Y)^c$, but details seem inappropriate at this point. Such an analysis does provide an alternative description of the integrals (G_1,G_2) as has been given, for the case of finite dimensional Y, by Shmulyan [18] and by Habib Salehi [16]: with finite dimensional Y, there is also available the complex inner product $q(G_1,G_2)$ = trace of (G_1,G_2) (*cf.* Remark 2 following Theorem 5).

THEOREM 16. *If each of* α *and* β *is in* Ω, $m_{\alpha\beta}$ *is the set to which* Γ *belongs only in case* Γ *is a function from* $R \times R$ *to* $L(Y)$ *such that* (i) *if* t *is in* R *then each of* $\Gamma(\cdot,t)$ *and* $\Gamma(t,\cdot)$ *is finitely additive and* (ii) *there is a nonnegative number* b *such that, if each of* x *and* y *is a function from a member* M *of* F *to* Y,

$$|\Sigma''_M\langle x(s),\Gamma(s,t)y(t)\rangle|^2 \leqslant b^2\Sigma_{s \text{ in } M}\|\alpha(s)^{1/2}x(s)\|^2\Sigma_{t \text{ in } M}\|\beta(t)^{1/2}y(t)\|^2$$

(with Σ''_M *denoting* $\Sigma_{\{s,t\} \text{ in } M\times M}$*), and* $T_{\alpha\beta}$ *is the space of all continuous linear transformations from* $\{H_\beta,Q_\beta\}$ *to* $\{H_\alpha,Q_\alpha\}$, *then the equations*

$$\Phi(C)(s,t)\eta = C(P_t\beta\cdot\eta)(s), \textit{for } C \textit{ in } T_{\alpha\beta}, \{s,t\} \textit{ in } R \times R, \textit{and } \eta \textit{ in } Y,$$

define a reversible linear transformation Φ *from* $T_{\alpha\beta}$ *onto* $m_{\alpha\beta}$ *such that, if the ordered pair* $\{C,\Gamma\}$ *belongs to* Φ *and* A *is the (adjoint) transformation from* H_α *to* H_β *determined by* $Q_\alpha(f,Cg) = Q_\beta(Af,g)$ *for* $\{f,g\}$ *in* $H_\alpha \times H_\beta$, *then*

(1) *in order that the nonnegative number* b *should satisfy condition (ii), it is necessary and sufficient that, for each* g *in* H_β, $N_\alpha(Cg) \leqslant b\, N_\beta(g)$,

(2) $\Gamma(s,t)^*\xi = A(P_s\alpha\cdot\xi)(t)$ *for each* $\{s,t\}$ *in* $R \times R$ *and* ξ *in* Y,

(3) *if* f *is in* H_α *then the equations* $h(t)(s) = [\alpha(t)^{-1/2}\Gamma(t,s)]^*\alpha(t)^{-1/2}f(t)$, *for* $\{s,t\}$ *in* $R \times R$, *define a function* h *from* R *to* H_β *such that* $Af = \int_{L/F} h$ *with respect to* N_β *so that, for each* s *in* R,

$$Af(s) = \int_{L/F}[\alpha^{-1/2}\Gamma(\cdot,s)]^*\alpha^{-1/2}f \textit{ with respect to } \|\cdot\|,$$

(4) *if* g *is in* H_β *then the equations* $h(t)(s) = [\beta(t)^{-1/2}\Gamma(s,t)^*]^*\beta(t)^{-1/2}g(t)$, *for* $\{s,t\}$ *in* $R \times R$ *define a function* h *from* R *to* H_α *such that* $Cg = \int_{L/F} h$ *with respect to* N_α *so that, for each* s *in* R,

$$Cg(s) = \int_{L/F} [\beta^{-1/2}\Gamma(s,\cdot)^*]^*\beta^{-1/2}g \textit{ with respect to } [\![\cdot]\!], \textit{ and}$$

(5) *in case* α *is* β *and each of* $\{C_1,\Gamma_1\}$ *and* $\{C_2,\Gamma_2\}$ *belongs to* Φ *and* $C = C_1C_2$, *if* $\{s,t\}$ *is in* $R \times R$ *and* η *is in* Y *then*

$$\Gamma(s,t)\eta = \int_{L/F} [\beta^{-1/2}\Gamma_1(s,\cdot)^*]^*[\beta^{-1/2}\Gamma_2(\cdot,t)]\eta \textit{ with respect to } [\![\cdot]\!].$$

INDICATION OF PROOF. To see this Theorem as in interpretation of Theorem 6 in the context of the present section, one may first note that if M is a finite subset of R then there is a member W of F such that each set in M is filled up by a subcollection of W: hence, the set $m_{\alpha\beta}$ described here is the $m_{\alpha\beta}$ from Theorem 6. Therefore, the computations (with s in R)

$$\langle A\Pi_\alpha(M)f(s),\eta\rangle = Q_\alpha(\Pi_\alpha(M)f,\Gamma(\cdot,s)\eta) = \Sigma_{t \text{ in } M}\langle\alpha(t)^{-1/2}f(t),\alpha(t)^{-1/2}\Gamma(t,s)\eta\rangle,$$

$$\langle\xi,C\Pi_\beta(M)g(s)\rangle = Q_\beta(\Gamma(s,\cdot)^*\xi,\Pi_\beta(M)g) = \Sigma_{t \text{ in } M}\langle\beta(t)^{-1/2}\Gamma(s,t)^*\xi,\beta(t)^{-1/2}g(t)\rangle,$$

for appropriate M, f, g, ξ, and η, serve to make Theorem 14 applicable and so all assertions through 16(5) are seen to be consequences of corresponding ones from 6.

COROLLARY 16.1. *If, in the context of Theorem 16,* Γ *is a function from* $R \times R$ *to* $L(Y)^c$ *then, in order that* Γ *should be such a member of* $m_{\alpha\beta}$ *that* $\Phi^{-1}(\Gamma)$ *is a linear isometry from* $\{H_\beta,Q_\beta\}$ *onto* $\{H_\alpha,Q_\alpha\}$, *it is necessary and sufficient that, for each* $\{s,t\}$ *in* $R \times R$ *and* $\{\xi,\eta\}$ *in* $Y \times Y$, $\Gamma(\cdot,s)\xi$ *be in* H_α, $\Gamma(t,\cdot)^*\eta$ *be in* H_β,

$$\int_{L/F}\langle\alpha^{-1/2}\Gamma(\cdot,s)\xi,\alpha^{-1/2}\Gamma(\cdot,t)\eta\rangle = \langle\xi,P_t\beta(s)\eta\rangle,$$

and

$$\int_{L/F}\langle\beta^{-1/2}\Gamma(s,\cdot)^*\xi,\beta^{-1/2}\Gamma(t,\cdot)^*\eta\rangle = \langle\xi,P_t\alpha(s)\eta\rangle.$$

Corollary 16.1 is an interpretation of Corollary 6.1 in the context of the present section, and may be argued from Theorem 16 even as Corollary 6.1 was shown to follow from Theorem 6.

REMARK. It was shown by F. Riesz in 1910 (*cf.* Lemma on page 75 of [14]) that the Lebesgue spaces of (equivalence classes of) square-summable measurable scalar functions may always be realized as Hellinger integral spaces. Thus, the present Corollary 16.1 may be seen to include S. Bochner's 1934 Theorem [14, page 291 f.] on representing the unitary transformations in such spaces: Bochner's cited Theorem corresponds to the case wherein (i) Y is the complex plane, (ii) R is the pre-ring of all bounded right-closed intervals on the real line L, and (iii) each of α and β is the restriction to R of Lebesgue measure, *i.e.*, is ordinary length.

COROLLARY 16.2. *If, in the context of Theorem 16 with* $\alpha = \beta$, Γ *is a function from* $R \times R$ *to* L(Y) *then, in order that* Γ *should be such a member of* $m_{\beta\beta}$ *that* $\Phi^{-1}(\Gamma)$ *is a* Q_β*-orthogonal projection, it is necessary and sufficient that, for each* $\{s,t\}$ *in* $R \times R$ *and* $\{\xi,\eta\}$ *in* $Y \times Y$, $\Gamma(\cdot,s)\xi$ *be in* H_β *and*

$$\int_{L/F} \langle \beta^{-1/2}\Gamma(\cdot,s)\xi, \beta^{-1/2}\Gamma(\cdot,t)\eta\rangle = \langle \xi, \Gamma(s,t)\eta\rangle.$$

Corollary 16.2 is an interpretation of Corollary 6.2 in the context of the present section, and may be argued from Theorem 16 with the help of the following: if Γ satisfies the indicated conditions then, for each function x from a finite subset M of R to Y, (with Σ''_M denoting $\Sigma_{\{s,t\} \text{ in } M \times M}$)

$$\Sigma''_M \langle x(s), \Gamma(s,t)\mathbf{x}(t)\rangle = Q_\beta(\Sigma_{s \text{ in } M}\Gamma(\cdot,s)x(s), \Sigma_{t \text{ in } M}\Gamma(\cdot,t)x(t)) \geqslant 0,$$

whence Γ maps $R \times R$ into $L(Y)^c$, $\Gamma(s,t)^* = \Gamma(t,s)$ for $\{s,t\}$ in $R \times R$, and if s is in R then $\Gamma(s,\cdot)$ is finitely additive.

REMARK. As an illustration of Corollary 16.2, with reference to Theorem 14, it may be noted that if β is in Ω and M is in F then, for each $\{s,t\}$ in $R \times R$,

$$\Phi(\Pi_\beta(M))(s,t) = \Sigma_{u \text{ in } M}[\beta(u)^{-1/2}P_u\beta(s)]^*[\beta(u)^{-1/2}P_u\beta(t)].$$

THEOREM 17. *If each of* α *and* β *is in* Ω, $m_{\alpha\beta}(P)$ *is the set to which* G *belongs only in case* (i) G *is a finitely additive function from* R *to* L(Y) *and* (ii) *there is a nonnegative number* b *such that*

$$|\langle \xi, G(t)\eta\rangle|^2 \leqslant b^2\langle \xi, \alpha(t)\xi\rangle\,\langle \eta, \beta(t)\eta\rangle \text{ for each } t \text{ in } R \text{ and } \{\xi,\eta\} \text{ in } Y \times Y,$$

and $T_{\alpha\beta}(P)$ *is the space of all continuous linear transformations* C *from* $\{\mathbf{H}_\alpha, Q_\beta\}$ *to* $\{H_\alpha, Q_\alpha\}$ *such that* $C(P_t g) = P_t(Cg)$ *for each* $\{t,g\}$ *in* $R \times H_\beta$, *then the equations*

$$\Psi(C)(t)\eta = C(P_t\beta\cdot\eta)(t), \text{ for } C \text{ in } T_{\alpha\beta}(P),\ t \text{ in } R, \text{ and } \eta \text{ in } Y,$$

define a reversible linear transformation Ψ *from* $T_{\alpha\beta}(P)$ *onto* $m_{\alpha\beta}(P)$ *such that, if the ordered pair* $\{C,G\}$ *belongs to* Ψ *and* A *is the (adjoint) transformation from* H_α *to* H_β *determined by* $Q_\alpha(f,Cg) = Q_\beta(Af,g)$ *for* $\{f,g\}$ *in* $H_\alpha \times H_\beta$, *then (with* Φ *as in Theorem 16)* $P_t G(s) = \Phi(C)(s,t)$ *for* $\{s,t\}$ *in* $R \times R$ *and the following hold:*

(1) *in order that the nonnegative number* b *should satisfy condition (ii), it is necessary and sufficient that, for each* g *in* H_β, $N_\alpha(Cg) \leqslant b\, N_\beta(g)$,

(2) A *is in* $T_{\beta\alpha}(P)$ *and* $G(t)^*\xi = A(P_t\alpha\cdot\xi)(t)$ *for each* t *in* R *and* ξ *in* Y,

(3) *if* f *is in* H_α *then the equations* $h(t)(s) = [\alpha(t)^{-1/2}P_tG(s)]^*\alpha(t)^{-1/2}f(t)$, *for* $\{s,t\}$ *in* $R \times R$, *define a function* h *from* R *to* H_β *such that* $Af = \int_{L/F} h$ *with respect to* N_β *so that, for each* s *in* R,

$$Af(s) = \int_{s/F}[\alpha^{-1/2}G]^*\alpha^{-1/2}f \textit{ with respect to } \|\cdot\|,$$

(4) *if* g *is in* H_β *then the equations* $h(t)(s) = [\beta(t)^{-1/2}P_tG(s)^*]^*\beta(t)^{-1/2}g(t)$, *for* $\{s,t\}$ *in* $R \times R$, *define a function* h *from* R *to* H_α *such that* $Cg = \int_{L/F} h$ *with respect to* N_α *so that, for each* s *in* R,

$$Cg(s) = \int_{s/F}[\beta^{-1/2}G\cdot{}^*]^*\beta^{-1/2}g \textit{ with respect to } \|\cdot\|, \textit{ and}$$

(5) *in case* α *is* β *and each of* $\{C_1,G_1\}$ *and* $\{C_2,G_2\}$ *belongs to* Ψ *and* $C = C_1C_2$, *if* t *is in* R *and* η *is in* Y *then*

$$G(t)\eta = \int_{t/F}[\beta^{-1/2}G_1\cdot{}^*]^*[\beta^{-1/2}G_2]\eta \textit{ with respect to } \|\cdot\|.$$

INDICATION OF PROOF. Suppose, first, that C is in the class $T_{\alpha\beta}$ from Theorem 16 with adjoint transformation A as there indicated, that $\Gamma = \Phi(C)$, and that G is the function from R to $L(Y)^c$ given by $G(t) = \Gamma(t,t)$ for t in R. If C belongs to $T_{\alpha\beta}(P)$ then, *seriatim*,

(i) for each $\{f,g\}$ in $H_\alpha \times H_\beta$ and t in R, $A(P_tf) = P_t(Af)$ since

$$Q_\alpha(P_tf,Cg) = Q_\alpha(f,CP_tg) = Q_\beta(Af,P_tg) = Q_\beta(P_tAf,g),$$

(ii) for each $\{f,\eta\}$ in $H_\alpha \times Y$, and s and t in R, $P_s\Gamma(\cdot,t) = P_t\Gamma(\cdot,s)$ since

$$Q_\alpha(f,P_t\Gamma(\cdot,s)\eta) = \langle AP_tf(s),\eta\rangle = \langle AP_sf(t),\eta\rangle = Q_\alpha(f,P_s\Gamma(\cdot,t)\eta),$$

(iii) for each member M of F filling up the member t of R,

$$G(t) = \Sigma_{v \text{ in } M}P_v\Gamma(\cdot,t)(v) = \Sigma_{v \text{ in } M}P_t\Gamma(\cdot,v)(v) = \Sigma_{v \text{ in } M}G(v),$$

so that G is finitely additive, and

(iv) $P_tG(s) = \Gamma(s,t)$ for $\{s,t\}$ in $R \times R$, as in the Proof of Theorem 8.

If, on the other hand, G is finitely additive and $P_tG(s) = \Gamma(s,t)$ for each $\{s,t\}$ in $R \times R$ then, for each $\{f,\eta\}$ in $H_\alpha \times Y$ and $\{s,t\}$ in $R \times R$,

(i) if s does not intersect t then $P_sP_tf = 0$ in H_α so that

$$\langle AP_tf(s),\eta\rangle = Q_\alpha(P_tf,P_sG\cdot\eta) = Q_\alpha(f,P_sP_tG\cdot\eta) = 0 = \langle P_tAf(s),\eta\rangle, \text{ but}$$

(ii) if M is a member of F filling up the common part of s and t then

$$\langle AP_tf(s),\eta\rangle = \Sigma_{v \text{ in } M}Q_\alpha(f,P_vG\cdot\eta) = \Sigma_{v \text{ in } M}\langle Af(v),\eta\rangle = \langle P_tAf(s),\eta\rangle,$$

so that $A(P_tf) = P_t(Af)$, whence C is in $T_{\alpha\beta}(P)$ as in the preceding argument (i).

Suppose, now, that G is a finitely additive function from R to L(Y) and that Γ is defined on $R \times R$ by $\Gamma(s,t) = P_tG(s)$. It is clear that if $b \geqslant 0$ and the condition (ii) for membership of Γ in $m_{\alpha\beta}$ (Theorem 16) holds then condition (ii) for membership of G in $m_{\alpha\beta}(P)$ holds - consider degenerate members of F. Suppose that G is in $m_{\alpha\beta}(P)$ and that b is a nonnegative number so that the condition (ii) of the present Theorem holds. Let each of x and y be a function from a member M of F to Y: with Σ''_M denoting $\Sigma_{\{s,t\} \text{ in } M\times M}$ as before,

$$\begin{aligned}|\Sigma''_M\langle x(s),\Gamma(s,t)y(t)\rangle|^2 &= |\Sigma_{u \text{ in } M}\langle x(u),G(u)y(u)\rangle|^2\\ &\leqslant \{\Sigma_{u \text{ in } M}b\|\alpha(u)^{1/2}x(u)\|\,\|\beta(u)^{1/2}y(u)\|\}^2\\ &\leqslant b^2\Sigma_{s \text{ in } M}\|\alpha(s)^{1/2}x(s)\|^2\Sigma_{t \text{ in } M}\|\beta(t)^{1/2}y(t)\|^2.\end{aligned}$$

The foregoing considerations may be arranged, along with those from the preceding paragraph, to produce an argument for Theorem 17 based on Theorem 16.

REMARK. Theorem 17 is an extended version (as reinforced by Theorem 14) of a theorem which I propounded in 1962 to P. H. Jessner, and for which he gave a proof in his Dissertation [6, Theorem 4.1]. One of Jessner's remarkable discoveries in this connection [6, Theorem 4.2] takes the following form in the present context. If α and β are finitely additive functions from R to $L(Y)^+$ then, in order that the space $\{H_\alpha,Q_\alpha\}$ should be approximately included in $\{H_\beta,Q_\beta\}$ (*cf.* Remark 1 following Theorem 3 of the present report), it is necessary and sufficient that there should be a finitely additive *Hermitian valued* function G from R to $L(Y)^c$ such that if s is in R and $\{\xi,\eta\}$ is in $Y \times Y$ then $P_sG\cdot\xi$ belongs to H_β and

$$\langle\xi,\alpha(s)\eta\rangle = \int_{s/F}\langle\beta^{-1/2}G\cdot\xi,\beta^{-1/2}G\cdot\eta\rangle;$$

if, moreover, each of the spaces $\{H_\alpha,Q_\alpha\}$ and $\{H_\beta,Q_\beta\}$ is approximately included in the other, then there exists a finitely additive function G from R to $L(Y)^+$ such that the preceding hold as stated and also hold with α and β interchanged. Hence, Corollary 17.1 provides a refinement of the latter result by identifying such a G from R to $L(Y)^+$ with a special kind of linear isometry from $\{H_\alpha,Q_\alpha\}$ onto $\{H_\beta,Q_\beta\}$.

COROLLARY 17.1. *If* G *is a finitely additive function from* R *to* $L(Y)^c$ *then (in the context of Theorem 17), in order that* G *should be such a member of* $m_{\alpha\beta}(P)$ *that* $\Psi^{-1}(G)$ *is a linear isometry from* $\{H_\beta,Q_\beta\}$ *onto* $\{H_\alpha,Q_\alpha\}$, *it is necessary and sufficient that, for each* s *in* R *and* $\{\xi,\eta\}$ *in* Y X Y, $P_sG\cdot\xi$ *belong to* H_α, $P_sG\cdot{}^*\eta$ *belong to* H_β,

$$\int_{s/F}\langle\alpha^{-1/2}G\cdot\xi,\alpha^{-1/2}G\cdot\eta\rangle = \langle\xi,\beta(s)\eta\rangle,$$

and

$$\int_{s/F}\langle\beta^{-1/2}G\cdot{}^*\xi,\beta^{-1/2}G\cdot{}^*\eta\rangle = \langle\xi,\alpha(s)\eta\rangle.$$

Corollary 17.1 may be argued from Corollary 16.1 and Theorem 17 with the help of the following: if G satisfies the indicated conditions and M is a member of F filling up the common part of the sets s and t in R then, for $\{\xi,\eta\}$ in Y X Y,

$$\begin{aligned}\langle\xi,P_t\beta(s)\eta\rangle &= \Sigma_{v \text{ in } M}\langle\xi,\beta(v)\eta\rangle = \Sigma_{v \text{ in } M}Q_\alpha(P_vG\cdot\xi,P_vG\cdot\eta)\\ &= \Sigma_{v \text{ in } M}Q_\alpha(P_sG\cdot\xi,P_vG\cdot\eta) = Q_\alpha(P_sG\cdot\xi,P_sP_tG\cdot\eta)\\ &= Q_\alpha(P_sG\cdot\xi,P_tG\cdot\eta)\end{aligned}$$

and, similarly, $\langle\xi,P_t\alpha(s)\eta\rangle = Q_\beta(P_sG\cdot{}^*\xi,P_tG\cdot{}^*\eta)$.

COROLLARY 17.2. *If* G *is a finitely additive function from* R *to* L(Y) *then (in the context of Theorem 17 with* $\alpha=\beta$*), in order that* G *should be such a member of* $m_{\beta\beta}(P)$ *that* $\Psi^{-1}(G)$ *is a* Q_β*-orthogonal projection, it is necessary and sufficient that, for each* s *in* R *and* $\{\xi,\eta\}$ *in* Y X Y, $P_sG\cdot\xi$ *belong to* H_β *and*

$$\int_{s/F}\langle\beta^{-1/2}G\cdot\xi,\beta^{-1/2}G\cdot\eta\rangle = \langle\xi,G(s)\eta\rangle.$$

Corollary 17.2 may be argued from Corollary 16.2 and Theorem 17 with the help of computations, similar to those indicated **for** Corollary 17.1, to show that if G is as indicated then $\langle\xi,P_tG(s)\eta\rangle = Q_\beta(P_sG\cdot\xi,P_tG\cdot\eta)$ for appropriate s,t,ξ,η .

REMARK. As an illustration of Corollary 17.2, it may be noted that if r is an element of R and C is the restriction to H_β of P_r then $\Psi(C)(t) = P_r\beta(t)$.

THEOREM 18. *If* α *and* β *are member of* Ω *then*

(1) H_α *is a subset of* H_β *only in case there is a nonnegative number* b *such that if* $\{t,\eta\}$ *is in* R $\times$ Y *then* $\langle\eta,\alpha(t)\eta\rangle \leqslant b\langle\eta,\beta(t)\eta\rangle$, *in which case the least such* b *is the least nonnegative number* c *such that*

$$\int_{L/F} \|\beta^{-1/2}f\|^2 \leqslant c \int_{L/F} \|\alpha^{-1/2}f\|^2 \textit{ for each } f \textit{ in } H_\alpha,$$

and, with $\pi(\alpha,\beta)$ *the transformation defined by* $Q_\alpha(f,\pi(\alpha,\beta)g) = Q_\beta(f,g)$ *for* $\{f,g\}$ *in* $H_\alpha \times H_\beta$, $\pi(\alpha,\beta)$ *belongs to* $m_{\alpha\beta}(P)$ *(as in Theorem 17) and if* g *is in* H_β *then*

$$\pi(\alpha,\beta)g(s) = \int_{s/F} [\beta^{-1/2}\alpha] * \beta^{-1/2}g \textit{ with respect to } \|\cdot\|, \textit{ for each } s \textit{ in } R,$$

(2) $H_{\alpha+\beta}$ *is the vector sum* $H_\alpha \dot{+} H_\beta$ *of* H_α *and* H_β *and, if* h *is in* $H_{\alpha+\beta}$, *then*

$$\int_{L/F} \|(\alpha+\beta)^{-1/2}h\|^2 = \text{minimum}\left\{ \int_{L/F} \|\alpha^{-1/2}f\|^2 + \int_{L/F} \|\beta^{-1/2}g\|^2 \right\}$$

for all f *in* H_α *and* g *in* H_β *such that* f+g = h, *and*

(3) *the equations, for* $\{t,\eta\}$ *in* R $\times$ Y,

$$\begin{aligned}(\alpha{:}\beta)(t)\eta &= \tfrac{1}{4}\left\{ (\alpha+\beta)(t)\eta - \int_{t/F} [(\alpha+\beta)^{-1/2}(\alpha-\beta)] * [(\alpha+\beta)^{-1/2}(\alpha-\beta)]\,\eta \right\} \\ &= \int_{t/F} [(\alpha+\beta)^{-1/2}\alpha] * [(\alpha+\beta)^{-1/2}\beta]\,\eta \textit{ with respect to } \|\cdot\|,\end{aligned}$$

define a finitely additive function $\alpha{:}\beta$ *from* R *to* $L(Y)^+$ *such that* $H_{\alpha:\beta}$ *is the common part* $H_\alpha H_\beta$ *of* H_α *and* H_β, *and if each of* f *and* g *is in* $H_{\alpha:\beta}$ *then*

$$\int_{L/F} \langle(\alpha{:}\beta)^{-1/2}f,(\alpha{:}\beta)^{-1/2}g\rangle = \int_{L/F} \langle\alpha^{-1/2}f,\alpha^{-1/2}g\rangle + \int_{L/F} \langle\beta^{-1/2}f,\beta^{-1/2}g\rangle.$$

Theorem 18 may be seen as a direct consequence of Theorem 3, as reinforced in the context of the present section by Theorems 14 and 17. No proof is offered.

REMARK. It may be argued, just as in the Remarks following Theorem 3, that the *parallel summation* of finitely additive functions from R to $L(Y)^+$ (indicated in Theorem 18(3) by $\alpha{:}\beta$ for α and β in Ω) is both commutative and associative.

COMMENT. In continuation of the Remark immediately preceding Corollary 17.1, P. H. Jessner's discoveries [6, Theorems 3.1 and 3.2], adapted to this context, yield a symmetric function J from $\Omega \times \Omega$ such that, for each $\{\alpha,\beta\}$ in $\Omega \times \Omega$, $J_{\alpha\beta}$ is a finitely additive function from R to $L(Y)^+$ with these properties: (i) if s is in R and ξ is in Y then $P_s J_{\alpha\beta}\cdot\xi$ belongs to $H_\alpha H_\beta$ and (ii) $\{H_\alpha,Q_\alpha\}$ is approximately included in $\{H_\beta,Q_\beta\}$ only in case, for each t in R and $\{\xi,\eta\}$ in Y $\times$ Y,

$$\langle\xi,\alpha(t)\eta\rangle = \int_{t/F}\langle\beta^{-1/2}J_{\alpha\beta}\cdot\xi,\beta^{-1/2}J_{\alpha\beta}\cdot\eta\rangle.$$

With reference to Remark 1 following Theorem 3, and with a slight extension of the π-notation from Theorem 18, the function J may be described as follows: if each of α and β is in Ω then $\pi(\alpha:\beta,\alpha+\beta) = \pi(\alpha,\alpha+\beta)\pi(\beta,\alpha+\beta)$ and, with $\pi(\alpha:\beta,\alpha+\beta)^{1/2}$ the square root of $\pi(\alpha:\beta,\alpha+\beta)$ which is Hermitian and nonnegative with respect to $Q_{\alpha+\beta}$, $P_tJ_{\alpha\beta}\cdot\eta = \pi(\alpha:\beta,\alpha+\beta)^{1/2}(P_t(\alpha+\beta)\cdot\eta)$ for each $\{t,\eta\}$ in $R \times Y$, and the function M, mentioned in Remark 1(3) after Theorem 3, satisfies $M(s,t) = P_tJ_{\alpha\beta}(s)$ on $R \times R$. It may be noted, on the basis of Theorem 7(1), that the space $H_{\alpha:\beta}$ is the image of $H_{\alpha+\beta}$ under the transformation $\pi(\alpha:\beta,\alpha+\beta)^{1/2}$ as described here. The failure (with Y infinite dimensional) of approximate inclusion to be transitive persists even in this context [10]: if the pre-ring R consists of the right closed intervals (0,1], (1,2], and (0,2], but Y is not finite dimensional, then there exist three finitely additive functions α, β, and γ from R to $L(Y)^+$ with the following properties:

(1) the function β is scalar valued and $\alpha(L) = \beta(L) = \gamma(L) = 1$,

(2) there exist finitely additive functions G_1 and G_2 from R to $L(Y)^+$ such that, for each set t in R,

$$\alpha(t) = \int_{t/F}[\beta^{-1/2}G_1]^*[\beta^{-1/2}G_1] \text{ and } \beta(t) = \int_{t/F}[\gamma^{-1/2}G_2]^*[\gamma^{-1/2}G_2], \text{ and}$$

(3) there does not exist a finitely additive Hermitian valued function G from R to $L(Y)^c$ such that if t is in R then $\alpha(t) = \int_{t/F}[\gamma^{-1/2}G]^*[\gamma^{-1/2}G]$.

THEOREM 19. *Suppose α and β are members of Ω such that H_α is a subset of H_β, A is in $T_{\alpha\alpha}$ and B is in $T_{\beta\beta}$ (in the notation of Theorem 16), and $\Gamma(\alpha)$ and $\Gamma(\beta)$ are functions from $R \times R$ to $L(Y)^c$ defined by*

$$\Gamma(\alpha)(s,t)\eta = A(P_t\alpha\cdot\eta)(s) \text{ and } \Gamma(\beta)(s,t)\eta = B(P_t\beta\cdot\eta)(s)$$

for $\{s,t\}$ in $R \times R$ and η in Y. With $\pi(\alpha,\beta)$ as described in Theorem 18(1),

(1) *in order that A should be a subset of B, it is necessary and sufficient that if $\{s,\xi\}$ is in $R \times Y$ then $\Gamma(\alpha)(s,\cdot)^*\xi = \pi(\alpha,\beta)(\Gamma(\beta)(s,\cdot)^*\xi)$, and*

(2) *if A is a subset of B then, in order that the adjoint of A with respect to Q_α should be a subset of the adjoint of B with respect to Q_β, it is necessary and sufficient that if $\{t,\eta\}$ is in $R \times Y$ then $\Gamma(\alpha)(\cdot,t)\eta = \pi(\alpha,\beta)(\Gamma(\beta)(\cdot,t)\eta)$.*

THEOREM 20. *Suppose α and β are members of Ω such that H_α is a subset of*

H_β, A *is in* $T_{\alpha\alpha}(P)$ *and* B *is in* $T_{\beta\beta}(P)$ *(in the notation of Theorem 17), and* $G(\alpha)$ *and* $G(\beta)$ *are functions from* R *to* $L(Y)^c$ *defined by*

$$G(\alpha)(t)\eta = A(P_t\alpha\cdot\eta)(t) \textit{ and } G(\beta)(t)\eta = B(P_t\beta\cdot\eta)(t)$$

for t *in* R *and* η *in* Y. *With* $\pi(\alpha,\beta)$ *as described in Theorem 18(1),*

(1) *in order that* A *should be a subset of* B, *it is necessary and sufficient that if* $\{s,\xi\}$ *is in* R $\times$ Y *then* $P_sG(\alpha)\cdot{}^*\xi = \pi(\alpha,\beta)(P_sG(\beta)\cdot{}^*\xi)$, *and*

(2) *if* A *is a subset of* B *then, in order that the adjoint of* A *with respect to* Q_α *should be a subset of the adjoint of* B *with respect to* Q_β, *it is necessary and sufficient that if* $\{t,\eta\}$ *is in* R $\times$ Y *then* $P_tG(\alpha)\cdot\eta = \pi(\alpha,\beta)(P_tG(\beta)\cdot\eta)$.

Proofs for Theorems 19 and 20 are readily available on the basis of Theorems 16, and 17, respectively, with the help of the Corollary to Theorem 7.

Operations on the Linear Span of a Family of Spaces. It is now supposed concerning the collection Ω of finitely additive functions from R to $L(Y)^+$ that if α and β are such members of Ω that neither of H_α and H_β is a subset of the other then both the *arithmetical sum* $\alpha+\beta$ and the *parallel sum* $\alpha{:}\beta$ as described in Theorem 18(2,3) belong to Ω; $S(\Omega)$ denotes the linear span of the spaces H_α for α in Ω, and π denotes a function from the subset of $\Omega \times \Omega$ to which $\{\alpha,\beta\}$ belongs only in case H_α is a subset of H_β, in which case $\pi(\alpha,\beta)$ denotes the linear transformation from H_β to H_α as described in Theorem 18(1). It may be seen here, just as in [12], that the ordered triple $\{H,Q,\pi\}$ determines an inverse limit system in the sense that if each of α, β, and γ is in Ω then (i) if H_α is a subset of H_β then $\pi(\alpha,\beta)$ is a continuous linear transformation from $\{H_\beta,Q_\beta\}$ to $\{H_\alpha,Q_\alpha\}$, (ii) if H_α is a subset of H_β and H_β is a subset of H_γ then $\pi(\alpha,\gamma)$ is the composite transformation $\pi(\alpha,\beta)\pi(\beta,\gamma)$, and (iii) if H_α is H_β then $\pi(\beta,\alpha)$ is the inverse of the transformation $\pi(\alpha,\beta)$. The *operational inverse limit space* determined by the triple $\{H,Q,\pi\}$ is the linear space to which V belongs only in case V is a function from Ω such that, for each α in Ω, $V(\alpha)$ is a finitely additive function from R to $L(Y)^c$ and, if ξ is in Y, $V(\alpha)\cdot{}^*\xi$ belongs to H_α and if β is such a member of Ω that H_α is a subset of H_β then $V(\alpha)\cdot{}^*\xi = \pi(\alpha,\beta)(V(\beta)\cdot{}^*\xi)$ - this operational inverse limit space is denoted by OPER-INV-LIM-$\{H,Q,\pi\}$.

OBSERVATION 1. Each collection Ω_0 of finitely additive functions from R to $L(Y)^+$ determines a collection Ω of the type supposed here: by Theorem 18, one may (perhaps generously) take Ω to be the common part of all collections Ω_1 such that (i) Ω_1 is a collection of finitely additive functions from R to $L(Y)^+$ of which Ω_0 is a subcollection and (ii) for each α and β in Ω_1, both $\alpha+\beta$ and $\alpha{:}\beta$ belong to Ω_1.

OBSERVATION 2. If, in the preceding Observation, each member α of Ω_0 has the property that if ξ is in Y then $\alpha\cdot\xi$ belongs to H_α, Theorem 18 may further be used to show that each member of Ω may also be assumed to have this property.

OBSERVATION 3. It would be possible, with minor changes in notation, to have a theory analogous to that presently contemplated but with the following type of convexity condition imposed on Ω: if α and β are such members of Ω that neither of H_α and H_β is a subset of the other then both the *arithmetic mean* $\frac{1}{2}(\alpha+\beta)$ and the *harmonic mean* $2(\alpha{:}\beta)$ belong to Ω (the latter terminology is consistent with the notion of the harmonic mean of two positive numbers, commonly the reciprocal of the arithmetic mean of their reciprocals). This will not be done, although such formulas as $H_{2(\alpha:\beta)} = H_\alpha H_\beta$ and $Q_{2(\alpha:\beta)} = \frac{1}{2}(Q_\alpha + Q_\beta)$ may be noted: these would have a special significance in the case that each of α and β is projection valued, inasmuch as $2(\alpha{:}\beta)$ is then also (orthogonal) projection valued. This latter fact is a simple consequence of the fact (noticed by Fillmore or Williams [4, page 279], as an extension of an Anderson-Duffin result [1] for finite dimensional Y) that if A and B are projections in $L(Y)^+$ then *their harmonic mean* $E = 2(A{:}B)$ is that projection in $L(Y)^+$ which maps Y onto the common part $A(Y)B(Y)$ of $A(Y)$ and $B(Y)$; a way of seeing this is to notice (with reference to Theorem 3^{sp}) that $E = E^2$, in consequence of the fact that if x belongs to $E^{1/2}(Y) = A(Y)B(Y)$ then

$$\|E^{-1/2}x\|^2 = \tfrac{1}{2}\{\|A^{-1/2}x\|^2+\|B^{-1/2}x\|^2\} = \tfrac{1}{2}\{\|Ax\|^2+\|Bx\|^2\} = \tfrac{1}{2}\{\|x\|^2+\|x\|^2\} = \|x\|^2$$

so that, in particular, if ξ is in Y and $x = E\xi$ then

$$\langle\xi,E\xi\rangle = \|E^{1/2}\xi\|^2 = \|E^{-1/2}x\|^2 = \|x\|^2 = \|E\xi\|^2 = \langle\xi,E^2\xi\rangle.$$

THEOREM 21. *If* D_0 *is the space of all linear functions* μ *from* $S(\Omega)$ *to* Y *such that, for each* α *in* Ω, *the restriction to* H_α *of* μ *is continuous from* $\{H_\alpha,Q_\alpha\}$ *to* $\{Y,\langle\cdot,\cdot\rangle\}$ *then the equations* $\sigma(\mu)(\alpha)(t)\eta = \mu(P_t\alpha\cdot\eta)$, *for* μ *in* D_0 *and* α *in* Ω *and* t *in* R *and* η *in* Y, *define a reversible linear transformation* σ *from* D_0 *onto all of* OPER-INV-LIM-$\{H,Q,\pi\}$ *such that if the ordered pair* $\{\mu,V\}$ *belongs to* σ *and* f *is in*

$S(\Omega)$ *then* $\mu(f)$ *is an integral in the following sense: for each* α *in* Ω *such that* f *belongs to* H_α, $\mu(f) = \int_{L/F} [\alpha^{-1/2} V(\alpha)\cdot{}^*]\, {}^*\alpha^{-1/2} f$ *with respect to the norm* $[\![\cdot]\!]$.

INDICATION OF PROOF. That σ is a reversible linear transformation from D_0 to a subspace of OPER-INV-LIM-$\{H,Q,\pi\}$ follows directly from Theorem 15, as does the indicated integral representation. It remains to be shown that the σ-image of D_0 is all of that operational inverse limit space. Suppose, now, that V is a point in OPER-INV-LIM-$\{H,Q,\pi\}$. If f is in $S(\Omega)$ and α and β are such members of Ω that f belongs to H_α and to H_β then one of the following conditions is satisfied:

(i) one of H_α and H_β is a subset of the other, in which case if ξ is in Y then

$$Q_\alpha(f, V(\alpha)\cdot{}^*\xi) = Q_\beta(f, V(\beta)\cdot{}^*\xi), \text{ or}$$

(ii) neither of H_α and H_β is a subset of the other, in which case $\alpha{:}\beta$ is in Ω, f belongs to $H_{\alpha:\beta}$ (which is the common part $H_\alpha H_\beta$), and if ξ is in Y then

$$Q_\alpha(f, V(\alpha)\cdot{}^*\xi) = Q_{\alpha:\beta}(f, V(\alpha{:}\beta)\cdot{}^*\xi) = Q_\beta(f, V(\beta)\cdot{}^*\xi).$$

Therefore, by Theorem 15, the indicated integral formulas define a function μ from $S(\Omega)$ to Y such that if α is in Ω then the restriction to H_α of μ is a continuous linear function from $\{H_\alpha, Q_\alpha\}$ to $\{Y, \langle\cdot,\cdot\rangle\}$. If f and g are functions belonging to $S(\Omega)$ and α and β are members of Ω such that f is in H_α and g is in H_β but not in H_α then one of the following conditions is satisfied:

(i) H_α is a subset of H_β, in which case if ξ is in Y then

$$Q_\alpha(f, V(\alpha)\cdot{}^*\xi) + Q_\beta(g, V(\beta)\cdot{}^*\xi) = Q_\beta(f, V(\beta)\cdot{}^*\xi) + Q_\beta(f, V(\beta)\cdot{}^*\xi), \text{ or}$$

(ii) H_α is not a subset of H_β, in which case $\alpha+\beta$ is in Ω, f+g belongs to $H_{\alpha+\beta}$ (which is the vector sum of H_α and H_β), and if ξ is in Y then

$$Q_\alpha(f, V(\alpha)\cdot{}^*\xi) + Q_\beta(g, V(\beta)\cdot{}^*\xi) = Q_{\alpha+\beta}(f, V(\alpha+\beta)\cdot{}^*\xi) + Q_{\alpha+\beta}(g, V(\alpha+\beta)\cdot{}^*\xi).$$

It follows, by symmetry, that μ is linear on $S(\Omega)$ and so belongs to the space D_0.

NOTATION. E_1 is the algebra of all linear transformations C from $S(\Omega)$ into $S(\Omega)$ such that, for some nonnegative number b, if α is in Ω then C maps H_α into H_α and $N_\alpha(Cf) \leqslant b\, N_\alpha(f)$ for each f in H_α, the least such b being the *norm* $|C|$ of C; E_2 is the set of all C in E_1 such that if t is in R then $C(P_t f) = P_t(Cf)$ for all f in $S(\Omega)$; E_3 is the set of all C in E_1 such that if α and β are members of Ω such that H_α is a subset of H_β then $C(\pi(\alpha,\beta)g) = \pi(\alpha,\beta)(Cg)$ for each g in H_β; E_4 is the common part of E_2 and

E_3.

REMARK. There is a natural norm-preserving involution in the algebra E_3: it is a consequence of Theorem 7 that if C is in E_1 then, in order that C should be in E_3, it is necessary and sufficient that there be a member A of E_1 such that if α is in Ω then the restriction to H_α of A is the adjoint with respect to Q_α of the restriction to H_α of C - this member A of E_3 is the *Ω-adjoint* C^a *of* C.

THEOREM 22. *There is a linear isomorphism* Φ *from* E_1 *onto the collection of all functions* Γ *from* Ω *such that* (i) *if* α *is in* Ω *then* $\Gamma(\alpha)$ *is a function from* $R \times R$ *to* $L(Y)^c$ *and, for each* t *in* R, *each of* $\Gamma(\alpha)(\cdot,t)$ *and* $\Gamma(\alpha)(t,\cdot)$ *is finitely additive,* (ii) *there is a nonnegative number* b *such that, for each* α *in* Ω *and* M *in* F, *if each of* x *and* y *is a function from* M *to* Y *then* (Σ''_M *is* $\Sigma_{\{s,t\} \text{ in } M \times M}$)

$$|\Sigma''_M\langle x(s),\Gamma(\alpha)(s,t)y(t)\rangle|^2 \leqslant b^2\Sigma_{s \text{ in } M}\|\alpha(s)^{1/2}x(s)\|^2\Sigma_{t \text{ in } M}\|\alpha(t)^{1/2}y(t)\|^2,$$

and (iii) *if* α *and* β *are such members of* Ω *that* H_α *is a subset of* H_β *then, for each* $\{s,\xi\}$ *in* $R \times Y$, $\Gamma(\alpha)(s,\cdot)^*\xi = \pi(\alpha,\beta)(\Gamma(\beta)(s,\cdot)^*\xi)$: *if* $\{C,\Gamma\}$ *is in* Φ *then*

(1) *for each* $\{\alpha,\eta\}$ *in* $\Omega \times Y$ *and* $\{s,t\}$ *in* $R \times R$, $\Gamma(\alpha)(s,t)\eta = C(P_t\alpha\cdot\eta)(s)$,

(2) *the norm* $|C|$ *of* C *is the least nonnegative number* b *such that (ii) holds,*

(3) *a necessary and sufficient condition for* C *to belong to* E_3 *is that if* α *and* β *are such members of* Ω *that* H_α *is a subset of* H_β *then, for each* $\{t,\eta\}$ *in* $R \times Y$, $\Gamma(\alpha)(\cdot,t)\eta = \pi(\alpha,\beta)(\Gamma(\beta)(\cdot,t)\eta)$, *and if* C *does belong to* E_3 *then, for each* α *in* Ω *and* $\{s,t\}$ *in* $R \times R$, $\Phi(C^a)(\alpha)(s,t) = \Gamma(\alpha)(t,s)^*$,

(4) *if* f *is in* $S(\Omega)$ *then* Cf *is an integral in the following sense: if* α *is in* Ω *and* f *is in* H_α *then the equations* $h(t)(s) = [\alpha(t)^{-1/2}\Gamma(\alpha)(s,t)^*]^*\alpha(t)^{-1/2}f(t)$, *for* $\{s,t\}$ *in* $R \times R$, *define a function* h *from* R *to* H_α *such that* $Cf = \int_{L/F} h$ *with respect to* N_α *so that, for each* s *in* R,

$$Cf(s) = \int_{L/F}[\alpha^{-1/2}\Gamma(\alpha)(s,\cdot)^*]^*\alpha^{-1/2}f \textit{ with respect to } \|\cdot\|, \textit{ and}$$

(5) *in case each of* $\{C_1,\Gamma_1\}$ *and* $\{C_2,\Gamma_2\}$ *belongs to* Φ *and* $C = C_1C_2$, *if* α *is in* Ω *and* $\{s,t\}$ *is in* $R \times R$ *and* η *is in* Y *then*

$$\Gamma(\alpha)(s,t)\eta = \int_{L/F}[\alpha^{-1/2}\Gamma_1(\alpha)(s,\cdot)^*]^*[\alpha^{-1/2}\Gamma_2(\cdot,t)]\eta \textit{ with resepct to } \|\cdot\|.$$

Theorem 22 may be proved, with the help of Theorems 16 and 19, with the type of argument indicated in support of Theorem 21.

THEOREM 23. *There is a linear isomorphism* Ψ *from* E_2 *onto the collection of all functions* G *from* Ω *such that* (i) *if* α *is in* Ω *then* $G(\alpha)$ *is a finitely additive function from* R *to* $L(Y)^c$, (ii) *there is a nonnegative number* b *such that, for each* α *in* Ω *and* t *in* R, *if* $\{\xi,\eta\}$ *is in* $Y \times Y$ *then*

$$|\langle\xi,G(\alpha)(t)\eta\rangle|^2 \leqslant b^2\langle\xi,\alpha(t)\xi\rangle\langle\eta,\alpha(t)\eta\rangle,$$

and (iii) *if* α *and* β *are such members of* Ω *that* H_α *is a subset of* H_β *then, for each* $\{s,\xi\}$ *in* $R \times Y$, $P_sG(\alpha)\cdot{*}\xi = \pi(\alpha,\beta)(P_sG(\beta)\cdot{*}\xi)$: *if* $\{C,G\}$ *is in* Ψ *then*

(1) *for each* $\{\alpha,\eta\}$ *in* $\Omega \times Y$ *and* t *in* R, $G(\alpha)(t)\eta = C(P_t\alpha\cdot\eta)(t)$,

(2) *the norm* $|C|$ *of* C *is the least nonnegative number* b *such that (ii) holds,*

(3) *a necessary and sufficient condition for* C *to belong to* E_4 *is that if* α *and* β *are such members of* Ω *that* H_α *is a subset of* H_β *then, for each* $\{t,\eta\}$ *in* $R \times Y$, $P_tG(\alpha)\cdot\eta = \pi(\alpha,\beta)(P_tG(\beta)\cdot\eta)$, *and if* C *does belong to* E_4 *then, for each* α *in* Ω *and* t *in* R, $\Psi(C^a)(\alpha)(t) = G(\alpha)(t)^*$,

(4) *if* f *is in* $S(\Omega)$ *then* Cf *is an integral in the following sense: if* α *is in* Ω *and* f *is in* H_α *then the equations* $h(t)(s) = [\alpha(t)^{-1/2}P_tG(\alpha)(s)^*]^*\alpha(t)^{-1/2}f(t)$, *for* $\{s,t\}$ *in* $R \times R$, *define a function* h *from* R *to* H_α *such that* $Cf = \int_{L/F} h$ *with respect to* N_α *so that, for each* s *in* R,

$$Cf(s) = \int_{s/F}[\alpha^{-1/2}G(\alpha)\cdot{*}]^*\alpha^{-1/2}f \textit{ with respect to } [] \cdot [], \textit{ and}$$

(5) *in case each of* $\{C_1,G_1\}$ *and* $\{C_2,G_2\}$ *belongs to* Ψ *and* $C = C_1C_2$, *if* α *is in* Ω *and* t *is in* R *and* η *is in* Y *then*

$$G(\alpha)(t)\eta = \int_{t/F}[\alpha^{-1/2}G_1(\alpha)\cdot{*}]^*[\alpha^{-1/2}G_2(\alpha)]\eta \textit{ with respect to } []\cdot[].$$

Theorem 23 may be proved, with the help of Theorems 17 and 20, with the type of argument indicated in support of Theorem 21.

REMARK. The condition (iii) on the member G of the Ψ-image of E_2 is readily seen to be equivalent to the condition that, for each $\{v,\xi\}$ in $R \times Y$,

$$G(\alpha)(v)^*\xi = \int_{v/F}[\beta^{-1/2}\alpha]^*\beta^{-1/2}G(\beta)\cdot{*}\xi \text{ with respect to } []\cdot[]$$

(inthe light of such formulas as $P_sG(\alpha)(t) = \Sigma_{v \text{ in } M}G(\alpha)(v)$ for M in F filling up $\{st\}$); the latter display may be rewritten, for each $\{v,\eta\}$ in $R \times Y$, as

$$G(\alpha)(v)\eta = \int_{v/F}[\beta^{-1/2}G(\beta)\cdot{*}]^*\beta^{-1/2}\alpha\cdot\eta \text{ with respect to } []\cdot[],$$

inasmuch as the latter integral is, by Theorem 23(4), $\Psi^{-1}(G)(P_v\alpha\cdot\eta)(v)$. Now, the condition (3) for $\Psi^{-1}(G)$ to belong to E_4 is: for each $\{v,\eta\}$ in R × Y,

$$G(\alpha)(v)\eta = \int_{v/F}[\beta^{-1/2}\alpha]*\beta^{-1/2}G(\beta)\cdot\eta \text{ with respect to } [\!]\cdot[\!].$$

Hence, it may be noted that if each member of Ω is scalar valued then E_2 is E_4, an observation consistent with Theorem 15 of [12]. Similar computations may be used to show that, if each member of Ω is scalar valued, OPER-INV-LIM-$\{H,Q,\pi\}$ is indeed a subset of the space previously denoted by INV-LIM-$\{H,Q,\pi\}$ and consisting of all functions U from Ω such that, if α is in Ω, $U(\alpha)$ is a finitely additive function from R to L(Y) and, for each ξ in Y, $U(\alpha)\cdot\xi$ belongs to H_α and if β is such a member of Ω that H_α is a subset of H_β then $U(\alpha)\cdot\xi = \pi(\alpha,\beta)(U(\beta)\cdot\xi)$ [12].

THEOREM 24. *Let ζ be a function from D_0 (of Theorem 21) such that if μ is in D_0 then $\zeta(\mu)$ is the linear transformation from $S(\Omega)$ to a set of functions on R to Y given by $\zeta(\mu)f(t) = \mu(P_tf)$ for f in $S(\Omega)$ and t in R, and D_1 be the subset of D_0 to which the member μ of D_0 belongs only in case $\zeta(\mu)$ belongs to E_1:*

(1) *if μ is in D_0 and f is in $S(\Omega)$ then $\mu(f) = \int_{L/F}\zeta(\mu)f$ with respect to $[\!]\cdot[\!]$,*

(2) *if μ is in D_0 then, for each s in R and f in $S(\Omega)$, $\zeta(\mu)(P_sf) = P_s(\zeta(\mu)f)$, so that the ζ-image of D_1 is a subset of E_2,*

(3) *if μ is in D_1 then $\Psi(\zeta(\mu)) = \sigma(\mu)$ (with Ψ as in Theorem 23) and*

(4) *in order that the ζ-image of D_1 should be all of E_2, it is necessary and sufficient that if α is in Ω and ξ is in Y then $\alpha\cdot\xi$ is in H_α, i.e., $\int_{L/F}\alpha\cdot\xi$ exists.*

Theorem 24 is a consequence of Theorems 21 and 23.

NOTATION. The direct sum of the spaces $\{H_\alpha,Q_\alpha\}$ (for α in Ω), with the usual inner product, is here denoted by $\{\Sigma_\Omega\{H,Q\},\hat{Q}_\Omega\}$, and A_0 denotes the algebra of all continuous linear transformations in this space. A_1 denotes the set of all B in A_0 with a representation Δ such that

$$\hat{Q}_\Omega(Bf,g) = \Sigma_{\alpha \text{ in } \Omega}Q_\alpha(\Delta(B)_\alpha f_\alpha,g_\alpha) \text{ for all f and g in } \Sigma_\Omega\{H,Q\}$$

where, for each α in Ω, $\Delta(B)_\alpha$ is a continuous linear transformation in $\{H_\alpha,Q_\alpha\}$ and if β is such a member of Ω that H_α is a subset of H_β then $\Delta(B)_\alpha$ is the restriction to H_α of $\Delta(B)_\beta$; A_2 is the set of all B in A_1 such that if α is in Ω and h is in H_α then $\Delta(B)_\alpha(P_th) = P_t(\Delta(B)_\alpha h)$ for each t in R; A_3 is the set of all B in A_1 such that if α and

β are members of Ω and H_α is a subset of H_β then, for each h in H_β, $\Delta(B)_\alpha(\pi(\alpha,\beta)h) = \pi(\alpha,\beta)(\Delta(B)_\beta h)$; A_4 is the common part of A_2 and A_3.

THEOREM 25. *Each of* A_1, A_2, A_3, *and* A_4 *is a weakly closed subalgebra of* A_0, *the member* B *of* A_1 *belongs to* A_3 *only in case* A_1 *contains the adjoint with respect to* $Q_{\hat{\Omega}}$ *of* B, *there is an isometric algebra-isomorphism* Z *from* E_1 *onto* A_1 *given by*

$$Q_{\hat{\Omega}}(Z(C)f,g) = \Sigma_{\alpha \text{ in } \Omega} Q_\alpha(Cf_\alpha,g_\alpha) \textit{ for } C \textit{ in } E_1,\ f \textit{ and } g \textit{ in } \Sigma_\Omega\{H,Q\},$$

the transformation Z *maps* E_j *onto* A_j *for* j = 2,3,4, *and the restriction to* E_3 *of* Z *is involution-preserving in the sense that if* C *is in* E_3 *then* $Z(C^a)$ *is the adjoint with respect to* $Q_{\hat{\Omega}}$ *of* Z(C).

A proof for Theorem 25 may be constructed, along the lines of that given for [12, Theorem 25], by considering special weak neighborhoods of members of A_0, then of A_1, and by applying appropriate consequences of Theorems 22 and 23.

REMARK 1. Theorem 25 may be used to describe the idea of a member C of E_3 or of E_4 as being, *e.g.*, Hermitian, normal, or unitary, in terms of the corresponding property of Z(C) in A_3 or in A_4: suitable reinforcements of Theorems 22 and 23 by Corollaries 16.1 and 16.2, and by Corollaries 17.1 and 17.2, respectively, may be used to give integral formulas involving "spectral resolutions" of such C.

REMARK 2. There is a sense in which the inner product $Q_{\hat{\Omega}}$ may be viewed as an integral. Consider the direct sum $\{Y^\Omega,\langle\cdot,\cdot\rangle_\Omega\}$ of Ω copies of $\{Y,\langle\cdot,\cdot\rangle\}$: Y^Ω is the set of all functions x from Ω to Y such that there is a nonnegative number b such that $\Sigma_{\alpha \text{ in } W}\|x_\alpha\|^2 \leqslant b$ for each finite subset W of Ω, and if each of x and y is in Y^Ω then $\langle x,y\rangle_\Omega = \Sigma_{\alpha \text{ in } \Omega}\langle x_\alpha,y_\alpha\rangle$ (*cf.* Remark 3 following Theorem 5). Now, as an extension of the notation indicated in Theorem 14, if M is a member of F then there is an orthogonal projection $\Pi(M)$ in the algebra A_0 determined by

$$\begin{aligned} Q_{\hat{\Omega}}(\Pi(M)f,g) &= \Sigma_{\alpha \text{ in } \Omega} Q_\alpha(\Pi_\alpha(M)f_\alpha,g_\alpha) \\ &= \Sigma_{\alpha \text{ in } \Omega}\Sigma_{t \text{ in } M}\langle \alpha(t)^{-1/2}f_\alpha(t),\alpha(t)^{-1/2}g_\alpha(t)\rangle, \end{aligned}$$

for f and g in $\Sigma_\Omega\{H,Q\}$, and this draws attention to functions x and y from R to Y^Ω determined by the equations,

$$x(t)_\alpha = \alpha(t)^{-1/2}f_\alpha(t) \text{ and } y(t)_\alpha = \alpha(t)^{-1/2}g_\alpha(t) \text{ for } \{t,\alpha\} \text{ in } R \times \Omega,$$

such that if M is in F then $Q\hat{_\Omega}(\Pi(M)f,g) = \Sigma_{t \text{ in } M}\langle x(t),y(t)\rangle_\Omega$. The following may be proved: if each of f and g is in $\Sigma_\Omega\{H,Q\}$ and $\epsilon > 0$, there is a finite subset W_0 of Ω and a member M_0 of F with the property that, for each finite subset W of Ω which includes W_0 and each member M of F such that each set in M_0 is filled up by a subcollection of M,

$$|Q\hat{_\Omega}(f,g) - \Sigma_{\alpha \text{ in } W}\Sigma_{t \text{ in } M}\langle\alpha(t)^{-1/2}f_\alpha(t),\alpha(t)^{-1/2}g_\alpha(t)\rangle| < \epsilon.$$

REMARK 3. The approximation process indicated in the preceding Remark might be further formalized as follows. With R′ the pre-ring consisting of all subsets of $\Omega \times L$ of the form $\{\alpha\} \times t$ for α in Ω and t in R ($\{\alpha\}$ denotes the degenerate subset of Ω of which α is the only member), let F′ denote the family of all finite subcollections M′ of R′ such that no element of $\Omega \times L$ belongs to two sets in M′. Now, the formulas $\omega(\{\alpha\} \times t) = \alpha(t)$ (for $\{\alpha,t\}$ in $\Omega \times R$) determine a finitely additive function ω from R′ to $L(Y)^+$; also, the transformation δ consisting of all $\{f,f'\}$ such that f is in $\Sigma_\Omega\{H,Q\}$, and f′ is the function from R′ to Y determined by $f'(\{\alpha\} \times t) = f_\alpha(t)$ for $\{\alpha,t\}$ in $\Omega \times R$, maps $\Sigma_\Omega\{H,Q\}$ onto a set of finitely additive functions from R′ to Y. If each of $\{f,f'\}$ and $\{g,g'\}$ belongs to δ and $\{\alpha,t\}$ is in $\Omega \times R$ and $u = \{\alpha\} \times t$ then

$$\langle\alpha(t)^{-1/2}f_\alpha(t),\alpha(t)^{-1/2}g_\alpha(t)\rangle = \langle\omega(u)^{-1/2}f'(u),\omega(u)^{-1/2}g'(u)\rangle.$$

Inasmuch as, for each finite subset W of Ω and each member M of F, F′ contains the collection M′ consisting of all $\{\alpha\} \times t$ for α in W and t in M, it may be shown that δ is a linear isometry from $\{\Sigma_\Omega\{H,Q\},Q\hat{_\Omega}\}$ onto that Hellinger integral space generated by ω (relatively, of course, to the pre-ring R′):

$$\Sigma_{\alpha \text{ in } \Omega}\int_{L/F}\langle\alpha^{-1/2}f_\alpha,\alpha^{-1/2}g_\alpha\rangle = \int_{(\Omega \times L)/F'}\langle\omega^{-1/2}\delta(f),\omega^{-1/2}\delta(g)\rangle$$

for all f and g in $\Sigma_\Omega\{H,Q\}$, on the basis of Remark 2, with the help of Theorem 10. This construction is not peculiar to the special assumptions on Ω in this section.

Miscellaneous Examples. Let $|\cdot|_c$ denote the usual norm for $L(Y)^c$, so that if A is in $L(Y)^c$ then $|A|_c$ is the least nonnegative number b such that $\|A\eta\| \leq b\,\|\eta\|$ for each η in Y. With reference to Theorems 23 and 24, the following Theorem may be proved.

THEOREM 26. *Suppose, of the collection Ω, that if α is in Ω and ξ is in* Y *then $\alpha\cdot\xi$ belongs to H_α. Then there is a norm $\|\cdot\|$ for $S(\Omega)$ with these properties:*

(1) *if* f *is in* $S(\Omega)$ *then* $||f||$ *is the least nonnegative number* b *such that, if* μ *is in* D_1, $\|\mu(f)\| \leqslant b\ |\zeta(\mu)|$, *and*

(2) *if* f *is in* $S(\Omega)$ *and* s *is in* R *then, for each* α *in* Ω *such that* f *is in* H_α,

$$\|f(s)\| \leqslant ||f|| \leqslant N_\alpha(f)|\textstyle\int_{L/F}\alpha|_c^{1/2}.$$

INDICATION OF PROOF. It may be recalled, from Theorem 24(4), that D_1 is a complete space with respect to the norm $|\zeta(\cdot)|$; moreover, a linear function δ from $S(\Omega)$ to the set of all linear transformations from D_1 to Y is determined by the equations $\delta(f)\mu = \mu(f)$, for f in $S(\Omega)$ and μ in D_1. To show that 26(1) does define a norm $||\cdot||$ for $S(\Omega)$, it is sufficient to show that δ is reversible and if f is in $S(\Omega)$ then $\delta(f)$ is continuous with respect to the ordered pair $\{|\zeta(\cdot)|, \|\cdot\|\}$ of norms on D_1 and Y, respectively. If f is a nonzero member of $S(\Omega)$ then there is a set s in R such that $f(s) \neq 0$; there is a member G of OPER-INV-LIM-$\{H,Q,\pi\}$ such that if α is in Ω then $G(\alpha) = P_s\alpha$ and, for each t in R and $\{\xi,\eta\}$ in $Y \times Y$,

$$|\langle\xi,G(\alpha)(t)\eta\rangle|^2 \leqslant \langle\xi,P_s\alpha(t)\xi\rangle\langle\eta,P_s\alpha(t)\eta\rangle \leqslant \langle\xi,\alpha(t)\xi\rangle\langle\eta,\alpha(t)\eta\rangle;$$

hence, G is in the Ψ-image of E_2 and $|\Psi^{-1}(G)| \leqslant 1$ and if $\mu = \sigma^{-1}(G)$ then it is true that $\delta(f)\mu = \mu(f) = f(s) \neq 0$: thus, the function δ is reversible. Suppose, now, that f is in $S(\Omega)$: if μ is in D_1 and $G = \sigma(\mu)$ then, for each α in Ω such that f belongs to H_α and each η in Y,

$$N_\alpha(G(\alpha)\cdot{}^*\eta)^2 = \int_{L/F}\|\alpha^{-1/2}G(\alpha)\cdot{}^*\eta\|^2 \leqslant |\zeta(\mu)|^2\int_{L/F}\|\alpha^{1/2}\cdot\eta\|^2 = |\zeta(\mu)|^2\langle\eta,\textstyle\int_{L/F}\alpha\cdot\eta\rangle$$

so that $N_\alpha(G(\alpha)\cdot{}^*\eta) \leqslant |\int_{L/F}\alpha|_c^{1/2}\|\eta\|\ |\zeta(\mu)|$, from which it follows that

$$|\langle\mu(f),\eta\rangle| = |Q_\alpha(f,G(\alpha)\cdot{}^*\eta)| \leqslant N_\alpha(f)N_\alpha(G(\alpha)\cdot{}^*\eta) \leqslant N_\alpha(f)|\textstyle\int_{L/F}\alpha|_c^{1/2}|\zeta(\mu)|\ \|\eta\|,$$

whence $\|\mu(f)\| \leqslant N_\alpha(f)|\int_{L/F}\alpha|_c^{1/2}|\zeta(\mu)|$. This establishes the aforementioned continuity of $\delta(f)$, yielding thus the norm $||\cdot||$ for $S(\Omega)$, and also serves to give the second indicated inequality in 26(2); as to the first inequality indicated in 26(2), that follows from the existence (as indicated earlier in this paragraph), for each s in R, of a μ in D_1 with $|\zeta(\mu)| \leqslant 1$ and $\mu(f) = f(s)$ for f in $S(\Omega)$.

TERMINOLOGY. Suppose the collection Ω in Theorem 26 consists of *all* finitely additive functions α from R to $L(Y)^+$ such that if ξ is in Y then $\alpha\cdot\xi$ is in H_α: a finitely additive function G *from* R *to* L(Y) is said [9, page 76] to be of bounded variation with respect to $\langle\cdot,\cdot\rangle$ provided there is a member $\{\alpha,\beta\}$ of $\Omega \times \Omega$ such that if $\{\xi,\eta\}$ is in $Y \times Y$ and t is in R then $|\langle\xi,G(t)\eta\rangle|^2 \leqslant \langle\xi,\alpha(t)\xi\rangle\langle\eta,\beta(t)\eta\rangle$, and in this case $\{\alpha,\beta\}$ is called a dominant pair for G (*cf.* Theorem 17). Thus, in case Ω is the aforementioned collection, it is consistent to refer to $S(\Omega)$ as the space of all functions (*from* R *to* Y) "of bounded variation with respect to $\langle\cdot,\cdot\rangle$," and to call $\|\cdot\|$ (from Theorem 26) the total variation norm corresponding to $\langle\cdot,\cdot\rangle$: it may be recalled from Theorem 10 that a finitely additive function f from R to Y belongs to $S(\Omega)$ only in case there is a member $\{\alpha,h\}$ of $\Omega \times \Omega$ such that h is real (scalar) valued and if $\{t,\eta\}$ is in $R \times Y$ then $|\langle f(t),\eta\rangle|^2 \leqslant h(t)\langle\eta,\alpha(t)\eta\rangle$.

Now, with the Supposition of Theorem 26, if α is in Ω then one might define a norm $\|\cdot\|_\alpha$ for H_α as follows: if f is in H_α, $\|f\|_\alpha$ is the least nonnegative number b such that if G is a finitely additive function from R to L(Y) and

$$|\langle\xi,G(t)\eta\rangle|^2 \leqslant \langle\xi,\alpha(t)\xi\rangle\langle\eta,\alpha(t)\eta\rangle \text{ for each t in R and } \{\xi,\eta\} \text{ in } Y \times Y$$

then $[\![\int_{L/F}[\alpha^{-1/2}G\cdot *]*\alpha^{-1/2}f]\!] \leqslant b$. As has been shown previously [12], if each member of Ω is (real) scalar valued and α is in Ω and f is in H_α then $\|f\|_\alpha = \|f\|$ and is the total variation of f with respect to the norm $[\![\cdot]\!]$: in particular, in this case, if α and β are members of Ω and f belongs to the common part $H_\alpha H_\beta$ then $\|f\|_\alpha = \|f\|_\beta$. That this latter can not be proved in general may be seen from the following Example, in which appeal is made to Theorem 6^{sp}, with Y two dimensional.

EXAMPLE 1. Let R be the pre-ring consisting of two mutually exclusive sets s and t, Y be (complex) two dimensional, B be a member of $L(Y)^+$ having eigenvalues 2 and 1/2, and Ω consist of the functions α and β defined on R to L(Y) as follows:

$$\alpha(s) = \beta(s) = 0,\ \alpha(t) = 1, \text{ and } \beta(t) = B.$$

It may be seen that $H_\alpha = H_\beta$ and consists of all functions f from R to Y such that $f(s) = 0$; let x be a member of Y such that $Bx = 1/2x$ and $[\![x]\!] = 1$, and f be the member of $S(\Omega)$ such that $f(s) = 0$ and $f(t) = x$. If D is in L(Y) and $|D|_c \leqslant 1$ and $G(s) = 0$ and $G(t) = D$ then $\int_{L/F}[\alpha^{-1/2}G\cdot *]*\alpha^{-1/2}f = Dx$: it may be seen from this that $\|f\|_\alpha = 1$. If G is a function from R to L(Y) such that $G(s) = 0$ then by Theorem 6^{sp}, in order that

$$|\langle\xi,G(t)\eta\rangle|^2 \leqslant \langle\xi,\beta(t)\xi\rangle\langle\eta,\beta(t)\eta\rangle \text{ for each } \{\xi,\eta\} \text{ in } Y \times Y,$$

it is necessary and sufficient that $G(t) = B^{1/2}DB^{1/2}$ for some D in L(Y) such that

$|D|_c \leqslant 1$, in which case

$$\int_{L/F}[\beta^{-1/2}G\cdot *]*\beta^{-1/2}f = [B^{-1/2}G(t)*]*B^{-1/2}x = [D*B^{1/2}]*B^{-1/2}x = B^{1/2}DB^{-1/2}x;$$

since there is a unitary member D of L(Y) such that BDx = 2 Dx, it may be seen from this that $\|f\|_\beta = 2 \neq \|f\|_\alpha$.

In case Y is finite dimensional it may be shown that, if β is a finitely additive function from R to $L(Y)^+$ such that $\beta\cdot\xi$ belongs to H_β for every ξ in Y, then β is of bounded variation with respect to the norm $|\cdot|_c$: hence, with the help of Theorem 18(1), there is a finitely additive scalar valued function γ from R to $L(Y)^+$ such that $\int_{L/F}\gamma$ exists and H_β is a subset of H_γ. In the foregoing situation, with particular reference to the representation in Theorem 15 and the Remarks 1 and 2 immediately thereafter, it might be of interest to show that there exists a finitely additive scalar valued function α from R to $L(Y)^+$ such that $H_\alpha = H_\beta$. This can not be proved. Indeed, for the foregoing situation with Y of finite dimension greater than 1, it can not be proved that there is a nontrivial finitely additive scalar valued function α from R to $L(Y)^+$ such that H_α is a subset of H_β. Consider the following Example.

EXAMPLE 2. Suppose Y is of finite dimension $n+1 > 1$, R is the pre-ring of all degenerate subsets of the set L of integers 0 through n, and $\{g_p\}_0^n$ is a simple ordering of an orthonormal set in $\{Y,\langle\cdot,\cdot\rangle\}$. Let β be defined from R to $L(Y)^+$ by

$$\beta(\{p\})\eta = \langle\eta,g_p\rangle g_p \text{ for each p in L and } \eta \text{ in Y},$$

and suppose α is a scalar valued function from R to $L(Y)^+$ such that H_α is a subset of H_β. By Theorem 18, there is a nonnegative number b such that if p is in L and η is in Y then $\alpha(\{p\})\|\eta\|^2 \leqslant b\langle\eta,\beta(\{p\})\eta\rangle$. Now the function α has only the value 0 since, for each p in L, there is a q in L different from p so that

$$0 \leqslant \alpha(\{p\}) = \alpha(\{p\})\|g_q\|^2 \leqslant b\langle g_q,\beta(\{p\})g_q\rangle = b\langle g_q,g_p\rangle\langle g_p,g_q\rangle = 0.$$

In the case that Y is infinite dimensional, it can not be proved that if β is a finitely additive function from R to $L(Y)^+$ such that $\int_{L/F}\beta\cdot\xi = \xi$ for each ξ in Y then either (i) there is a nontrivial finitely additive scalar valued function α from R to $L(Y)^+$ such that H_α is a subset of H_β or (ii) there is a finitely additive scalar valued function γ from R to $L(Y)^+$ such that H_β is a subset of H_γ. Consider the following Example.

EXAMPLE 3. Supposing that Y is infinite dimensional, there is a member B of $L(Y)^+$ with spectrum the number interval $[0,1]$: let R be the pre-ring consisting of all such subsets t of $L = [0,1]$ that either t is L or, for some numbers p and q with $0 < p < q \leqslant 1$, t is the interval $[0,p]$ or t is the right-closed interval $(p,q]$, and let β be the restriction to R of the spectral resolution of B. If α is a finitely additive scalar valued function from R to $L(Y)^+$ such that H_α lies in H_β then, by Theorem 18, there is a nonnegative number b such that if t is in R and η is in Y then $\alpha(t)\|\eta\|^2 \leqslant b\langle\eta,\beta(t)\eta\rangle$: as in Example 2, for each s in R different from L, there are a member t of R which does not intersect s and a member η of Y such that $\|\eta\| = 1$ and $\beta(t)\eta = \eta$ so that $\beta(s)\eta = 0$ and

$$0 \leqslant \alpha(s) = \alpha(s)\|\eta\|^2 \leqslant b\langle\eta,\beta(s)\eta\rangle = b\langle\eta,0\rangle = 0,$$

whence α has only the value 0. If there were a finitely additive scalar valued function γ from R to $L(Y)^+$ such that H_β is a subset of H_γ then, by Theorem 18, there would be a nonnegative number b such that if t is in R and η is in Y then $\langle\eta,\beta(t)\eta\rangle \leqslant b\,\gamma(t)\|\eta\|^2$: this would imply that $|\beta(t)|_c \leqslant b\,\gamma(t)$ for each t in R but, since $|\beta(t)|_c = 1$ for each t in R, this would involve a contradiction. It may be shown that the equations $\lambda(\xi)(t) = \beta(t)\xi$, for $\{\xi,t\}$ in $Y \times R$, define a linear isometry λ from the space $\{Y,\langle\cdot,\cdot\rangle\}$ onto the space $\{H_\beta,Q_\beta\}$; *cf.* Theorem 4.

In one of the cases previously considered [12], that Ω is the collection of all finitely additive scalar valued functions α from R to $L(Y)^+$ such that $\int_{L/F}\alpha$ exists, $S(\Omega)$ is the space S_0 of all finitely additive functions from R to Y which are of bounded variation with respect to $\|\cdot\|$, and $\|\cdot\|$ is the total variation norm on $S(\Omega)$. Moreover, in that case, $\{D_1,|\zeta(\cdot)|\}$ is the space E (normed in the usual manner) of *all* linear functions from $S(\Omega)$ to Y which are continuous with respect to the ordered pair $\{\|\cdot\|,\|\cdot\|\}$ of norms on $S(\Omega)$ and Y, respectively: by considering the following Example, one may see that this can not be proved in general.

EXAMPLE 4. Suppose that Y is infinite dimensional and separable with respect to $\|\cdot\|$, and let $\{g_p\}_0^\infty$ be a simple ordering of a maximal orthonormal set in the space $\{Y,\langle\cdot,\cdot\rangle\}$; let R be the pre-ring of all degenerate subsets of the set L of all nonnegative integers; as in Example 2, let β be defined from R to $L(Y)^+$ by

$$\beta(\{p\})\eta = \langle\eta,g_p\rangle g_p \text{ for each p in L and } \eta \text{ in Y.}$$

Let the collection Ω consist of β together with the zero function from R to $L(Y)^+$, so that $S(\Omega)$ is simply H_β and $\|\cdot\| = \|\cdot\|_\beta$ as described in the paragraph preceding Example 1. It may be shown here, as suggested in Example 3, that the equations $\lambda(\xi)(t) = \beta(t)\xi$, for $\{\xi,t\}$ in $Y \times R$, define a linear isometry λ from $\{Y,\langle\cdot,\cdot\rangle\}$ onto the space $\{H_\beta,Q_\beta\}$. Suppose μ is in D_1: if p is in L then $\sigma(\mu)(\beta)(\{p\})$ maps the $\beta(\{p\})$-image of Y into itself and so may be realized as a complex scalar c_p, whence $|\zeta(\mu)| = \sup_{n \text{ in } L}|c_n|$ and if f is in $S(\Omega)$ then

$$\mu(f) = \int_{L/F}[\beta^{-1/2}\sigma(\mu)(\beta)\cdot *]*\beta^{-1/2}f = \lim \Sigma_{n\geqslant 0}c_n\langle\lambda^{-1}(f),g_n\rangle g_n$$

and $\|\mu(f)\|^2 = \lim \Sigma_{n\geqslant 0}|c_n|^2|\langle\lambda^{-1}(f),g_n\rangle|^2$. It follows from this that the norm $\|\cdot\|$ is $\|\lambda^{-1}(\cdot)\|$, which is N_β. Therefore the space E is D_0, which consists of all composites $B\lambda^{-1}$ for B in $L(Y)^c$, and so includes D_1 as a proper subset.

Pursuant to Remark 3 following Theorem 14, concerning the interpretation of a Hellinger integral space as a completion of a linear space of equivalence classes of R-simple functions from L to Y, there arises a kind of differential equivalence notion. With β a finitely additive function from R to $L(Y)^+$, and the set L itself assumed to belong to the pre-ring R, suppose k is a function from R to Y (such as, *e.g.*, a composite $\xi[c]$ for some R-simple ξ from L to Y and some choice function c from R to L) and the finitely additive function f from R to Y is given by integral formulas $f(t) = \int_{t/F}\beta\cdot k$ for t in R. Attention is directed to conditions on k so that f should belong to H_β and $\int_{L/F}\|\beta^{-1/2}f - \beta^{1/2}k\|^2 = 0$. The following Theorem provides an Example of such conditions on the function k (a preliminary version of this was announced in Abstract 623-25, Notices Amer. Math. Soc., 12(1965), 357).

THEOREM 27. *Suppose that the pre-ring* R *contains the set* L, β *is a finitely additive function from* R *to* $L(Y)^+$, *and* k *is a function from* R *to* Y *such that if* t *is in* R *then* $f(t) = \int_{t/F}\beta\cdot k$ *exists weakly in* $\{Y,\langle\cdot,\cdot\rangle\}$. *If for each set* t *in* R, $\int_{t/F}\|\beta^{1/2}k\|^2 = \int_{t/F}\langle k,f\rangle$ *then* f *belongs to* H_β *and* $\int_{L/F}\|\beta^{-1/2}f - \beta^{1/2}k\|^2 = 0$ *and, for each* t *in* R *and* g *in the space* H_β, $\int_{t/F}\langle k,g\rangle = \int_{t/F}\langle\beta^{-1/2}f,\beta^{-1/2}g\rangle$.

INDICATION OF PROOF. Assuming R, β, k, and f as indicated, suppose that if t is in R then $h_1(t) = \int_{t/F}\langle k,f\rangle$ and $h_2(t) = \int_{t/F}\|\beta^{1/2}k\|^2$. If the member M of F fills up the member t of R then (by the inequalities established in Theorem 9)

$$\|\beta(t)^{-1/2}\Sigma_{v \text{ in } M}\beta(v)k(v)\|^2 \leqslant \Sigma_{v \text{ in } M}\|\beta(v)^{1/2}k(v)\|^2$$

whence, for each η in Y,

$$|\langle\Sigma_{v \text{ in } M}\beta(v)k(v),\eta\rangle|^2 \leqslant \Sigma_{v \text{ in } M}\|\beta(v)^{1/2}k(v)\|^2\langle\eta,\beta(t)\eta\rangle,$$

so that $|\langle f(t),\eta\rangle|^2 \leqslant h_2(t)\langle\eta,\beta(t)\eta\rangle$. Therefore, by Theorem 10, f belongs to H_β and $N_\beta(f)^2 \leqslant h_2(L)$. Again, if M is a member of F filling up the member t of R,

$$|\Sigma_{v \text{ in } M}\langle k(v),f(v)\rangle|^2 \leqslant \Sigma_{v \text{ in } M}\|\beta(v)^{1/2}k(v)\|^2\Sigma_{v \text{ in } M}\|\beta(v)^{-1/2}f(v)\|^2$$

whence $|h_1(t)|^2 \leqslant h_2(t)\int_{t/F}\|\beta^{-1/2}f\|^2 \leqslant h_2(t)^2$: thus, the assumption that h_1 is h_2 implies that $h_2(t) = \int_{t/F}\|\beta^{-1/2}f\|^2$ for each t in R. Suppose, now, that h_1 is h_2: if the member M of F fills up the member t of R then

$$\Sigma_{v \text{ in } M}\|\beta(v)^{-1/2}f(v) - \beta(v)^{1/2}k(v)\|^2$$

$$= \Sigma_{v \text{ in } M}\|\beta(v)^{-1/2}f(v)\|^2 - 2 \operatorname{Re} \Sigma_{v \text{ in } M}\langle k(v),f(v)\rangle + \Sigma_{v \text{ in } M}\|\beta(v)^{1/2}k(v)\|^2$$

so that $\int_{t/F}\|\beta^{-1/2}f - \beta^{1/2}k\|^2 = N_\beta(P_tf)^2 - 2\, h_1(t) + h_2(t) = 0$. The argument may be completed by noting that, for appropriate g and M and t,

$$|\Sigma_{v \text{ in } M}\langle\beta(v)^{-1/2}f(v),\beta(v)^{-1/2}g(v)\rangle - \Sigma_{v \text{ in } M}\langle k(v),g(v)\rangle|^2$$

$$\leqslant \Sigma_{v \text{ in } M}\|\beta(v)^{-1/2}f(v) - \beta(v)^{1/2}k(v)\|^2\Sigma_{v \text{ in } M}\|\beta(v)^{-1/2}g(v)\|^2.$$

In the light of Theorem 14, there is another interpretation of the condition, relating f and k, to which attention has now been drawn. Consider the following:

THEOREM 28. *If β is a finitely additive function from* R *to* $L(Y)^+$ *and* f *is in the space* H_β *and* k *is a function from* R *to* Y *and* h *is a function from* R *to* H_β *such that* $h(t)(s) = P_t\beta(s)k(t)$ *for each* $\{s,t\}$ *in* R × R, *then the assertion that* $\int_{L/F}\|\beta^{-1/2}f - \beta^{1/2}k\|^2 = 0$ *is the assertion that* $f = \int_{L/F}h$ *with respect to* N_β.

On the basis of Theorem 14 it may be seen that, with g the function from R to H_β given by $g(t)(s) = [\beta(t)^{-1/2}P_t\beta(s)]^*\beta(t)^{-1/2}\{f(t) - \beta(t)k(t)\}$ for each $\{s,t\}$ in R × R, if M is in F then $\Sigma_{t \text{ in } M}g(t) = \Pi_\beta(M)f - \Sigma_{t \text{ in } M}h(t)$ and

$$\Sigma_{t \text{ in } M}\|\beta(t)^{-1/2}f(t) - \beta(t)^{1/2}k(t)\|^2 = N_\beta(\Pi_\beta(M)f - \Sigma_{t \text{ in } M}h(t))^2;$$

therefore, Theorem 28 may be proved as a direct consequence of Theorem 14.

As has been noted elsewhere [8], the evaluation kernels arising with the space Y one dimensional (*i.e.*, the complex plane) are the "positive matrices" of E. H. Moore's General Analysis [13]. In the context of Theorems 1-6, omitting the special pre-ring hypothesis on R, suppose that $\{h,q\}$ is a complete inner product space of complex functions on the set R and k is a complex function on $R \times R$ such that if t is in R then $k(\cdot,t)$ is in h and $q(f,k(\cdot,t)) = f(t)$ for each f in h: it may be shown that the function K from $R \times R$ to $L(Y)^c$, $K(s,t)\eta = k(s,t)\eta$ for η in Y and $\{s,t\}$ in $R \times R$, satisfies the first system of inequalities indicated in Theorem 2. Hence, the space $\{H,Q\}$ (in which K is the evaluation kernel) may be viewed as a "vectorization" (relatively, of course, to the space $\{Y,\langle\cdot,\cdot\rangle\}$) of the original space $\{h,q\}$. As one simple instance of this, if $\{h,q\}$ is the usual Hardy space of complex analytic functions on the (open) unit disc of the complex plane with $k(s,t) = (1-st^*)^{-1}$ for $\{s,t\}$ in $R \times R$, it is easily seen that $\{H,Q\}$ is the space of analytic functions f from R to Y with convergent $\Sigma_{n \geqslant 0}(\|f^{(n)}(0)\|/n!)^2$, and Q the inner product given by $Q(f,g) = \lim \Sigma_{n \geqslant 0}\langle f^{(n)}(0),g^{(n)}(0)\rangle/(n!)^2$ for $\{f,g\}$ in $H \times H$. Here is an Example to illustrate how certain familiar spaces may be considered as arising from the aforementioned vectorization procedure.

EXAMPLE 5. Let R be the space Y itself, and K be the function from $Y \times Y$ to $L(Y)^c$ given by $K(s,t)\eta = \langle s,t\rangle\eta$ for $\{s,t\}$ in $Y \times Y$ and η in Y. Now, if x is a function from a finite subset M of Y to Y then, for each maximal orthonormal set G in the space $\{Y,\langle\cdot,\cdot\rangle\}$,

$$\Sigma_{\{s,t\} \text{ in } M\times M}\langle x(s),K(s,t)x(t)\rangle = \Sigma_{\{u,v\} \text{ in } G\times G}|\Sigma_{t \text{ in } M}\langle t,u\rangle\langle v,x(t)\rangle|^2;$$

let $\{H,Q\}$ be the complete inner product space of functions from Y to Y such that $\{K,Y,H,Q\}$ is a kernel system. A function f from Y to Y belongs to H only in case there is a nonnegative number b such that, for each finite subset M of Y and each function x from M to Y,

$$|\Sigma_{t \text{ in } M}\langle f(t),x(t)\rangle|^2 \leqslant b\, \Sigma_{\{s,t\} \text{ in } M\times M}\langle x(s),x(t)\rangle\langle t,s\rangle,$$

in which case $Q(f,f)$ is the least such number b. Let G be a maximal orthonormal set in the space $\{Y,\langle\cdot,\cdot\rangle\}$: it may be proved that H consists of all members f of $L(Y)^c$ such that $\Sigma_{u \text{ in } G}\|fu\|^2$ exists, and that $Q(f,g) = \Sigma_{u \text{ in } G}\langle fu,gu\rangle$ for each $\{f,g\}$ in $H \times H$, so that if f is in H and $\{t,\eta\}$ is in $Y \times Y$ then

$$Q(f,K(\cdot,t)\eta) = \Sigma_{u \text{ in } G}\langle fu,\langle u,t\rangle\eta\rangle = \Sigma_{u \text{ in } G}\langle t,u\rangle\langle fu,\eta\rangle = \langle t,f^*\eta\rangle = \langle ft,\eta\rangle.$$

This space $\{H,Q\}$ is the space of "Hilbert-Schmidt operators" [3], earlier known as linear transformations of finite norm [20, page 66], in the space $\{Y,\langle\cdot,\cdot\rangle\}$.

Here is a final Example which may serve to illustrate the determination of a kernel system, conceptually simple relatively to the space $\{Y,\langle\cdot,\cdot\rangle\}$, which seems not to arise by the vectorization procedure to which attention has now been drawn.

EXAMPLE 6. Suppose R is a subset of $L(Y)^c$ containing at least one nonzero transformation, and K is the function from $R \times R$ to $L(Y)^c$ given by $K(s,t) = st^*$ for $\{s,t\}$ in $R \times R$. If x is a function from a finite subset M of R to Y then

$$\Sigma_{\{s,t\} \text{ in } M\times M}\langle x(s),K(s,t)x(t)\rangle = \|\Sigma_{t \text{ in } M} t^*x(t)\|^2 \geqslant 0;$$

let $\{H,Q\}$ be the complete inner product space of functions from R to Y such that $\{K,R,H,Q\}$ is a kernel system. A function f from R to Y belongs to H only in case there is a nonnegative number b such that, for each finite subset M of R and each function x from M to Y,

$$|\Sigma_{t \text{ in } M}\langle f(t),x(t)\rangle|^2 \leqslant b\|\Sigma_{t \text{ in } M} t^*x(t)\|^2,$$

in which case $Q(f,f)$ is the least such number b. Let Z be the $\|\cdot\|$-closure of the linear span of the $t^*(Y)$ for t in R: it may be proved that a function f from R to Y belongs to H only in case there is a member ξ of Z such that $f(s) = s\xi$ for each s in R, and that if $\{\xi,\eta\}$ is in $Z \times Z$ and $\{f,g\}$ is the member of $H \times H$ such that $f(s) = s\xi$ and $g(s) = s\eta$ for each s in R then $Q(f,g) = \langle\xi,\eta\rangle$, whence if $\{t,z\}$ is a member of $R \times Y$ then $Q(f,K(\cdot,t)z) = \langle\xi,t^*z\rangle = \langle f(t),z\rangle$.

TERMINAL COMMENT. As an alternative to the present setting, but a chapter in what could properly be called General Analysis in Hilbert Spaces, one might have a set R and a collection Ω of evaluation kernels in spaces of functions on R to Y - so that if $\{K_1,R,H_1,Q_1\}$ and $\{K_2,R,H_2,Q_2\}$ are kernel systems, with K_1 and K_2 in Ω, and neither of H_1 and H_2 is a subset of the other then both $K_1{:}K_2$ and K_1+K_2 belong to Ω - with analogous results for the linear span $S(\Omega)$ of the family of spaces H.

REFERENCES

1. W. N. Anderson and R. J. Duffin, *Series and parallel addition of matrices,* J. Math. Anal. Appl., 26(1969), 576-594.
2. R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space,* Proc. Amer. Math. Soc., 17(1966), 413-415.
3. N. Dunford and J. T. Schwartz, *Linear Operators Part II,* Interscience, New York, 1963.
4. P. A. Fillmore and J. P. Williams, *On operator ranges,* Advances in Math., 7(1971), 254-271.

5. M. R. Hestenes, *A ternary algebra with applications to matrices and linear transformations,* Arch. Rational Mech. Anal., 11(1962), 138-194.
6. P. H. Jessner, *Interrelated kernel systems,* Dissertation, University of North Carolina, 1962.
7. J. S. Mac Nerney, *Investigation concerning positive definite continued fractions,* Duke Math. J., 26(1959), 663-678.
8. ———, *Hellinger integrals in inner product spaces,* J. Elisha Mitchell Sci. Soc., 76(1960), 252-273.
9. ———, *Hermitian moment sequences,* Trans. Amer. Math. Soc., 103(1962), 45-81.
10. ———, *Dense embeddings of Hilbert spaces,* Proc. Amer. Math. Soc., 24(1970), 92-94.
11. ———, *Continuous embeddings of Hilbert spaces,* Rend. Circ. Mat. Palermo, 19(1970), 109-112.
12. ———, *Linear operations on a space of functions of bounded variation,* Finitely Additive Set Functions II.
13. E. H. Moore, *General Analysis,* Memoirs of the American Philosophical Soc., Part I, (1935), and Part II, (1939).
14. F. Riesz and B. Sz.-Nagy, *Functional Analysis,* Ungar, New York, 1955, (translation of second French edition, 1952).
15. Walter Rudin, *Functional Analysis,* McGraw-Hill, New York, 1973.
16. Habib Salehi, *The Hellinger square-integrability of matrix-valued measures with respect to a non-negative Hermitian measure,* Ark. Mat., 7(1968), 299-303.
17. Yu. L. Shmulyan, *An operator Hellinger integral,* Mat. Sb. (N.S.), 49(1959), 381-430, (Russian); Amer. Math. Soc. Translations 22(1962), 289-337.
18. ———, *Reduction of the Hellinger operator integral to a Lebesgue integral,* Izv. Vyss. Ucbn. Zaved. Matematika, 33(1963), 164-175, (Russian); Math. Rev., 33(1967), page 43, No. 248.
19. ———, *Two-sided division in the ring of operators,* Mat. Zametki, 1(1967), 605-610, (Russian); translated as Math. Notes, 1(1967), 400-403.
20. M. H. Stone, *Linear Transformations in Hilbert Space,* Amer. Math. Soc. Colloq. Publ., 15, New York, 1932.
21. B. Sz.-Nagy, *A moment problem for self-adjoint operators,* Acta Math. Acad. Sci. Hungar., 3(1952), 285-293.
22. ———, *Extensions of linear transformations in Hilbert space which extend beyond this space,* Ungar, New York, 1960, (trans. of 1955 Appendix to [14]).
23. J. von Neumann, *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren,* Math. Ann., 102(1930), 49-131.
24. ———, *Uber Adjungierte Funktionaloperatoren,* Ann. of Math., 33(1932), 294-310.
25. ———, *Functional Operators Vol. II,* Annals of Math. Studies, 22(1950), Princeton University Press, Princeton.

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