Survey 1979

HOUSTON JOURNAL OF MATHEMATICS

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Published quarterly by the Department of Mathematics, University of Houston, Houston, Texas 77004. The price per volume is $35.00; outside of the United States and Canada, $40.00. Each volume consists of four quarterly issues. Subscription orders should be addressed to the Managing Editor.

EQUATIONAL LOGIC

Walter Taylor

This is a survey of existing work of many authors in *equational logic* or varieties of algebras. Our primary interest is in equations for general algebraic systems, and we will not report in detail on equations in special systems (e.g., fields, where equations began). As a branch or "fragment" of general first order logic, this subject has two aspects, one focusing attention on the formal expressions (in this case, the equations), and the other focusing on the models of these equations. We give slightly more attention to the first of these, focusing, until §13, on sets of equations. This survey owes much to an expository article of Tarski [413] in 1968, and to some unpublished notes of D. Pigozzi (ca. 1970). Our exposition will be self-contained for the general mathematician, the only special prerequisite being a rudimentary understanding of the term "decidable." We include no proofs. This survey originated in a series of talks at the 1975 Summer Research Institute of the Australian Mathematical Society. Some valuable suggestions about this article were made by G. Bergman, W. J. Blok, S. Comer, B. Csákány, B. Davey, A. Day, G. Grätzer, W. Hodges, B. Jónsson, R. McKenzie, G. McNulty, J. Mycielski, E. A. Palyutin, D. Pigozzi, A. Pixley, and A. D. Taĭmanov.

Our objective is to make more mathematicians aware of this subject and to provide a readable introduction to its examples, its theorems, and what they mean in a fairly broad context. At the same time we hope to provide a reasonably complete survey of the literature which will be helpful to specialists. For these reasons (and others) we have omitted all proofs, and so perhaps we should warn the reader of one aspect of the subject on which we have not commented in detail: which of these results were most difficult to prove.

But here we may mention one of the attractions of equational logic: there are hard and interesting theorems which are very easy to state (a property of all attractive forms of mathematics from the Theorem of Pythagoras onward). While the maturity and value of mathematical logic are unquestioned nowadays, we hope the reader will also gain an insight into the present-day vigor (if not yet maturity) of general algebra. This subject has been clouded by a skepticism ranging from Marczewski's [283] sympathetic warning:

> [In subjects like general topology and general algebra] it is easy to get stranded in trivial topics, and caught in the net of overdetailed conditions, of futile generalizations.

to the outright malediction: "nobody should specialize in it" ([184], [64]). This injunction is certainly out of date (if indeed it ever was valid), and we hope this survey will be adequate evidence of the successes of specialists in universal algebra. And perhaps this survey will help dispel another (closely related) myth, which is epitomized by Baer's remark [16, page 286], "The acid test for [a wide variety of methods in universal algebra] will always be found in the theory of groups." Many interesting results and ideas here either collapse completely or become hopelessly complicated when applied to groups; but there is no lack of interesting classes of algebras defined by equations to which the theories may be applied, as we shall see. And this again is one of the attractions of the subject.

The writing of this survey was supported, in part, at various times, by the University of Colorado, the Australian-American Educational Foundation and the National Science Foundation. An early ancestor of this survey was a report of the Australian S. R. I. lectures which appeared in the proceedings of the Szeged Universal Algebra conference of 1975.

Although we believe the material unfolds rather naturally in the order we present it, only § §1, 2, 3, 5 are essential to read first.

EQUATIONAL LOGIC

CONTENTS

1. Early history and definitions. For a very readable history of algebra, consult Birkhoff [51], [52], [53]. The role of algebraic equations was pronounced from the start (e.g. duplication of a cube, solvability of third and fourth degree equations, unsolvability of fifth degree equations, etc.). We are concerned with the *identical* satisfaction of an equation - e.g. the *associative law*

$$x + (y + z) = (x + y) + z$$

holds for *all* real numbers x, y, z. The importance of equations holding identically emerged with the axiomatic approach to group theory and ring theory, and later with Boolean algebra (late 19^{th} century) and lattice theory (early 20^{th} century). The first *general* result on identities was Birkhoff's 1935 theorem [48] which is stated in detail in §3 below.

A *type* is a family $(n_t)_{t\in T}$ of natural numbers $(0 \leqslant n_t < \omega)$ (where ω = first infinite ordinal). (Typically, in our discussion the type $(n_t)_{t\in T}$ is arbitrary but fixed.)

An *algebra* [*of type* $(n_t)_{t\in T}$] is a structure of the form $\mathbf{A} = (A,F_t)_{t\in T}$, where for each $t \in T$, F_t is a function

$$F_t\colon A^{n_t} \to A$$

(sometimes called an n_t-ary operation). (Remember that A^0 is a singleton, and so if $n_t = 0$, then F_t has one-point range, i.e. F_t can be thought of as a designated element - a "constant" - of A.)

A *homomorphism* $\varphi\colon \mathbf{A} \to \mathbf{B}$ between algebras $\mathbf{A} = (A,F_t)_{t\in T}$ and $\mathbf{B} = (B,G_t)_{t\in T}$ (of the same type) is a function $\varphi\colon A \to B$ such that always

$$\varphi F_t(a_1,a_2,\ldots) = G_t(\varphi a_1,\varphi a_2,\ldots).$$

If φ is onto then $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$; if φ is the inclusion map of $A \subseteq B$, then $\mathbf{A}$ is a *subalgebra* of $\mathbf{B}$. If $\mathbf{A}_i = (A_i,F_{it})_{t\in T}$ $(i \in I)$ are algebras all of the same type, then we define the *product*

$$\prod_{i\in I} \mathbf{A}_i = (\Pi A_i,F_t),$$

where

$$F_t(\alpha_1,\alpha_2,\ldots) = \langle F_{it}(\alpha_{1i},\alpha_{2i},\ldots)\colon i \in I\rangle.$$

A *congruence relation* on $\mathbf{A}$ is any equivalence relation θ on A given by $(a,b) \in \theta$ iff $\varphi(a) = \varphi(b)$ for some fixed homomorphism $\varphi\colon \mathbf{A} \to \mathbf{B}$.

2. The existence of free algebras. Let V be any class of algebras of fixed type. By definition a *V-free algebra on the set* X (denoted $\mathbf{F}_V(X)$) is an algebra $\mathbf{B} = (B, F_t)_{t \in T}$ such that

(1) $\mathbf{B} \in V$;

(2) $X \subseteq B$;

(3) if $\mathbf{A} \in V$ and $\varphi_0: X \to A$ is any function, then there exists a unique homomorphism $\varphi: \mathbf{B} \to \mathbf{A}$ with $\varphi \restriction X = \varphi_0$.

One easily checks that if $\mathbf{B}$ and $\mathbf{C}$ are each V-free on X, then there exists a unique isomorphism $\varphi: \mathbf{B} \to \mathbf{C}$ with φ the identity on X. (Thus we may say "the" V-free algebra on X.)

THEOREM. [48]. *If V is any non-trivial class of algebras closed under the formation of subalgebras and products, then $\mathbf{F}_V(X)$ exists for every* X.

A proof can be found in any of our references on general algebra. Birkhoff's original proof has been abstracted in category theory to yield the "adjoint functor theorem." See e.g., [143, page 84]. For some other adjointness results on classes V see e.g., [432] and [346, pages 148-149]. Such generalized free objects (e.g. tensor algebras and universal enveloping algebras) were important in Lawvere's development of an invariant approach to this subject (see §7 below).

Familiar examples of free algebras are free groups and free Abelian groups. Thus in some (but not all) cases, elements of free algebras can be written as "words." (We will return to this point in §12 below.) This more concrete description of free algebras has also caught the attention of category theorists; see e.g. Gray [170].

It is unclear whether there is a successful generalization of the Theorem to more general classes V. Grätzer [163, Chapter 8] proposed such a generalization for classes V defined by a set Σ of first order sentences. But unfortunately his "free algebra" was not independent of the choice of axiomatization Σ [Colorado Logic Seminar, Spring, 1969, unpublished].

Perhaps the most remarkable recent result on free algebras is Shelah's [397]: if λ is a singular cardinal, every subalgebra of $\mathbf{A}$ with $< \lambda$ generators is free, and $\mathbf{A}$ has cardinality λ, then $\mathbf{A}$ is a free algebra.

Notational convention henceforth: $\boldsymbol{K}$ is the class of *all* algebras of the fixed type

$(n_t)_{t\in T}$. By the theorem, $\mathbf{F}_{\boldsymbol{K}}(X)$ always exists. It is sometimes called "absolutely free."

A *term* is an element of the absolutely free algebra $\mathbf{F}_{\boldsymbol{K}}(X)$, an *equation* is a pair of terms (σ,τ), usually written $\sigma = \tau$ or similarly. In our context (namely, the identical satisfaction of equations), equations are often referred to as *identities* or *laws* (or sometimes even by the astronomical term "syzygy" - see e.g. [440]).

To know terms more explicitly, we should have a more explicit representation of the absolutely free algebra $\mathbf{F}_{\boldsymbol{K}}(X)$. (One such representation is adequate, since all absolutely free algebras on X are isomorphic *via* a unique isomorphism over X.) To do this, we will first redefine "term" by the following recursive scheme for generating formal expressions (regarding the members of X and the symbols $F_t (t \in T)$ as belonging to an "alphabet"):

(1) x is a term whenever $x \in X$;

(2) $F_t\alpha_1 \cdots \alpha_{n_t}$ is a term whenever $\alpha_1,\ldots,\alpha_{n_t}$ are terms.

Let $\boldsymbol{T}$ be the set of all terms, and define operations $\hat{F}_t\colon \boldsymbol{T}^{n_t} \to \boldsymbol{T}$ via

$$\hat{F}_t(\alpha_1,\ldots,\alpha_{n_t}) = F_t\alpha_1 \cdots \alpha_n.$$

EXERCISE. $(\boldsymbol{T},\hat{F}_t)_{t\in T}$ is $\boldsymbol{K}$-free on X.

EXAMPLE. If we are dealing with one binary operation F, then the following are in $\boldsymbol{T}$ (and hence are terms):

$$Fxy,\ Fyx,\ FxFyz,\ FFxyz.$$

And so the following are equations:

$$Fxy = Fyx$$

$$FxFyz = FFxyz,$$

which are readily recognized as the usual commutative and associative laws for F.

3. **Equationally defined classes.** We say that the *equation* $\sigma = \tau$ *holds identically in the algebra* **A**, in symbols

$$\mathbf{A} \models \sigma = \tau,$$

iff $\varphi(\sigma) = \varphi(\tau)$ for every homomorphism $\varphi\colon \mathbf{F}_{\boldsymbol{K}}(X) \to \mathbf{A}$. (N.b. recall that $\varphi\colon \mathbf{F}_{\boldsymbol{K}}X \to \mathbf{A}$ is given exactly by $\varphi_0\colon X \to A$. Thus our definition is easily seen to be a precise formulation of the idea that σ and τ "come out the same" nomatter what elements of

A are taken as values of the "variables" $x \in X$ appearing in σ and τ, i.e. the usual idea for familiar equations like the associative law.) An *equationally defined* class of algebras, alias a *variety*, is a class V for which there exists a set Σ of equations with

$$V = \text{mod } \Sigma = \{ \mathbf{A}\text{: for all } e \in \Sigma, \mathbf{A} \models e \}.$$

(Here "mod Σ" abbreviates "the class of all **models** of Σ.")

THEOREM. (Birkhoff 1935 [48]). *V is a variety if and only if V is closed under formation of products, homomorphic images and subalgebras.*

This enormously important result, in a style almost unheard of at its time, effectively began "model theory." (See e.g. Tarski [411].) It can be considered the ancestor of almost all research described in this survey. And yet in his history of modern algebra [52], Birkhoff alludes to it in half a line only!

This theorem has been followed over the years by many others of a similar format - sometimes called "preservation theorems" since Birkhoff's theorem (together with compactness) has the corollary that if a sentence φ is preserved under formation of homomorphic images, subalgebras and products, then φ is equivalent to a conjunction of equations. For instance Keisler and Shelah proved that *a class **L** of structures is definable by a set of first order sentences iff **L** is closed under the formation of isomorphic structures, ultraproducts and ultraroots.* (Keisler proved this assuming the G.C.H., and Shelah [396] without. See also [7].) For many other preservation theorems, see e.g. [276]. More in keeping with the algebraic results of this survey are the following three theorems.

THEOREM. ([228]; see also [366]). *V is definable by regular equations if and only if V is closed under the formation of products, subalgebras, homomorphic images and sup-algebras.*

(An equation is *regular* if and only if exactly the same variables appear on both sides. The *sup-algebra* of type $(n_t)_{t\in T}$ (unique within isomorphism) is the algebra $(\{0,1\}, F_t)_{t\in T}$, where for each t,

$$F_t(a_1,\dots,a_{n_t}) = \begin{cases} 0 & \text{if } a_1 = \cdots = a_{n_t} = 0 \\ 1 & \text{otherwise.)} \end{cases}$$

THEOREM. ([151], [55]; see also [395]). *V is definable by linear equations if* *and only if V is closed under the formation of products, subalgebras, homomorphic images and complex algebras.*

(An equation is *linear* iff each side has at most one occurrence of every variable. If $\mathbf{A} = (A,F_t)_{t\in T}$ is any algebra, the *complex algebra* of $\mathbf{A}$ is $\mathbf{B} = (B,G_t)_{t\in T}$, where B is the set of non-empty subsets of A, and

$$G_t(u_1,\ldots,u_{n_t}) = \{ F_t(a_1,\ldots,a_{n_t}) \colon a_i \in u_i \ (1 \leqslant i \leqslant n_t)\} .)$$

The next important result really goes back to J.C.C. McKinsey [303] in 1943 (he proved a theorem which, in combination with the above theorem of Keisler and Shelah, immediately yields our statement). The present formulation was probably first given by A. I. Malcev [280, page 214], [279, page 29]; many other proofs have been independently given [166], [307], [394], [145], [28] - although the precise formulation differs from author to author. See also [85, Theorem 6.2.8, page 337], and for related results [33], [239],[186] and [187].

THEOREM. *V is definable by equational implications iff V is closed under the formation of products, subalgebras and direct limits.*

An *equational implication* is a formula of the form

$$(e_1 \mathbin{\&} e_2 \mathbin{\&} \cdots \mathbin{\&} e_n) \rightarrow e,$$

where $e,e_1,\ldots,e_n$ are equations, for example the formula

$$(xy = xz \rightarrow y = z)$$

defining *left-cancellative* semigroups among all semigroups. For *direct limits* see [163], [143] (or any other book on category theory). For some interesting classes defined by equational implication, see [417] and [40]. For some infinitary analogs of Birkhoff's theorem see [402], and of McKinsey's theorem, see [186]. In the next result, infinitary formulas are in a sense forced upon one, even though it is a result about ordinary finitary algebras. A *generalized equational implication* is a formula

$$\bigwedge_{i\in I} e_i \rightarrow e,$$

where e, e_i $(i \in I)$ are equations (possibly infinitely many). The next theorem was perhaps first stated in [33], although maybe some other people knew of it.

THEOREM. *V is definable by a class of generalized equational implications iff V* *is closed under the formation of products and subalgebras.*

Fisher has in fact shown [138] that we can always take this class of formulas to be a ***set*** iff *Vopěnka's principle* holds. (This is one of the proposed "higher" axioms of set theory.) Some less conclusive results about classes closed under the formation of products and subalgebras occur in [196], [177], and [178]. Some mistakes of [155] are corrected in [327].

For some other infinitary (in this case, topological) analogs of Birkhoff's theorem, see [112], [108], and [428]. (A unified treatment appears in [109].) For instance, the condition

(*) $n!\,x \to 0$

defines a class of topological Abelian groups (here $\to$ means "converges to"), which contains all finite discrete groups but not the circle group. In [428] there is a theory of classes defined by conditions similar to (*); these classes are called "varieties" of topological algebras.

See [57] for another analog of Birkhoff's theorem which goes beyond pure algebra.

4. Generation of varieties and subdirect representation. Let V_0 be any collection of algebras of the same type. Since the intersection of any family of varieties is again a variety, there exists a *smallest* variety $V \supseteq V_0$. It clearly follows from §3 that

$$V = \text{Mod Eq } V_0,$$

where Eq V_0 means the set of equations holding identically in V_0. It is also very easy to prove (using Birkhoff's theorem of §3) that

$$V = \mathbf{HSP}\ V_0,$$

since, as one easily checks, the R.H.S. is **H**-, **S**- and **P**-closed. Problem 31 of Grätzer's book [163] asks whether this fact implies the axiom of choice. (For M any class of algebras, **HM**, **SM**, **PM** denote the classes of algebras isomorphic to homomorphic images, subalgebras and products of algebras in M.) In practice, **S** and **P** seem natural enough, affording a "coördinate" representation of [some] algebras in V using algebras of V_0. But **H** seems less natural, and one hopes in favourable circumstances to avoid it, arriving at

(*) $V = \mathbf{SP}\, V_0$,

an equation which one can occasionally prove for a given V_0, or, less difficult, given V, one can look for a manageable V_0 for which (*) is true. For example it is historical that if V is the variety of vector spaces over a fixed field, then (*) holds for V_0 containing only a single one-dimensional space. And Stone's (1936) representation theorem for Boolean algebras said (in part) that if V is the variety of Boolean algebras, then (*) holds for V_0 consisting of only a two-element algebra.

The key to understanding (*) in general is Birkhoff's (1944) subdirect representation theorem [49], which in fact is a generalization of Stone's theorem above. An algebra $\mathbf{A}$ is said to be *subdirectly irreducible* iff it cannot be non-trivially embedded in a product of other algebras, i.e. any family of homomorphisms separating points of $\mathbf{A}$ must contain some one-one homomorphism.

THEOREM. (G. Birkhoff). *Every algebra* $\mathbf{A} \cong$ *a subalgebra of a product of subdirectly irreducible algebras, each a homomorphic image of* $\mathbf{A}$.

COROLLARY 1. *(*) holds iff* $\mathbf{S}V_0$ *contains all subdirectly irreducible algebras of* V.

Birkhoff's Theorem above makes essential use of the fact that all operations are finitary (i.e. $n_t < \aleph_0$ for all $t \in T$). For some counterexamples in the domain of infinitary algebra, see [36] and [113]. (Such counterexamples are implicit in Grätzer and Lampe [Notices A.M.S., 19(1972), A-683].)

Grätzer proved that this theorem implies the axiom of choice, answering a question of Rubin and Rubin. See [163, Exercise 102, page 160].

(Notice that every simple group is subdirectly irreducible, and so Corollary 1 tells us that (*) cannot hold for the variety of groups unless V_0 is already a proper class. In 14.8 below we will return to the distinction between varieties which have "good" subdirect representation theories, and those which do not.)

COROLLARY 2. *If two varieties contain exactly the same subdirectly irreducible algebras, then they are the same.*

EXERCISE. [49]. *If* R *is a subdirectly irreducible commutative ring without non-zero nilpotents, then* R *is a field.*

Consult [76] and [31] for a general treatment of subdirect irreducibility in model theory, and [84] for a treatment in the Bourbaki framework of mathematics. Consult [357], [323] and [25] for a general theory of manipulation of **H**, **S** and **P**.

5. **Equational theories.** Birkhoff's theorem of §3 sets up a one-one correspondence between varieties V and certain sets Σ of equations

$$V \leftrightarrow \Sigma$$

via

$$V \mapsto \operatorname{Eq} V$$

$$\operatorname{Mod} \Sigma \mapsfrom \Sigma.$$

The sets Σ appearing here (as Eq V) are called *equational theories*. One easily sees that Σ is an equational theory iff $e \in \Sigma$ whenever $\Sigma \models e$; i.e., e is true in every model of Σ, i.e. e is a *consequence* of Σ. Birkhoff's next result was to *axiomatize* the consequence relation, as follows:

(1) $\sigma = \sigma$ is always an axiom.

(2) From $\sigma = \tau$, deduce $\tau = \sigma$.

(3) From $\rho = \sigma$ and $\sigma = \tau$, deduce $\rho = \tau$.

(4) From $\sigma_i = \tau_i$ $(1 \leqslant i \leqslant n_t)$, deduce

$$F_t(\sigma_1,\ldots,\sigma_n) = F_t(\tau_1,\ldots,\tau_n).$$

(5) From $\sigma(x_1,\ldots,x_n) = \tau(x_1,\ldots,x_n)$, deduce

$$\sigma(\rho_1,\ldots,\rho_n) = \tau(\rho_1,\ldots,\rho_n).$$

(In (5), σ, τ, $\rho_1,\ldots,\rho_n$ are any terms and $\sigma(\rho_1,\ldots,\rho_n)$ is defined as the image of σ under a homomorphism of $\mathbf{F}_K(X)$ given by mapping $x_i \mapsto \rho_i$ $(1 \leqslant i \leqslant n)$ - as one may check, this is a precise expression of a naive idea of *substitution*.) We write $\Sigma \vdash e$ if there exists a (finite) proof of e starting from Σ and using only the rules (1) - (5).

THEOREM. (Birkhoff, [48]). $\Sigma \models e$ *iff* $\Sigma \vdash e$.

It is sometimes useful to know refined versions of this "completeness" theorem, which state a similar result for different (usually more restrictive) variations on the notion of $\vdash$. See for instance [74, page 40] for one; similar methods go back to Tarski. For the main result of Tarski [414] (stated at the beginning of §11 below), it is important to know that (4) can be replaced by some rules of proof which have only one antecedent (as is not too difficult to see). We will describe briefly one other derivation system for "equational logic" which has been useful for proving many of the undecidability results of §12. For *semigroups* it was originated by Kuroš (see [251]); see also [305], [306]. We first assume Σ is closed under forming substitutions. By a "derivation" we mean a sequence $(\sigma_1,\ldots,\sigma_n)$ such that for $i = 1,2,\ldots,n-1$, there exists $(\alpha = \beta) \in \Sigma$ such that α (or β) is a subterm of σ_i and σ_{i+1} results from σ_i by replacing the subterm α by β (resp. α). Then if we define $\Sigma \vdash \sigma = \tau$ to mean that there exists a derivation $(\sigma_1,\ldots,\sigma_n)$ with $\sigma_1 = \sigma$ and $\sigma_n = \tau$, then the above theorem of Birkhoff remains true.

Proof calculi have also been useful in establishing some interpolation and definability results [142], [200], [187]. Some (more computational) rules are given in Knuth and Bendix [248] and used in Glennie [158]. See also [43, §10].

Theorems parallel to this completeness theorem of Birkhoff are not numerous. There is of course Gödel's complete set of rules of proof for first order logic. (For second order logic, no such set of rules can exist, but see Karp [238], Keisler [242] for some rules which go beyond first order logic.) A. Selman [393] has given a set of rules for equation implications (independently discovered by D. Kelly [unpublished]), and Słomiński gave an infinitary analog of Birkhoff's theorem in [402]. G. McNulty has asked [307] whether a simple set of rules exists for the class of all *positive* sentences. See any logic book (e.g. [267] or [85]) for more information on $\vdash$. A large portion of this survey is concerned with the relation $\vdash$; in the next section we will specifically illustrate Birkhoff's completeness theorem.

6. Examples of $\models$. Let P denote the axioms of commutative associative ring theory. We will show that

(*) $P, x^{48} = x \models x^2 = x.$

(Definition in §5.) For this it is enough to prove that every ring obeying $x^{48} = x$ also obeys $x^2 = x$, and by Corollary 2 in §4, it is enough to check subdirectly irreducible rings obeying $x^{48} = x$. Such a ring clearly has no non-zero nilpotent elements, and so by the exercise in §4, must be a field, having $q \leqslant 48$ elements. One easily checks that for some m, $m(q-1) + 1 = 48$, i.e. $m(q-1) = 47$ and so either $(q-1) = 47$, i.e. $q = 48$, impossible since q must be a prime power; or $q - 1 = 1$, i.e. $q = 2$, yielding the two-element field, which does obey the law $x^2 = x$, proving (*). And so by Birkhoff's completeness theorem (§5),

$$P, x^{48} = x \vdash x^2 = x.$$

(It is an interesting exercise to try to perform this deduction directly, using any of the methods of §5.) (Cf. e.g. [176].)

The number of places where equational deductions occur in the mathematical literature is too great to be catalogued. An interesting example concerns some ring-theoretic identities of Hilbert (see e.g. [114]). For an interesting mistake, see [422]. Conway's book on "machines" contains whole chapters of equational deductions [92]. And "Baxter algebra" (a kind of abstraction of probability theory) proceeds partly *via* equational deductions [384]. For some interesting and nontrivial deductions in lattice theory see [291], and in general algebra [342]. Some papers, such as [192], consist entirely of a single equational deduction.

The strength of equational deduction can be well appreciated from the words of Chin and Tarski [86] on *relation algebras* (see 9.23 below) "it has even been shown that every problem concerning the derivability of a mathematical statement from a given set of axioms can be reduced to the problem of whether an equation is identically satisfied in every relation algebra. One could thus say that, in principle, the whole of mathematical research can be carried out by studying identities in the arithmetic of relation algebras." This idea has been carried further in Tarski's forthcoming book [415]. (But the interest here is clearly theoretical, not practical – it is easier to examine mathematical problems directly than to translate them to identities.)

And so $\vdash$ seems decidedly non-trivial (a fact to be more firmly established in §12 below). Almost all equational deductions in the literature proceed *via* an informal mix of $\vdash$ and $\models$ (i.e. using the framework of $\models$, but also applying rules of $\vdash$ whenever obvious or convenient). For example, it is a familiar exercise that

$$\Gamma, x^2 = e \models xy = yx$$

(where Γ stands for (equationally expressed) axioms of group theory). Pursuing any "naive" proof of this should show one how to write a "formal" proof using $\vdash$. A more difficult exercise ([406], [259]) is Γ, $(x^{n_1}y^{n_1}) = (xy)^{n_1}, x^{n_2}y^{n_2} = (xy)^{n_2}, \ldots, x^{n_t}y^{n_k} = (xy)^{n_k} \models xy = yx$ iff g.c.d. $\{(n_1^2 - n_1), \ldots, (n_k^2 - n_k)\} = 2$.

One might also consider Albert's deduction [5] of full power-associativity of "algebras" (multiplicative vector spaces) over fields of characteristic $\neq 2,3,5$ from the laws $xy = yx$ and $(x^2x)x = x^2x^2$.

7. **Equivalent varieties.** We mention two of the many possible ways of axiomatizing group theory equationally (not to mention non-equational forms such as "for all x there exists y $(xy = e)$").

$$\begin{aligned} \Gamma_1:\ & x(yz) = (xy)z \\ & u\cdot(xx^{-1}) = (y^{-1}\cdot y)\cdot u = u \\ \Gamma_2:\ & (xy)z = x(yz) \quad ex = xe = x \\ & x/x = e \quad x/y = x(e/y) \\ & u(e/u) = (e/u)u = e, \end{aligned}$$

(where / denotes "division"). Clearly Γ_1 and Γ_2 do not define the *same* variety, for they are of different types - (2,1) and (2,2,0). But examination of the models of Γ_1 and the models of Γ_2 will convince one that there is no *essential* difference between a non-empty Γ_1-group and a non-empty Γ_2-group. To make this sameness precise we introduce equations which will serve as *definitions*:

$$\begin{aligned} \Delta_1:\ & x/y = x\cdot y^{-1} \\ & e = x\cdot x^{-1} \\ \Delta_2:\ & x^{-1} = e/x. \end{aligned}$$

Now one may check that

(*) $\Gamma_1, \Delta_1 \vdash \Gamma_2$ and $\Gamma_2, \Delta_2 \vdash \Gamma_1$.

One more point is important. If we take one of the Δ_1 definitions of an operation F, i.e. $F = \alpha$, and substitute into α all the Δ_2 definitions, we get $F = \alpha[\Delta_2]$; then one should have

(**) $\Gamma_2 \vdash F = \alpha[\Delta_2]$ and likewise with the roles of Γ_1,Γ_2; Δ_1,Δ_2 reversed.

(E.g. Δ_1 says $x/y = x\cdot y^{-1}$. Upon substituting the Δ_2 definitions, we get $x/y = x\cdot(e/y)$, and this is indeed provable from Γ_2.) Now generally, equational theories Γ_1,Γ_2 are said to be *equivalent* iff there exist sets of definitions Δ_1,Δ_2 such that (*) and (**) hold.

(There is one intrinsic difference: Γ_1 has an empty model, but Γ_2 does not. Nonetheless, Γ_1 and Γ_2 are generally regarded as equivalent. To this extent, empty algebras do not matter, and some writers save themselves this and related considerations by always taking algebras to be non-empty. See Friedman [142] for a more general theory of definition within varieties. Operations may be implicitly definable (i.e., specified by the other operations) in V, but not explicitly definable except by arbitrarily complex formulas of first order logic.)

Equivalence has its model-theoretic aspect, too. *Varieties* V_1 *and* V_2 *are equivalent* (i.e. Eq V_1 and Eq V_2 are equivalent in the above sense) *iff there exists an isomorphism of categories* Φ: $V_1 \to V_2$ *which commutes with the forgetful functor to sets* (i.e. ΦV_1 has the same universe as V_1 and a similar fact holds for homomorphisms). (These categories are formed from the *non-empty* models of a variety and all the homomorphisms between them. Cf. the remarks in the 2nd paragraph on page 52 of [425]. For various references and remarks on this theorem of A. I. Malcev, see [420, page 355].) Perhaps the first historical example of an equivalence of varieties is the well known natural correspondence between *Boolean algebras* and *Boolean rings* (with unit). Also consider the correspondence between the varieties of *Abelian groups* and *Z-modules* – here the equivalence is so easy that some people write as if it were an equality. Some other interesting examples of equivalence may be found in [96].

Very close to the idea of equivalence (in it model theoretic form) is the idea of *weak isomorphism* as developed in Wrocław. This together with an emphasis on *independent sets* over free algebras gave equational logic a somewhat different direction and flavor in that school. See Marczewski [282] for an introduction to these ideas. Briefly, algebras **A** and **B** are weakly isomorphic iff there is a bijection φ: A $\to$ B such that the algebraic operations of **A** are exactly the same as the operations $\varphi^{-1}F(\varphi x_1,\ldots,\varphi x_n)$ where $F(x_1,\ldots,x_n)$ is an algebraic operation of **B**. (Here, by the family of algebraic operations, we mean the closure under composition of the family of all operations F_t together with all projection functions.) Then two varieties are equivalent iff they have weakly isomorphic generic algebras (see §8 for "generic").

Properties of varieties seem more natural and interesting if they are equivalence-invariant, if only because then they do not force us to make any "unnatural" choice between, say Γ_1 and Γ_2 above. For example, the similarity type

(2,1) is obviously not intrinsic to the idea of a group. Many of the properties considered below are (obviously) equivalence-invariant, but a few, such as being "one-based" (§10) are not, as we shall see, comparing 10.3 and 10.10 below. Moreover, certain results cannot even be stated without mentioning equivalence (14.2 and 14.5 below).

It is possible to define equational classes so as to make all expressable properties automatically equivalence-invariant, i.e. to give no preference to any of the possible equivalent forms of a given variety. This amounts to considering the *set* (rather than a sequence $(F_t)_{t \in T}$) of *all* possible operations defined by V-terms. This idea goes back to P. Hall (see [91, pages 126-132]), and has been worked out independently in detail by W. D. Neumann [332] and F. W. Lawvere [254] (see page 362 of [420] for more detailed historical remarks, and pages 390-392 for a proof - independently found by W. Felscher - of the equivalence of these two approaches). Some of these ideas were presented independently by Claude Chevalley in a speech at Stanford in November, 1962. Certainly Lawvere's approach came much sooner than Neumann's and has obtained a much wider following. We will not describe his invention, "algebraic theories," except to say that they contain precisely the right amount of information to describe varieties without allowing any individual operations to play a special role. For further references see [420, loc. cit.]; see also [272].

Despite some enthusiastic claims (see e.g. the dustjacket or Chapter 3 of [346] or [443, page 121]) that these category-theoretic ideas would take over the study of universal algebra, this hasn't really happened by 1979. Their significant role, so far, has been to suggest analogies outside pure algebra (e.g. compact Hausdorff spaces). But they have had almost no impact yet in the study of ordinary varieties (i.e. the kinds of subjects discussed in this survey), with one interesting exception: the study of *Malcev conditions* (§15 below) was facilitated by viewing it as a study of morphisms between algebraic theories (see [420] and [427]). Other possible directions are given in [347], [40], [58], and [244]. Some useful remarks are found in Lawvere [255]. The reason that direct application of "algebraic theories" to equational logic is difficult (or unnecessary) lies mostly in its model theory; to see this, let us notice (as many others have before) that the passage groups with a specific presentation $\longrightarrow$ abstract groups

is closely analogous to the passage

varieties, as defined by a set of laws $\longrightarrow$ algebraic theories.

Although this analogy is perfect for operation symbols and laws, unfortunately the models (i.e., the algebras in a given variety) fit more conveniently with the L.H.S., and somewhat spoil the analogy (although they correspond, very roughly, to the structure which is to be preserved in forming a group of automorphisms). Moreover equivalence-invariance of model theoretic properties is almost always transparent; and such properties can usually be discussed in a very simple invariant way (by considering the set of all operations - see e.g. [282]); cf. §§14-16 below.

8. **Bases and generic algebras.** As seen in §5, if Σ_0 is any set of sentences, the smallest equational theory $\supseteq \Sigma_0$ is

$$\text{Eq Mod } \Sigma_0 = \{e\colon \Sigma_0 \vdash e\},$$

and in this case we say that Σ_0 is a *set of axioms*, or an *equational base* for Σ. Several of the next sections are concerned with the problem of finding (various sorts of) bases Σ_0.

Here we consider what amounts to some concrete examples of Birkhoff's theorem of §3, namely we look for a base Σ_0 for a single algebra **A**, i.e., we want

$$\text{Mod } \Sigma_0 = \mathbf{HSP\ A}.$$

Actually, given Σ_0, **A** may be regarded as unknown. Here we refer to **A** as *generic* for the variety Mod Σ_0 or for the theory Σ = Eq Mod Σ_0. Using **P** one can easily see that every variety V has a generic algebra, i.e.

$$\text{for all } V \text{ there exists } \mathbf{A}(V = \mathbf{HSP\ A}).$$

One such **A** is the V-free algebra on $\aleph_0$ generators; see also [408]. We mention here a few examples of such **A** and Σ_0.

8.1. The ring Z of integers is generic for the theory of commutative rings.

8.2. The 2-element Boolean ring [with unit] is generic for the theory of Boolean rings [with unit], given by the laws for rings [with unit] together with the law $x^2 = x$. (Similarly for Boolean algebras, by remarks in §7; this fact may be interpreted as a completeness theorem for propositional logic.)

8.3. The algebra $(A; \cap, \cup, -, ^{-})$, where A is the family of all subsets of the Euclidean plane and $^{-}$ denotes topological closure, is a generic *closure algebra* (McKinsey and Tarski [304]; see also [408]).

8.4. Any non-commutative totally ordered ring is generic for the theory of rings, by a theorem of Wagner [435]. (Such rings were first constructed by Hilbert.) There is a long history of investigation of which rings can obey non-trivial polynomial identities (PI-rings); see [6], also [41], [42].

8.5. Each of the two 8-element non-commutative groups is generic for the variety of groups defined by the laws

$$x^4 = 1$$

$$[x^2, y] = 1$$

(where [,] denotes group commutator) [270].

8.6. The group of rigid motions of the plane is generic for the variety of groups defined by the law

$$[[x,y],[u,v]] = 1$$

(L. G. Kovács and M. F. Newman - from [331]).

8.7. The rotation group of a 2-sphere is generic for the variety of all groups (Hausdorff). (Notice that this statement has a meaning obviously invariant under equivalence, and so I do not have to state whether I mean e.g., Γ_1 or Γ_2 of §7. Similar remarks are applicable throughout §8.)

8.8. The group of all monotone permutations of $(R, \leqslant)$ is a generic *lattice-ordered group* (Holland [188]). (Here R denotes the set of real numbers, and $\leqslant$ its usual ordering.)

8.9. For fixed $p \in R$, the algebra

$$(R, px + (1 - p)y)$$

is generic for the laws

$$xx = x$$

$$(xy)(zw) = (xz)(yw)$$

iff p is transcendental. (Fajtlowicz and Mycielski [136].)

8.10. The algebra (ω, x^y) is generic for the law

$$(x^y)^z = (x^z)^y$$

(Martin [286, page 56]). (Here $\omega = \{0,1,2,...\}$ and x^y denotes ordinary exponentiation with $0^0 = 1$.) The proof is surprisingly long. Cf. 9.20 below.

8.11. The algebra (ω,xy,x^y) is generic for the laws

$$xy = yx$$
$$(xy)z = x(yz)$$
$$(xy)^z = x^z y^z$$
$$(x^y)^z = x^{yz}$$

(Martin [286, page 78]).

8.12. The algebra $(\Omega,+)$ is generic for the laws

$$(x+y) + z = x + (y+z)$$
$$x + y + x + y = y + x + x + y$$
$$x + y + z + x + y = y + x + z + x + y$$
$$x + y + x + z + y = y + x + x + z + y$$
$$x + y + z + x + w + y = y + x + z + x + w + y$$

(J. Karnofsky [unpublished] - see [286, page 31]). Here + denotes addition on the class Ω of all ordinals - the algebra $(\Omega,+)$ could be replaced by a countable one.

8.13. The algebras $(\omega;A_n)_{n\geqslant 3}$ and $(\omega;0_n)_{n\geqslant 4}$ are each generic for the variety of *all* algebras of type (2,2,2,..) (i.e., for $\Sigma_0 = \emptyset$). (Martin [286, page 131, page 134].) Here A_n $(n \geqslant 3)$ are the Ackermann operations beyond exponentiation, and the 0_n are some related operations invented by Doner and Tarski [110], who conjectured a somewhat stronger statement.

PROBLEM 1. Do these three equations form an axiom base for Boolean algebras?

$$x \vee y = y \vee x$$
$$x \vee (y \vee z) = (x \vee y) \vee z$$
$$((x \vee y)' \vee (x \vee y')')' = x.$$

(See [354] for a history of this problem.) All finite models of these equations are Boolean algebras.

PROBLEM 2. Do the following eleven equations form an axiom base for $(\omega,1,x+y,xy,x^y)$?

1. $x + y = y + x$

2. $xy = yx$
3. $x + (y+z) = (x+y) + z$
4. $x(yz) = (xy)x$
5. $x(y+z) = xy + xz$
6. $x^{y+z} = x^y x^z$
7. $(xy)^z = x^z y^z$
8. $(x^y)^z = x^{(yz)}$
9. $x \cdot 1 = x$
10. $x^1 = x$
11. $1^x = 1$.

(Tarski - see [286].) Tarski has called this the "high school identity problem (with unit)." (Incidentally, Martin [286] has remarked that it follows easily from the methods of Richardson [377] that

$$\mathrm{Eq}(\omega,1,x+y,xy,x^y) = \mathrm{Eq}(R^+,1,x+y,xy,x^y),$$

where R^+ is the set of non-negative real numbers.)

We could go on and on with interesting examples (see e.g. [4], [104], [135], [153], [188], [265], and [408]), but we will stop here. In place of finding a base Σ_0 of a given **A**, one can often be content with the knowledge that a *finite* Σ_0 exists or does not exist, as the case may be: this is the idea of the next section. (Although sometimes explicit - but complicated - bases are found in the researches reported in §9 and §10. The methods of §14.5 and §15 also sometimes lead to finding Σ_0 and **A** as in this section.) The reverse problem, of finding a generic **A** for a given Σ_0, is less well defined. As remarked above, **A** always exists, but finding a "known" (i.e. familiar) or simple generic algebra can be very elusive, e.g. for modular lattices. (The problem of "simply" describing a free algebra is really a *word problem* - see §12 below - and the word problem for free modular lattices is not solvable.)

9. **Finitely based theories.** We say that an equational theory Σ is *finitely based* iff there exists a finite set Σ_0 of axioms for Σ. (The definitions in §7 should make it clear that this is an equivalence-invariant property of all Σ which have finitely many operations.) Evidently many familiar theories are finitely based - groups, Boolean algebras, rings, lattices, etc.; see also the various examples in §8. Here we list some

algebras A with Eq {A} known to be ***finitely based***:

9.1. Any two-element algebra (Lyndon [265]). (But Cf. 9.16 below.)

9.2. Any finite group (Oates and Powell [336]).

9.3. Any commutative semigroup (Perkins [349]). (In other words, every variety of commutative semigroups is finitely based. This is also proved in [124].) Also, any 3-element semigroup [349]. (Cf. 13.5 below.)

9.4. Any idempotent semigroup (Fennemore [137], Biryukov [54], Gerhard [152]). (A semigroup is *idempotent* iff it obeys the law $x^2 = x$.)

9.5. Any finite, simple, 2-generated quasigroup (McKenzie [297]).

9.6. Any finite ring (Kruse [250], Lvov [263]).

9.7. The ring $M_2(k)$ of 2×2 matrices over a field k of characteristic 0. (Razmyslov [374].) (Cf. 8.4 above.) (For $n \geqslant 3$, this is open.)

9.8. Any nilpotent ring; any commutative ring (Bang and Mandelberg [37]).

9.9. Any finite (non-associative) ring without zero-divisors (Lvov [264]).

9.10. Any finite lattice (possibly with operators) (McKenzie [291]). (Answering Problem 45 in Grätzer's book [163].) More generally:

9.11. Any finite algebra which generates a congruence-distributive variety (see §15 below) (Baker [17] - see also [277], [426] and [226]). The special case (of 9.10 - 9.11) of primal algebras was known much earlier (Rosenbloom [379], Yaqub [444]; also Yablonskiĭ in the mid-fifties - see [319]).

9.12. Any finite simple algebra with no proper subalgebras except one-element subalgebras which generates a congruence-permutable variety (McKenzie [299]).

9.13. If V has only finitely many subdirectly irreducible algebras, all of them are finite, and V has definable principal congruence relations, then V is finitely based. As a corollary, if V is a locally finite variety and there exist $A_1,\ldots,A_k \in V$ so that every finite $A \in V$ is isomorphic to some $A_1^{n_1} A_2^{n_2} \cdots A_k^{n_k}$, then V is finitely based. (McKenzie [300].) Thus the *para-primal* varieties of Clark and Krauss are finitely based.

9.14. Any finite $\otimes$-product of finitely based theories is finitely based (see [420], pages 357-358]; [424, pages 266-267] for $\otimes$ - which corresponds to taking the product of the algebraic theories described in §7). Pursuing the analogy at the end of

§7, this easy result corresponds to the fact that the product of two finitely presented groups is finitely presented.

9.15. Recently Murskiĭ has proved [319] that "almost all" finite algebras have a finite base for their identities (i.e., for fixed type, the fraction of such algebras among all algebras of power k approaches 1 as $k \to \infty$ - or even, for fixed k, as the number of operations approaches ∞). (The unary case is easy - all are finitely based; in the non-unary case he in fact proves much more: almost all are quasi-primal - cf. §15 below, and also 9.11 and 10.7.)

For some further remarks about finite algebras with finite bases, consult [235]. We now turn to equational theories which are not finitely based. Of course it is almost trivial to construct such theories using infinitely many operations $F_t (t \in T)$, even some which are equivalent to the (finitely based!) theory with no operations. As G. Bergman pointed out, non-finitely based theories with *finite* T arise almost automatically if we consider a semigroup S which is finitely generated (say by $F \subseteq S$), but not finitely related. Our theory can be taken to have unary operations $\hat{f}$ for $f \in F$ and laws $\hat{f}_1 \hat{f}_2 \cdots \hat{f}_k x = \hat{f}_{k+1} \cdots \hat{f}_s x$ whenever $f_1 \cdots f_k = f_{k+1} \cdots f_s$ in S. Some more interesting research has centered on finding less obvious, but more important, examples of theories and algebras which have T finite and are still ***not finitely based***:

9.16. The algebra with universe {0,1,2} and binary operation:

	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

(Murskiĭ [318], following Lyndon [266]).

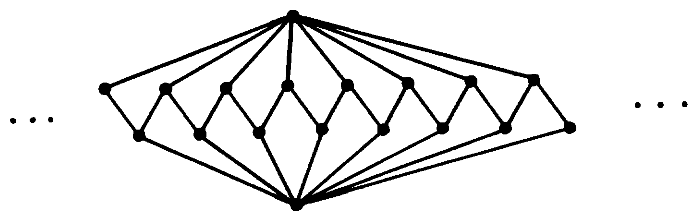
9.17. The six-element semigroup

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

(with ordinary matrix multiplication). (Perkins [349].) Earlier Austin [11] gave some other varieties of semigroups which are not finitely based. C. C. Edmunds has recently shown that six is as small as possible for a semigroup with zero and unit.

9.18. Some varieties of groups (Ol'shanskiĭ [337], Vaughan-Lee [434]) (cf. 13.3 below for a more complete discussion). The existence of such varieties of groups was open for a long time.

9.19. The infinite lattice



(McKenzie [291]).

9.20. The algebras $(\Omega,+,\cdot)$ (Martin [286, page 210] [287]) and (Ω,x^y) [286, page 211]. See 8.10 and 8.12 for the definitions and some comparisons. Note also that $(\omega,+,\cdot)$ is obviously finitely based (cf. 8.1). Also cf. §12 below.

9.21. The algebra $(\omega,x+y,xy,x^y)$ (Martin [286, page 118]). (Cf. 8.11.) This is a negative solution to one version of Tarski's "high school identities problem" - he described a set of 8 familiar identities (namely the first 8 of Problem 2 of §8), and asked if these formed an equational base. For another version of this problem, see Problem 2 below and Problem 2 of §8.

9.22. Any lattice-ordered ring which is an ordered field (and all of these have the same equational theory) (Isbell [199]). (An infinite basis is indirectly described [*loc. cit.*].)

9.23. The variety of representable relation algebras (Monk [312]) and for $n \geqslant 3$ that of representable cylindric algebras of dimension n (Monk [314]). (Roughly speaking, cylindric algebras are to full logic what Boolean algebras are to logic without quantifiers "for all," "there exists." Relation algebras are intermediate in strength.) Representable algebras (of either type) have a very natural semantic definition; the definition of the entire class of cylindric or relation algebras amounts to selecting a (necessarily rather arbitrary, no matter how utilitarian) finite subset of the equational theory of representable algebras. Monk's results indicate that there is really no natural finite set of axioms.

9.24. The variety of disassociative groupoids (Clark [88]). (The axioms of this theory consist of all two-variable consequences of the associative law for a single binary operation. Cf. 14.4 below.)

9.25. The two theories axiomatized by T_0 and T_1 (of Tarski [413]):

$$T_0 = \{F^{n+1}yx_1\cdots x_ny = F^{n+1}yx_2\cdots x_nx_1y\colon n \in \omega\}$$

$$T_1 = T_0 \cup \{F^nyx_1\cdots x_n = F^{n+1}yx_1\cdots x_nFyy\colon n \in \omega\}.$$

(Here F is a binary operation and F^n is defined recursively *via* $F^{n+1}x_1\cdots x_{n+2} = F(F^nx_1\cdots x_{n+1})x_{n+2}$.)

PROBLEM 1. Is the algebras $(R,xy,1\text{-}x)$ finitely based? (Here R denotes the real numbers.) (J. Mycielski [136]; R. McKenzie discovered a non-trivial identity of this algebra - again see [136].)

PROBLEM 2. Is $(\omega,1,x+y,xy,x^y)$ finitely based? (Tarski) Cf. 9.21 and Problem 2 of §8. For a discussion of this problem see Henkin [181].

PROBLEM 3. Is **A** finitely based if **A** is finite and all subdirectly irreducible algebras in **HSP A** are in **HS A**? (B. Jónsson).

We close with three "problems" which are no longer problems - they were solved just as final preparations were made on this survey. Pigozzi showed that the answer to Problem 4 is "no"; his example is actually generated by a finite algebra.

S. V. Polin has answered negatively Problems 5 and 6. His work (see supplemental bibliography) has been replicated and improved by M. R. Vaughan-Lee. Their example is a non-associative ring of characteristic 2 having 64 elements. Other examples have since been found by I. V. Lvov, Yu. N. Mal'tsev and V. A. Parfenov.

"PROBLEM" 4. Is every equationally complete (see §13 below) locally finite variety finitely based? (McKenzie [299]).

"PROBLEM" 5. Is every finite algebra which generates a congruence-permutable variety finitely based? (McKenzie [299]).

"PROBLEM" 6. Is every finite algebra which generates a congruence-modular variety finitely based? (Macdonald [269]). (Cf. 9.11 above.)

10. One-based theories. Taking Σ_0 and Σ as in §8, we say that Σ (or $V = \text{mod}\ \Sigma$) is *one-based* iff there exists a set of axioms Σ_0 with $|\Sigma_0| = 1$. Here are some algebras or ***theories which are one-based***:

10.1. The variety of all lattices (McKenzie [291]). McKenzie's original proof yields a single equation of length about 300,000 with 34 variables. Padmanabhan [341] has reduced it to a length of about 300, with 7 variables. Here we mean lattices formulated as usual with meet and join. Cf. 10.8 and 10.9 below. More generally:

10.2. Any variety which has a polynomial m obeying

$$m(x,x,y) = m(x,y,x) = m(y,x,x) = x$$

(a "majority polynomial") and is defined by "absorbtion identities," i.e., equations of the form $x = p(x,y,\ldots)$. (McKenzie [291]; see also [341].)

10.3. Any finitely based variety V of Γ_1-groups (see the beginning of §7) (Higman and Neumann [185]). Tarski got this for V = all Abelian groups (see [413]). (Cf. 10.10 below.) For a recent proof, see [236].

10.4. Certain varieties of rings (with operators) (Tarski [413]). For some more general formulations of 10.3 and 10.4, see Tarski [413].

10.5. Boolean algebras. (Grätzer, McKenzie and Tarski) (see [165, page 63]). (Cf. [401].) (Also cf. 10.6 and 10.7.)

10.6. Any two-element binary algebra except (within isomorphism) as in 10.11 below. (Potts [368].)

10.7. Every finitely based variety with permutable and distributive congruences (McKenzie [296]; Padmanabhan and Quackenbush [342]). By 9.11 this applies to any finite algebra which generates a variety with permutable and distributive congruences, e.g. a *quasi-primal* algebra (see [362], [369]). Primal algebras were already known to Grätzer and McKenzie [168]. ([296] contains some very interesting special one-based varieties.)

Here are some theories (and algebras) which are 2-based but ***not 1-based***:

10.8. The variety of all lattices given in terms of the single quaternary operation $Dxyzw = (x \vee y) \wedge (z \vee w)$ (McKenzie [291]). (Cf. 10.1.)

10.9. Any finitely based variety of lattices other than the variety of all lattices and the trivial variety defined by $x = y$ (McKenzie [291]). (Here again we mean the usual lattice operations.)

10.10. Any non-trivial finitely based variety of Γ_2-groups (defined at the beginning of §7) (Green and Tarski [172], [413]). (Cf. 10.3 above.)

10.11. $\mathbf{A} = (\{0,1\}, \vee)$ and $\mathbf{A} = (\{0,1\}, \rightarrow)$ with

$$\begin{array}{c|cc} \vee & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} \rightarrow & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}$$

(Potts [368]).

10.12. If Σ is a finitely based theory of type (m_1,m_2) with $m_1,m_2 \geqslant 2$ in which F_1 and F_2 are each idempotent, i.e.

$$\Sigma \vdash F_i(x,x,\ldots,x) = x \qquad (i = 1,2),$$

then Σ is 2-based (and may also be 1-based) (Padmanabhan [340]).

10.13. If Σ is a finitely based theory with a majority polynomial (as in 10.2 above), then Σ is 2-based (and sometimes 1-based) (Padmanabhan and Quackenbush [342]). (McKenzie [291] had this result for varieties in which lattices are definable.)

Isolated results: Lattices are definable in 2 equations using only 3 variables (Padmanabhan [339]). Two variables will not suffice for lattices (see [165, page 62]), nor for Boolean algebras (Diamond and McKinsey [107]). Cf. also [372] and 14.4. If Σ_0 is

$$x(yz) = (xy)z$$

$$xx = x \ ,$$

then every theory $\supseteq \Sigma_0$ has a basis consisting of $\Sigma_0 \cup \{\alpha\}$, i.e. Σ_0 together with one more axiom (Biryukov [54], Fennemore [137], and Gerhard [152]). (Cf. 13.6 below.)

PROBLEM. (Specht). Does there exist a non-finitely based variety of rings?

11. **Irredundant bases.** Σ_0 is an irredundant base for Σ iff Σ_0 is a base for Σ but no proper subset of Σ_0 is a base. Tarski [413] has defined

$$\nabla(\Sigma) = \{\, |\Sigma_0| \colon \Sigma_0 \text{ is an irredundant base of } \Sigma \}\,.$$

(Here $|\ |$ denotes cardinality.) Tarski's interpolation theorem [413], [414] states that $\nabla(\Sigma)$ is always an interval (see [309] for a connection between this and some other interpolation theorems, especially in graph theory; see also [157]). One easily checks that (at least for a type $(n_t)_{t\in T}$ with T finite), either $\nabla(\Sigma) = \emptyset$, $\nabla(\Sigma) = \{\aleph_0\}$ or $\nabla(\Sigma)$

is an interval of natural numbers. All these cases can occur. Referring to 9.25 above (from [413]),

$$\nabla(T_0) = \{\aleph_0\}$$

$$\nabla(T_1) = \emptyset.$$

For some other infinite irredundant bases, see [11], [78], [120], [198], and [349]. Distinct subsets of an irredundant base define distinct subtheories of Σ, and so infinite irredundant bases are useful in proving that some lattices of varieties have cardinality $2^{\aleph_0}$; see 13.3, 13.4 and 13.11 below.

McKenzie proved that $\nabla(\Sigma)$ can be any interval, and Ng showed that Σ can be found with one binary operation (see [413]). For example, if

$$\Sigma = \{F^5x_1\cdots x_6 = F^5x_2\cdots x_6x_1\},$$

then $\nabla(\Sigma) = \{1,2\}$, essentially because the cyclic group C_6 has both a single generator and an irredundant set of two generators. $\nabla(\Sigma)$ is an unbounded interval if $\Sigma \vdash \tau = x$, where τ contains x at least twice (Tarski [413]), strengthened by McNulty [305] to the case where τ has at least one operation of rank $\geqslant 2$. On the other hand, if Σ is defined by balanced equations, and Σ is finitely based, then $\nabla(\Sigma)$ is a bounded interval [305]. (An equation $\sigma = \tau$ is *balanced* iff each variable, each nullary operation symbol and each unary operation symbol occurs equally often in σ and τ.) T. C. Green got irredundant bases of power n (any $n \in \omega$) for groups ([172], see also [413]). $\nabla(\Sigma)$ was also defined by G. Grätzer [163, Problem 34], who asked for a characterization of it. Finally, note that ∇ is *not* an equivalence invariant (§7), as can be seen from 10.1 and 10.8 or from 10.3 and 10.10.

12. **Decidability question.** We assume given a finite alphabet A and a fixed way of interpreting all our variables, function symbols, terms, equations, etc. unambiguously as words in A. We assume that the reader knows what is meant for a collection W_0 of words to be decidable (relative to a collection $W \supseteq W_0$ of words). For this (non-numerical) notion of decidability, probably the Turing-machine approach ("computability") is easiest. For readable brief descriptions, see [431], [217], [267], or [416, pages 12-14]. Usually W will be obvious (such as the collection of all finite sequences of equations) and we will not mention it. As is well known [*op. cit.*] a relatively easy analysis of Cantor's diagonal argument yields *undecidable sets*; the

greatest example of an undecidable set is given by Gödel's incompleteness theorem [*op. cit.*], about which we will say no more. Here we will discuss decidability properties of equations. When we say that a *property* P of finite sets Σ of equations is (un)decidable, we mean that

$$W_0 = \{\Sigma : P(\Sigma)\}$$

is (un)decidable.

We may first ask when a theory Σ itself is a decidable set of equations, or, as it is frequently put, "the word problem for free Σ-algebras is solvable." (See the discussion of word problems below.) There are two common methods for showing that Σ is a decidable equational theory, the first being to find a recursive procedure to convert every term σ to a unique ("normal form") term σ' with $(\sigma = \sigma') \in \Sigma$ and such that if σ and τ are distinct normal forms then $(\sigma = \tau) \notin \Sigma$. (The decision procedure then reduces to comparison of normal forms - and conversely, a decision procedure for Σ obviously implies the existence of normal forms.) E.g. every group term reduces uniquely to either 1 or

$$x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_k}^{n_k},$$

where $x_{i_1} \neq x_{i_2} \neq \cdots \neq x_{i_k}$. Several of the best known equational theories are decidable, as may be seen similarly. See e.g. Margaris [284] for implicative semilattices, following work of McKay and Diego. Eq$(\Omega,+)$ is decidable (in fact its full first order theory is decidable, by Ehrenfeucht and Büchi [70]) - a simple method given by Selman and Zimbarg-Sobrinho [unpublished] is closely related to Karnofsky's identities 8.12 above. Martin [286] gave a decision procedure for $(\Omega,+,\cdot)$ with normal forms (cf. 9.20 above). Richardson [377] gave normal forms for $(\omega,1,x+y,xy,x^y)$ (cf. Problem 2 in §8). Finally, we remark that the Birkhoff-Witt theorem yields a procedure for finding normal forms for (free) Lie algebras and rings, as observed by P. Hall [175]. See Bergman [43] for some detailed methods for finding normal forms, mainly in ring theory; also see [158] and [248].

Notice that representing free algebras uniquely *via* terms (as we did for $\mathbf{F}_K(X)$ in §2) really requires a normal form. Often a normal form is required for finding the cardinality of $\mathbf{F}_V(X)$, a topic we will come to in 14.5 below. For some other results related to normal forms, see Hule [194].

The second method for decidability: if Σ has a finite (or, more generally, a recursive) base, and $V = \text{mod}\ \Sigma$ is generated by its finite algebras (equivalently, if the V-free algebras are residually finite), then Σ is a decidable equational theory. (Evans [122].) For instance, the variety of lattices has this property, although the word problem for free lattices was explicitly solved by Whitman [441] (also see [100]). And G. Bruns and J. Schulte-Mönting have recently given explicit solutions to the word problem for free ortholattices although it was known earlier that this variety is generated by its finite members (see [67]).

Ralph Freese has very recently shown that modular lattices do not have a decidable equational theory. It is known that the variety of modular ortholattices is not generated by its finite members [67]. These questions remain open for orthomodular lattices [67].

Tarski [410] proved that the equational theory of relation algebras is undecidable (this is more or less immediate from the ideas of Tarski mentioned in §6; cf. also 9.23 above) - in fact, it is *essentially* undecidable (see [416, page 4] for a definition); thus e.g. representable relation algebras (9.23) also have an undecidable equational theory.

For some other undecidable equational theories, consult Evans [122], Perkins [350] Malcev [281] and especially Murskiĭ [317] for a finitely based variety of semigroups.

PROBLEM 1. Does there exist a finitely based equational theory of groups which is undecidable?

One can also ask whether the entire first order theory of a variety V is decidable. The answer is yes for Boolean algebras (Tarski [409]), and more generally, for any variety generated by a quasiprimal algebra (Burris and Werner [81]), but no for groups (Tarski - see [416]), distributive lattices (Grzegorczyk) and a certain finitely based locally finite variety of semigroups with zero (Friedman [141]).

WORD PROBLEMS. Enlarge our type $(n_t)_{t\in T}$ to include constants $(C_i)_{i\in I}$. Let Σ_0 be a fixed finite set of equations. The *word problem* for Σ_0 consists of the decision problem for the set of equations

$$\{e\colon \Sigma_0 \vdash e \text{ and } e \text{ has no variables}\}.$$

(Typically Σ_0 is of the form $\Sigma_1 \cup \Sigma_2$, where Σ_1 is a set of laws not involving $(C_i)_{i\in I}$ and Σ_2 is a set of equations with no variables, viewed as "relations" on the "generators" $C_i (i \in I)$. One then speaks of "the word problem for [this presentation of] the algebra $\mathbf{F}_{\Sigma_1}(C_i)/(\Sigma_2)$", where (Σ_2) means the smallest congruence containing all pairs of terms in Σ_2.) Post [367] and Markov [285] proved that there exists a semigroup with undecidable word problem (i.e. that one may take Σ_1 to be the associative law). Much more difficult was the 1955 result of Boone and Novikov [61], [334] (see also [65] and [302]) that there exists a group with unsolvable word problem. Notice that all the results on (un)decidability of equational theories mentioned above are really a special kind of word problem result (with $|I| = \aleph_0$, $\Sigma_2 = \emptyset$).

G. Hutchinson [J. Algebra , 26(1973), 385-399] and independently L. Lipshitz [Trans. Amer. Math. Soc., 193(1974), 171-180] have shown the existence of a finitely presented modular lattice with unsolvable word problem. Hutchinson later gave an example with five generators and one relation [Alg. Univ., 7(1977), 47-84]. As we mentioned above, R. Freese subsequently reduced the number of relations to zero.

Evans [116],[117] proved that the word problem is solvable for Σ_1 (i.e., uniformly for all Σ_2) iff it is decidable whether a finite partial algebra obeying the laws of Σ_1 (insofar as they can be evaluated) can be embedded in a full algebra obeying Σ_1 (see also [163, §30]). Thus lattices have solvable word problem. Clearly the condition holds if finitely generated algebras in Σ_1 are always finite, or if finitely presented algebras are always residually finite (i.e. embeddable in a product of finite algebras), and for this last case there is a "local" version: the word problem is solvable for $\mathbf{A} = \mathbf{F}_{\Sigma_1}(C_i)/(\Sigma_2)$ with $\{C_i\}$, Σ_2 finite, if $\mathbf{A}$ is residually finite (Evans [121]). For applications see e.g. [148], [260].

Following work of Boone and Higman in group theory [63], Evans [127] recently proved that an algebra $\mathbf{A}$ has solvable word problem iff $\mathbf{A}$ can be embedded in a finitely generated *simple* algebra $\mathbf{B}$ which is recursively presented. (Generally we cannot demand that $\mathbf{B} \in \mathbf{HSP}\,\mathbf{A}$ or $\mathbf{B} \in V$ for V any preassigned variety containing $\mathbf{A}$.) (Cf. 14.8 and 17.1 below.) Also see [129].

PROBLEMS ON FINITE ALGEBRAS. Kalicki proved [232] that it is decidable, for arbitrary finite algebras **A**, **B** of finite type whether

$$\mathbf{HSP\ A = HSP\ B}.$$

And Scott proved [390] that it is decidable whether finite **A** is equationally complete. Almost all interesting decision problems about finite **A** are open, for instance:

PROBLEM 2. (Tarski). Is it decidable whether a finite algebra **A** of finite type has a finite base for its equations? (Cf. §9.) (And see Perkins [348].)

PROBLEM 3. (Pixley, et al.). For a finite **A** of finite type, is it decidable whether

$$\mathbf{HSP\ A = SPHS\ A},$$

or whether

$$\mathbf{HSP\ A = SP\ A}?$$

(For the relevance of this question, consult §§4,15.)

PROBLEMS ON FINITE SETS OF EQUATIONS. Many undecidability results are known on finite sets Σ of equations; for a full report we must refer to the chart in the introduction of McNulty [306]. Here we list a few ***undecidable properties of*** Σ.

12.1. $\Sigma \vdash x = y$ (Perkins [348]).

12.2. Σ has a finite non-trivial model (McKenzie [296]).

12.3. There exists finite **A** with Σ a base for Eq{**A**} (Perkins [348]).

12.4. Σ is equationally complete (cf. §13 below)(Perkins [348]).

12.5. Σ is one-based (cf. §10) (Smith [403], McNulty [306]).

12.6. The equational theory deduced from Σ (i.e., Eq Mod $\Sigma = \{e\colon \Sigma \vdash e\}$) is decidable. (Perkins [348].)

12.7. Mod Σ has the amalgamation property (defined as in group theory - see 14.6 below) (Pigozzi [356]).

12.8. Σ has the "Schreier property," i.e., subalgebras of free algebras are free (see 14.12 below). (Pigozzi [356].)

12.9. (For certain fixed theories Γ, e.g., Γ = group theory) Σ is a base for Γ. (McNulty [306] Murskiǐ [316]). (But it *is* decidable whether Σ is a base for $xy = yx$ (Ng, Tarski - see [413]).

12.10. Mod Σ has distributive congruences (cf. §15 below) (McNulty [306]).

12.11. Mod Σ is residually finite (McNulty [306]).

12.12. Mod Σ is residually small (cf. 14.8 below) (McNulty [306]).

PROBLEM 4. (Mycielski - see [197]). Is it decidable whether a finite set of terms is jointly κ-universal (κ a fixed cardinal)?

(Terms $\tau_i(x_1,\ldots,x_{n_i})$ $(1 \leqslant i \leqslant n)$ in operations $F_1,\ldots,F_s$ are jointly κ-universal iff for any operations $G_i\colon \kappa^{n_i} \to \kappa$, the operations $F_1,\ldots,F_s$ can be defined on κ so that the term τ_i defines G_i $(1 \leqslant i \leqslant n)$. This notion of universality has been very useful in the study of undecidability of properties of sets of equations - see McNulty [305].)

Finally, we mention that Burris and Sankappanavar [80] have investigated undecidability properties of congruence lattices and lattices of subvarieties (§13 just below). A sample result: in a similarity type with at least one operation of rank $\geqslant 2$, the lattice Λ of all equational theories has a hereditarily undecidable first order theory.

13. The lattice of equational theories. For a fixed type $(n_t)_{t \in T}$, order the family Λ of all equational theories by inclusion. One easily sees (from Birkhoff's Theorem of §5 or directly from the definition at the beginning of §5), that Λ is closed under arbitrary intersections - and so from purely lattice theoretic considerations, Λ has arbitrary joins as well, and so is a *complete lattice*. More specifically,

$$\bigvee_{i \in I} \Sigma_i = \{e\colon \bigcup_{i \in I} \Sigma_i \vdash e\} = \mathrm{Eq}(\bigcap_{i \in I} \mathrm{Mod}\ \Sigma_i).$$

From the proof-theoretic characterization of $\vee$ it follows that Λ is an algebraic closure system and hence an *algebraic lattice* (see [91] [163] or [165]). Specifically, the compact elements of Λ are the finitely based theories of §9 above, and every element is the join (actually the union) of all its finitely based subtheories. Obviously the join of two finitely based theories is finitely based (this holds for compact elements in any lattice); but the meet (i.e., intersection) of two finitely based theories can fail to be finitely based. We present an example of Karnofsky (unpublished - see [354]) (here and below we will sometimes express a theory by one of its finite bases without further mention):

$$\begin{array}{ll} \Sigma_1\colon x(yz) = (xy)z & \Sigma_2\colon x(yz) = (xy)z \\ (xyz)^2 = x^2y^2z^2 & x^3y^3 = y^3x^3 \\ x^3y^3z^2w^3 = y^3x^3z^2w^3. & \end{array}$$

The theory $\Sigma_1 \wedge \Sigma_2$ is not finitely based, for (we omit the proof), every basis must contain equations essentially the same as

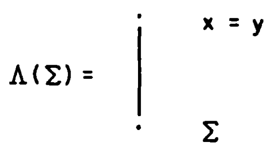
$$x^3y^3v_0^2 \cdots v_{2k}^2w^3 = y^3x^3v_0^2 \cdots v_{2k}^2w^3 \quad (k = 1,2,3,\ldots).$$

B. Jónsson [223] found two finitely based equational theories of lattices whose meet is not finitely based (also found by K. Baker - unpublished, but see [354]). Whether there exist such theories of groups is unknown. There do exist such theories among "modal logics" but not among "intermediate logics" (W. J. Blok thesis).

By the one-one correspondence between varieties and equational classes set up at the beginning of §5, we could equally well have described Λ as the lattice of all varieties under *reverse* inclusion, and sometimes it is helpful to view Λ this way. (And sometimes Λ is taken to be ordered by (non-reversed) inclusion of varieties - we will not do this here.)

It is of interest to know what the lattices Λ look like. It has become clear that they are very complicated, as we will see. Burris [74] and Ježek [207] have proved that if the type $(n_t)_{t\in T}$ has some $n_t \geqslant 2$ or if $n_t \geqslant 1$ for two values of t, then Λ contains an infinite partition lattice, and hence obeys no special lattice laws at all.

Thus, it has proved fruitful to proceed by studying some (often simpler) special sublattices of Λ, namely for fixed Σ, the lattice $\Lambda(\Sigma)$ of all equational theories $\supseteq \Sigma$. (Equivalently, as above, the lattice of all subvarieties of Mod Σ.) There is only one Σ with $|\Lambda(\Sigma)| = 1$, namely $\Sigma = \{x = y\}$. Theories Σ with $|\Lambda(\Sigma)| = 2$, i.e.



are called *equationally complete*. Since every Λ is an algebraic closure system and $\{x = y\}$ is finitely based, every theory has an equationally complete extension, and thus the top of Λ consists wholly of replicas of the above picture. An algebra **A** is *equationally complete* iff Eq **A** is equationally complete. It has been determined that there exist many equationally complete theories (and algebras), in two senses. First, Kalicki proved [233] that in a type with one binary operation there exist $2^{\aleph_0}$ distinct equationally complete theories (and the corresponding number has been evaluated for all types by Burris [74] and Ježek [207], answering Problem 33 of Grätzer [163]). Second, Bolbot [59] and Ježek [207] proved that (given at least two unary operations or one operation of rank $\geqslant 2$) Λ is dually pseudo-atomic, i.e. the zero of Λ (i.e. $\Sigma_0 = \emptyset$) is the meet of all dual atoms (i.e. equationally complete theories). But see some of the examples below for varying numbers of equationally complete theories in various $\Lambda(\Sigma)$.

Kalicki and Scott [234] found all equationally complete semigroups; there are only $\aleph_0$ of them. McNulty used their description in reproving Perkins' result that it is decidable whether

$$\Sigma, x(yz) = (xy)x \vdash x = y.$$

All equationally complete rings were found by Tarski [412]; again, there are $\aleph_0$ of them. See also [343]. We cannot begin to cover all the information presently known on equational completeness. For further information, consult Grätzer [163, Chapter 4], or Pigozzi [354, Chapter 2]. Here we sample just a few very recent results.

THEOREM. (Pigozzi [360]). *There exists an equationally complete variety which does not have the amalgamation property.* (Answering a question of S. Fajtlowicz.) (See 14.6 below for the amalgamation property.)

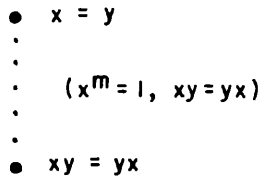
THEOREM. (Clark and Krauss [89]). *If V is a locally finite congruence-permutable equationally complete variety, then V has a plain paraprimal direct Stone generator.* (See [89] for the meaning of these terms - roughly speaking, this means that V is generated in the manner either of Boolean algebras or of primary Abelian groups of exponent p.)

Some examples of known or partly known $\Lambda(\Sigma)$ for $|\Lambda(\Sigma)| > 2$:

13.1. For $\Sigma = \emptyset$ in a type with just one unary operation F, Jacobs and Schwabauer [203] gave a complete description of Λ. For unary operations and constants, see Ježek [206].

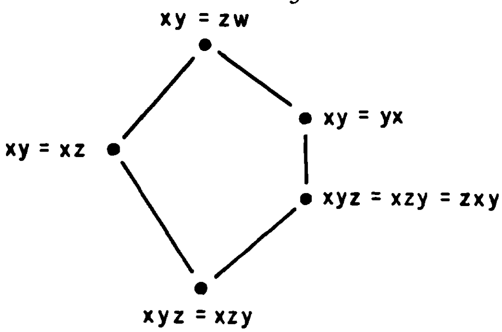
13.2. If $\Sigma = \mathrm{Eq}\,\mathbf{A}$ for a finite algebra $\mathbf{A}$ in a finite similarity type, then Scott showed [390] that $\Lambda(\Sigma)$ has only finitely many co-atoms (i.e. equationally complete varieties). If $\mathbf{A}$ generates a congruence-distributive variety, then Jónsson's lemma (see §15 below) easily implies that $\Lambda(\Sigma)$ is finite. If $\mathbf{A}$ is quasiprimal (see [369]) then $\Lambda(\Sigma)$ is a finite distributive lattice with a unique atom ($= \mathbf{HSP}\{\mathbf{B}:\mathbf{B} \subsetneq \mathbf{A}\}$), and, conversely, every finite distributive lattice with unique atom can be represented in this way (H. P. Gumm, unpublished). T. Evans informed the author that there exists a finite commutative semigroup with $\Lambda(\Sigma)$ infinite.

13.3. For $\Gamma =$ group theory, $\Lambda(\Gamma)$ is modular, but very complicated, and moreover can be given structure beyond the lattice-theoretic (see [330] and 17.9 below). At and near the top $\Lambda(\Gamma)$ contains



with $m \geqslant 1$, ordered by divisibility - the Abelian part. It turned out to be difficult to prove that $|\Lambda(\Gamma)| = 2^{\aleph_0}$. This was established by Vaughan-Lee [434] who found $\aleph_0$ irredundant equations (as in §11), and independently, by Ol'shanskiĭ [337] who found $\aleph_0$ "independent" subdirectly irreducible groups. (Cf. 9.18 above.) The variety of 3-nilpotent groups has been completely described - see Jónsson [220] or Remeslennikov [376].

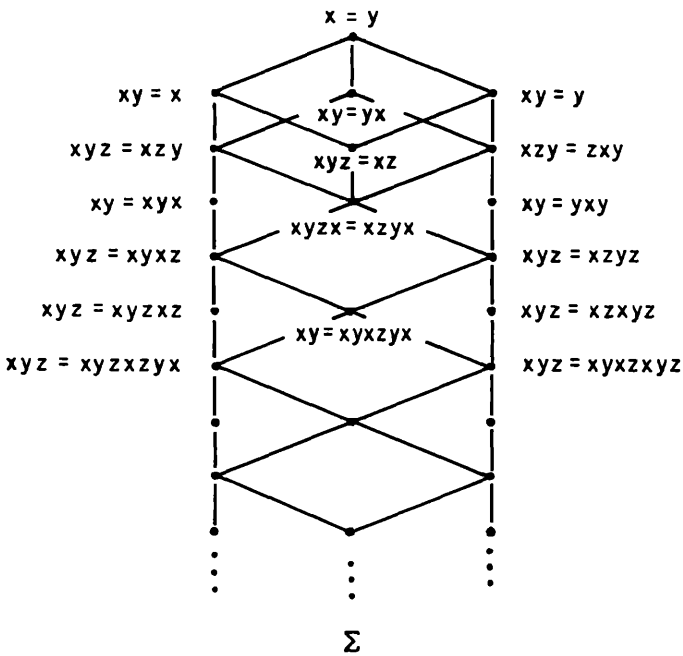
13.4. For $\Sigma =$ semigroup theory, many results are known; consult the survey by Evans [124] for more detailed information. Biryukov (1965) and later Evans [120] first proved that $|\Lambda(\Sigma)| = 2^{\aleph_0}$ (also see [198] for a nice infinite irredundant set of semigroup laws). Dean and Evans [106] proved that $x(yz) = (xy)z$ is finitely meet-irreducible in Λ, i.e., that $\Lambda(\Sigma)$ has a finitely meet-irreducible least element. Burris and Nelson [79] (and later Ježek [211]) proved that $\Lambda(\Sigma)$ contains a copy of Π_∞, the lattice of partitions on an infinite set, and hence obeys no special lattice laws. The fact that $\Lambda(\Sigma)$ is non-modular can be seen from this sublattice - due to Ježek - isomorphic to the smallest non-modular lattice, N_5:



(All theories intended as extensions of $\Sigma = \{x(yz) = (xy)z\}$.)

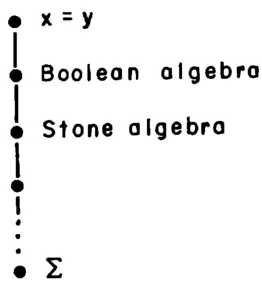
13.5. For commutative semigroups, see [349], [179], [325], [388], and [78]. Perkins proved that every commutative semigroup theory is finitely based (cf. 9.3 above), and hence this lattice is countable. And so it cannot contain Π_∞, but it does contain every Π_m (Burris-Nelson [78]) and hence obeys no special lattice laws; Schwabauer had earlier proved [388] that the lattice of commutative semigroup theories is nonmodular. For semigroups with zero, consult [324], [83], and with unit [179]. For related work see [344], [350], [351].

13.6. For $\Sigma = \{(xy)z = x(yz),\ x^2 = x\}$ ("idempotent semigroups"), $\Lambda(\Sigma)$ has been completely described by Biryukov [54], Fennemore [137] and Gerhard [152]. In this picture, the diamond pattern repeats indefinitely in the obvious way:



(all theories are intended as extensions of Σ). Note that this lattice is countable, distributive and of width three. The situation is very different for $\Sigma' = \{x(yz) = (xy)z,\ x^2 = x^3\}$: Burris and Nelson [79] proved that $\Pi_\infty \subseteq \Lambda(\Sigma')$.

13.7. For Σ the equational theory of *distributive lattices with pseudocomplementation*, $\Lambda(\Sigma)$ is an infinite chain:



(Lee [256]; see also [165]). A similar result holds for *one-dimensional polyadic algebras* (Monk [315]).

13.8. If $\Sigma = \{xy = yx,\ (xy)(zw) = (xz)(yw)\}$, then $\Lambda(\Sigma)$ is uncountable, even above $\Sigma \cup \{x^2 = y^2\}$. But $\Lambda(\Sigma \cup \{x^2 = x\})$ is countable (and explicitly described). See [213].

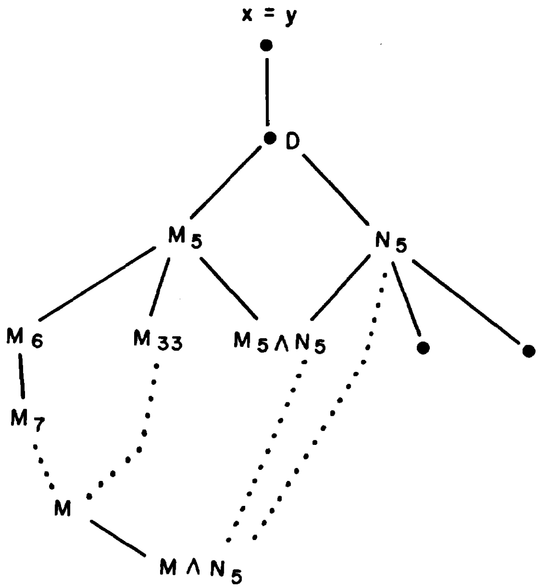
13.9. For Heyting algebras, consult Day [102]; and for the closely related Brouwerian algebras, see Köhler [249]. Also closely related are interior algebras; their varieties correspond to modal logic extensions of Lewis' S4 (W. J. Blok, thesis).

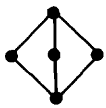
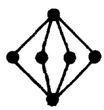
13.10. For *lattice-ordered* groups see Martínez [288]; their lattice has a surprising superficial similarity to that of Heyting algebras (above). Holland showed [188] that every proper extension of the theory of lattice-ordered groups contains (an identity equivalent to the implication)

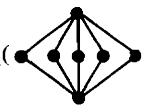
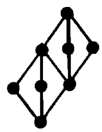
$$(1 \leqslant a)\ \&\ (1 \leqslant b) \Rightarrow ab \leqslant b^2a^2,$$

and thus this lattice has a unique atom. See Scrimger [389] for a study of the theories just below the theory of Abelian lattice-ordered groups.

13.11. $\Lambda(L)$ has been extensively studied for L = lattice theory (see e.g. [93]). It is a distributive lattice which contains, in part



where $M_5 = \mathrm{Eq}($  $)$, $M_6 = \mathrm{Eq}($  $)$

$M_7 = \mathrm{Eq}($  $)$, $N_5 = \mathrm{Eq}($  $)$ and $U_8 = \mathrm{Eq}($  $)$.

D is the theory of distributive lattices and M that of modular lattices. There are $2^{\aleph_0}$ equational theories of lattices. This was established by McKenzie [291] who found $\aleph_0$ irredundant equations and Baker [19] who found $\aleph_0$ independent subdirectly irreducible lattices, namely, for each prime p the lattice of subspaces of a 3-dimensional space over GF(p). (And Baker's lattice theories are all modular.) Even more is true: there is an interval in this lattice which is isomorphic to the Boolean algebra of all subsets of a countably infinite set [291], [19]. It was conjectured by McKenzie [295] and proved by Jónsson and Rival [229] that N_5 has exactly sixteen dual covers: one is $M_5 \wedge N_5$ and each of the other fifteen (•...• in our picture) is generated by a single subdirectly irreducible lattice.

We close this section with some remarkable general results of McKenzie on the full lattice $\Lambda = \Lambda(n_t\colon t \in T)$, where, temporarily, we make the type explicit.

THEOREM. (McKenzie [292]). *From the isomorphism type of* $\Lambda(n_t\colon t \in T)$ *one can recover the function*

$$\langle |\{t \in T\colon n_t = m\}| \colon m \in \omega\rangle.$$

(In other words, one can recover the type $(n_t\colon t \in T)$ up to renaming of all the operations.)

THEOREM. (McKenzie [292]). (Appropriate $(n_t\colon t \in T)$.). *There exists a first order formula* $\varphi(x)$ *with one free variable in the language of lattice theory such that the unique element of* Λ *satisfying* $\varphi(x)$ *is the equational theory of groups.* (Resp. semigroups, lattices, distributive lattices, commutative semigroups, Boolean algebras.)

This last theorem had a precursor in Ježek [209]: the variety of commutative semigroups obeying $x^2y = xy$ is definable (in a similar fashion).

PROBLEM. (McKenzie [292]). Does Λ possess any non-obvious automorphisms?

PROBLEM. (McKenzie and Maltsev). Which lattices L are isomorphic to some $\Lambda(\Sigma)$? McKenzie has conjectured that the following easy necessary condition is also sufficient: L is algebraic and its largest element is compact.

(Ježek has proved [211] that L is isomorphic to an interval in $\Lambda((2))$ iff L is algebraic and has only countably many compact elements.)

PROBLEM. (McKenzie [292]). When does an element $\alpha \in \Lambda(\Sigma)$ have a "dual cover," i.e., $\beta < \alpha$ such that there exists no γ with $\beta < \gamma < \alpha$? (E.g. the dual covers of $(x = y)$ are the equationally complete theories.) The answer is yes for semigroups [430] and lattices [211]; see [225] for a dual cover of modular lattices. The result of Ježek just cited implies that there exists $\delta < \alpha$ such that there is no dual cover β with $\delta < \beta < \alpha$. In fact δ and α can be taken to be equational theories of modal algebras (W. J. Blok, Notices Amer. Math. Soc. abstract 77T-A232).

PROBLEM. (Pigozzi [354] - q.v. for details.). Is $\Lambda(\Sigma)$ always a "partial Boolean ring"?

MODEL-THEORETIC QUESTIONS

Sections 5-13 have emphasized properties of equations *per se* (although of course, every property of [a set of] equations is a property of varieties, and *vice versa*). We now turn to some investigations which have emphasized the models of the equations. Here the lines are less sharp. Completely general descriptions of models in equationally defined classes do not exist - but on the other hand the very complete existing investigations of models of special equational theories (e.g. groups, lattices, Boolean algebras) must be omitted from this survey on the ground of space alone! For studies in the (special) model theory of various less well known varieties, one need only look in almost any issue of Algebra Universalis. Most of the results we will report on will be of medium generality, i.e. valid for an interesting class of varieties, but not for all varieties. It is also hard to say where the model theory of equational classes stops and general model theory begins.

14. Some further invariants of the equivalence class of a variety. See §7 above for equivalence; with the exception of §§5, 8, 10, 11, 12, we have been writing of the equivalence class of a variety V. In this section we survey very quickly some other equivalence invariants which have been studied.

14.1. The *spectrum* of V: $\quad \mathbf{spec}\, V = \{n \in \omega\colon (\text{there exists } A \in V)|A| = n\}$.

Clearly $1 \in \mathbf{spec}\, V$, and since V is closed under the formation of products, **spec** V is multiplicatively closed. Grätzer [161] proved that, conversely, any multiplicatively closed set containing 1 is the spectrum of some variety (see also [119]). Froemke and Quackenbush [144] showed that this variety need have only one binary operation. The characterization of sets **spec** V for V *finitely based* seems to be much more difficult. McKenzie proved [296] that if $K \subseteq \omega$ is the spectrum of any first order sentence, then there exists a single identity $\sigma = \tau$ such that the multiplicative closure of $K \cup \{1\}$ is the spectrum of $(\sigma = \tau)$. (See also [328].) Characterizations of first order spectra are known in terms of time-bounded machine recognizability - see [130] for detailed statements and further references. Note that the definition of **spec** can be extended to mean the image of any forgetful functor (or any pseudo-elementary class) - see [130] [131] for more details. For example, we can consider

$$T(V) = \{A\colon A \text{ is a topological space and there exists } (A,F_t) \in V \text{ with all } F_t \text{ continuous}\}.$$

(Some preliminary investigations on $T(V)$ appear in [429]; cf. §16 below.) Of course, many descriptions of individual varieties in the literature yield **spec** (V). A certain amount of attention has focused on the condition **spec** $(V) = \{1\}$. (See e.g. [231], [9] and remarks and references given in [420, page 382]; cf. 12.2 above.) For instance, Austin's equation [9]

$$((y^2 \cdot y) \cdot x)((y^2 \cdot (y^2 \cdot y)) \cdot z) = x$$

has infinite models but no nontrivial finite models.

Mendelsohn [310] has shown that if V is an idempotent binary variety given by 2-variable equations, then **spec** V is ultimately periodic.

14.2. The *fine spectrum* of V is the function

$$f_V(n) = \text{the number of non-isomorphic algebras of power } n \text{ in } V.$$

Characterization of such functions seems hopeless. A typical theorem is that of Fajtlowicz [133] (see also [424, pages 299-300] for a proof): if $f_V(\mathbf{n}) = 1$ for all cardinals $\mathbf{n} \geqslant 1$, then V must be (within equivalence) one of two varieties: "sets" (no operations at all) or "pointed sets" (one unary operation f which obeys the law $fx = fy$). For some related results see Taylor [424], Quackenbush [371], McKenzie [300] and Clark and Krauss [90].

PROBLEM. [424]. Does the collection of all fine spectra form a closed subset of ω^ω (power of a discrete space)?

14.3. *Categoricity in power.* Varieties obeying the condition $f_V(\mathbf{n}) = 1$ for all infinite $\mathbf{n} \geqslant$ the cardinality of the similarity type of V have been characterized (within equivalence) by Givant [156] and Palyutin [345]. For a detailed statement, also see e.g. [424, page 299]. For example if V is defined by the laws (of Evans [118])

$$(*)\quad \begin{aligned} &d(x,x) = x \\ &d(d(x,y),d(u,v)) = d(x,v) \\ &c(c(x)) = x \\ &c(d(x,y)) = d(cy,cx), \end{aligned}$$

then every algebra in V is isomorphic to an algebras with "square" universe $A \times A$ on which

$$c((\alpha,\beta)) = (\beta,\alpha)$$

$$d((\alpha,\beta),(\gamma,\delta)) = (\alpha,\delta).$$

(Cf. 16.5 below.)

Givant's complete list of varieties categorical in power is formed, roughly speaking, as a mix of (*) with ideas of linear algebra. The idea behind (*) is very general, leading to varieties whose members are "k^{th} powers" of the algebras in other varieties (Neumann [332], Fajtlowicz [132], McKenzie [296] - see [424, page 268] for further references and a detailed statement and history of these ideas). Also see [27].

14.4. *Varietal chains.* For any variety V we may define

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V \subseteq \cdots \subseteq V^2 \subseteq V^1,$$

with

$$\cup V_n = \cap V^n = V,$$

as follows. $V_n = \mathbf{HSP}(\mathbf{F}_V(n))$ (see §§2,4 for this notation), and V^n is the variety defined by all n-variable identities holding in V, i.e. $\mathbf{A} \in V^n$ iff every n-generated subalgebra of $\mathbf{A}$ is in V. (See 9.24 above for an example of V^2, and see [91, page 173] for a general exposition and some more examples. See Quackenbush [372] (or [370]) for B^2 with B the variety of Boolean algebras (or Boolean groups).) As an equivalence-invariant, one may take the set of proper inclusions in either of these two chains. Jónsson, McNulty and Quackenbush prove [227] that, with a few possible exceptions, almost any sequences of proper inclusions can occur. There exists a variety V of groups with $V \neq$ any V^i (i.e., with infinitely many gaps in the descending sequence) [337] [434].

14.5. The *size of free algebras* is a subject with a long history: precisely, define the invariant:

$$\omega = \omega(V) = (\omega_n(V)\colon n = 0,1,2\ldots)$$
$$= (|\mathbf{F}_V(n)|\colon n = 0,1,2,\ldots)$$

(i.e., the cardinalities of V-free algebras). (This notation was introduced in Marczewski [282], and the first extensive description of $\omega(V)$ occurred in Grätzer [162]; see also [169], [364], [134], [392].) In Problem 42 of his book [163], Grätzer asked for a complete characterization of the functions $\omega(V)$ - our references represent only a partial solution.

This invariant has been explicitly evaluated for only a few of the better known varieties: vector spaces over a q-element field ($\omega_n = q^n$), Boolean algebra ($\omega_n = 2^{2^n}$), semilattices ($\omega_n = 2^n-1$), the variety of groups given by the law $x^3 = 1$ (see e.g. [330]), the variety of Heyting algebras defined by "Stone's identity" (see [189]), and some varieties of interior algebras (W. J. Blok, thesis). The quasi-primal varieties of Pixley et al. often have very easily calculated ω (see e.g. [362] or [369] for quasi-primal varieties); see the chart on page 291 of [424] for some explicit calculations. (But in the eight line, 1 should be 0.)

But for most common varieties, the invariant $\omega(V)$ is either trivial (because infinite) or hopelessly complicated. Sometimes special cases can be calculated. Dedekind found in 1900 that the free modular lattice on 3 generators has 28 elements ("free algebra" had not yet been defined) (see [50, page 63]). For some other special calculations (distributive lattices, etc.), see [50, page 63], [437], [46] and [47]. The class of all finite $\omega(V)$ is closed under (coördinatewise) multiplication (see e.g. [424, 0.5(4), page 266]), and it forms a closed set in the space ω^ω (Świerczkowski - see [282, page 181]).

The famous *Burnside Problem* asked whether $\mathbf{F}_V(n)$ is always finite for V the variety of groups defined by the law $x^m = 1$. The negative solution by Novikov and Adjan [335] stated that $|\mathbf{F}_V(2)| = \aleph_0$ when e.g. m = 4381. The proof in [66] is said to be false (see [3]). (Now 4381 has been reduced to 665 [3].) For related results in semigroups, see [171].

In general algebra, a typical theorem is that of Płonka [364] : if $\omega_n(V) = n\cdot 2^{n-1}$, then V must be equivalent to one of four varieties, namely those given by Σ_1 - Σ_4:

$$\Sigma_1:\ xx = x$$
$$(xy)z = x(zy)$$
$$x(yz) = x(zy)$$
$$\Sigma_2:\ xx = x$$
$$(xy)z = (xz)y$$
$$x(yz) = xy$$
$$(xy)y = xy$$
$$\Sigma_3:\ xx = x$$
$$(xy)z = (xz)y$$
$$x(yz) = xy$$
$$(xy)y = x$$

$$\Sigma_4\colon (xyz)uv = x(yzu)v = xy(zuv)$$
$$xyy = x$$
$$xyz = xzy.$$

(Where Σ_4 has a single ternary operation, denoted by juxtaposition.) Although there exist isolated results such as this of Płonka and that of Fajtlowicz above, we are obviously a long way from a solution of this

PROBLEM. Find a class of numerical invariants of V which determine V within equivalence. (Perhaps only in special circumstances.)

We close with an interesting calculation of this invariant for an infinitary V. If V is the variety of *complete Boolean algebras* (which has n-ary operations for every cardinal **n**), then $\mathbf{F}_V(\aleph_0)$ is a *proper class* (Hales [174], Gaifman [146]). (See also [404].) Similarly, $\mathbf{F}_V(3)$ is a proper class for V the variety of complete lattices [174]. Compare the "free Borel algebra" [50, page 257].

14.6. The *amalgamation property* (AP) for V generalizes the existence (due to Schreier) of amalgamated free products in group theory. (One form of) the AP states that given $\mathbf{A},\mathbf{B},\mathbf{C} \in V$ and embeddings f: $\mathbf{A} \to \mathbf{B}$, g: $\mathbf{A} \to \mathbf{C}$, there exists $\mathbf{D} \in V$ and embeddings f′: $\mathbf{B} \to \mathbf{D}$, g′: $\mathbf{C} \to \mathbf{D}$ such that $f'f = g'g$. A general investigation of AP began with Jónsson [218], [219] and now there is an extensive literature - e.g., see Pigozzi [355], Bacsich [13] [14], Baldwin [23], Dwinger [111], Hule and Müller [195], Forrest [139], MacDonald [268], Simmons [399], Schupp [386], Yasuhara [445], Bacsich and Rowlands-Hughes [15]. Varieties known to have AP are relatively rare, but include groups, lattices, distributive lattices and semilattices (see 14.9 below). (No other variety of modular lattices has AP - Grätzer, Lakser and Jónsson [167], and AP fails for semigroups - Howie [191], Kimura [245].) We cannot begin to mention all results on the AP, but one representative theorem comes from Bryars [69] (also see [15]): V has the AP iff for any universal formulas $\alpha_1(x_1,x_2,\ldots)$, $\alpha_2(x_1,x_2,\ldots)$ such that $V \models \alpha_1 \vee \alpha_2$, there exist existential formulas β_1,β_2 such that $V \models \beta_i \to \alpha_i$ $(i = 1,2)$ and $V \models \beta_1 \vee \beta_2$.

For a related property, see [215]. See also 12.7 above.

14.7. A variety V has the *congruence extension property* (CEP) iff every congruence θ on a subalgebra $\mathbf{B}$ of $\mathbf{A} \in V$ can be extended to all of $\mathbf{A}$, i.e. there exists a congruence ψ on **A** such that $\theta = \psi \cap B^2$. Abelian groups and distributive lattices have CEP, but groups and lattices do not. See e.g. Banaschewski [30], Pigozzi [355], Davey [97], Day [101], [104], Fried, Grätzer and Quackenbush [140], Bacsich and Rowlands-Hughes [15], Magari [275], Mazzanti [289] (where one will find some other references to the Italian school - in Italian usage, "regolare" means "having the CEP"). In [15] there is a syntactic characterization of CEP in the style of that for AP in 14.6 above, although these two properties are really rather different. This characterization is closely related to Day [101]. Notice that in Boolean algebras the CEP can be checked rather easily because every algebra **B** is a subalgebra of some power $\mathbf{A}^I$, where **A** is the two-element algebra and every congruence θ on **B** is of the form

$$\langle a_i\rangle \theta \langle b_i\rangle \text{ iff } \{i\colon a_i = b_i\} \in \mathcal{F}$$

for some filter $\mathcal{F}$ of subsets of I. (And, of course, the same filter $\mathcal{F}$ may be used to extend θ to larger algebras.) A variety in which congruences can be described by filters in this manner is called *filtral* - e.g. [140], [289], [275] and especially [39]. But, e.g., semilattices form a non-filtral variety which has CEP. It is open whether filtrality implies congruence-distributivity (§15 below). For CEP see also Stralka [405].

PROBLEM. (Grätzer [165, page 192]). If V satisfies

$$(\text{for all } X \subseteq V)\ \mathbf{HS}\,X = \mathbf{SH}\,X,$$

then does V have the CEP? This is true for lattice varieties (Wille).

14.8. A variety V is *residually small* iff V contains only a set of subdirectly irreducible (s.i.) algebras (§4), i.e. the s.i. algebras do not form a proper class, i.e. there is a bound on their cardinality. It turns out that this bound, if it exists, may be taken as 2^n, where $n = \aleph_0 +$ the number of operations in V (see [419]). V is residually small iff V_0 can be taken to be a *set* in (*) of §4, which is to say that V has a "good" coordinate representation system. For some other conditions equivalent to residual smallness, see [419] and [34]; also see [24] where e.g. finite bounds on s.i. algebras are considered. McKenzie and Shelah [301] consider bounds on the size of *simple* algebras in V and obtain a result analogous to that on 2^n just above. (An algebra is simple iff it is non-trivial and has no proper homomorphic image other than a trivial algebra. Every non-trivial variety has at least one simple algebra [273]; but see [326] for the failure of this fact for varieties of infinitary algebras.)

Some residually small varieties: Abelian groups, commutative rings with a law $x^m = x$ (cf. §6), semilattices, distributive lattices, various "linear" varieties (as in 14.3 above); also if V = **HSP A** for **A** finite and V has distributive congruences (e.g. **A** = any finite lattice), then V is residually small (by Jónsson's Theorem in §15 below). Also any ⊗-product of two residually small varieties ⊗ as in 9.14) is residually small. (Similarly for AP and CEP.) Some non-residually small varieties: groups, rings, pseudocomplemented semilattices (see [216], [382]), modular lattices, and **HSP A** for **A** either 8-element non-Abelian group (both generate the same variety - see 8.5 above). Also cf. 12.12 above.

A variety V is residually small iff every **A** in V can be embedded in an equationally compact algebra **B** [419]. Mycielski [321] defined **B** to be *equationally compact* iff every set Γ of equations with constants from **B** is satisfiable in **B** if every finite subset of Γ is satisfiable in **B**. Here is an example [321] of *failure* of equational compactness in the group of integers:

$$\begin{aligned} 3x_0 + x_1 &= 1 \\ x_1 &= 2x_2 \\ x_2 &= 2x_3 \\ &\vdots \end{aligned}$$

One can solve any finite subset of these equations in integers simply by solving $3x_0 + 2^{n-1}x_n = 1$, always possible; but clearly the entire set implies $3x_0 = 1$, impossible. Equational compactness is implied by topological compactness (use the finite intersection property for solution sets), but not conversely. Thus we are led to this problem, which has been settled positively for many V.

PROBLEM. [419]. If V is residually small, can every algebra in V be embedded in a compact Hausdorff topological algebra? (See §16 below.)

There is a large body of research on equational compactness which we cannot begin to cover here. See the survey review [423], or [72] or [425] for references. Also see G. H. Wenzel's appendix to the new edition of [163].

Among the equationally compact **B** $\supseteq$ **A** there is one which is "smallest," i.e., a "compactification of **A**" - see [419, page 40] or [29]. Węglorz [439] proved that this compactification is always in **HSP A**.

14.9. The conjunction of AP, CEP and residual smallness (14.6 - 14.8) is equivalent to the purely category-theoretic property of *injective completeness* (see Banaschewski [30]). (Pierce [353] noticed that injective completeness implies AP.) V is *injectively complete* (or, "has enough injectives") iff every algebra in V is embeddable in a V-injective, i.e. an algebra $\mathbf{A} \in V$ such that whenever $\mathbf{B} \subseteq \mathbf{C} \in V$ and f: $\mathbf{B} \to \mathbf{A}$ is a homomorphism, there exists an extension of f to g: $\mathbf{C} \to \mathbf{A}$. (See e.g. [143, §6.2] but remember that most varieties are not Abelian categories.) The variety of semilattices has enough injectives (Bruns and Lakser [68], Horn and Kimura [190]) and so does that of distributive lattices (Banaschewski and Bruns [32], Balbes [22]). For a theory of injective hulls in varieties, see [418, page 411]. See also [154]. We know examples to show that AP, CEP and residual smallness are completely independent properties, except for one case:

PROBLEM. AP and Res. Small $\Rightarrow$ CEP?

For further information on varieties with enough injectives, see Day [99] and García [150]. In 14.10 just below we will mention another category-theoretic property of varieties. Yet another one is that of being a *binding* category, investigated for varieties in [398] and [180].

14.10. Unique factorization of finite algebras (UFF) in V, i.e., if $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n \cong \mathbf{B}_1 \times \cdots \times \mathbf{B}_s \in V$ is finite and no $\mathbf{A}_i$ or $\mathbf{B}_j$ can be further decomposed as a product of smaller factors, then n = s and after suitably renumbering, $\mathbf{A}_1 \cong \mathbf{B}_1$, $\mathbf{A}_2 \cong \mathbf{B}_2, \ldots, \mathbf{A}_n \cong \mathbf{B}_n$. (Historically, this problem has been approached independently of any mention of V, but the results obtained often have an equational character.) Birkhoff proved [50, page 169] that V has UFF if V has a constant term a such that

(*) $V \models F(a,\ldots,a) = a$ for all operations F of V

and V has permutable congruences, and Jónsson [222] improved "permutable" to "modular" (see §15 for these terms). Jónsson and Tarski [230] proved that V has UFF if V has a constant term a obeying (*) and a binary operation + such that

$$V \models x + a = x = a + x .$$

McKenzie showed [294] that the variety of idempotent semigroups (see 13.6) has UFF, but the variety of commutative semigroups does not (see [50, page 170]). UFF has an influence on the fine spectrum (cf. 14.2) - see [424, pages 285-6]. For the closely related subject of "cancellation," see Lovász [262].

14.11. Universal varieties. We must refer the reader to the papers [359], [361] of Pigozzi for this relatively new notion which promises to be quite important. V is *universal* iff for every similarity type there exist V-terms α_t corresponding to the operations of this type such that for each **A** of this type there exists $\mathbf{B} \in V$ such that **A** is a subalgebra of $(B; F_t^B)_{t\in T}$ and $(B; F_t^B)_{t\in T}$ obeys exactly the same laws as **A**. (E.g., the variety of quasigroups is universal.) Many of the undecidability and lattice-theoretic results of §§12,13 above extend to universal varieties.

14.12. The *Schreier* property (all subalgebras of free algebras are free) is investigated in Meskin [311], Kelenson [243], Aust [8], Ježek [210] and Budkin [71]; cf. 12.8 above. Neumann and Wiegold (supplementary bibliography) showed that the only Schreier varieties of groups are all groups, all Abelian groups, and all Abelian groups of exponent p (prime). (Schreier earlier proved that the variety of all groups has this property.) T. Evans gave a parallel result for semigroups (see supplementary bibliography).

15. Malcev conditions and congruence identities. Malcev proved [278] that a variety V has *permutable congruences* iff there is a ternary term $p(x,y,z)$ such that

$$V \models p(x,x,y) = p(y,x,x) = y.$$

(For binary relations φ, ψ, define $\varphi \cdot \psi = \{(a,c)$: there exists $b(a,b) \in \varphi$ and $(b,c) \in \psi\}$ and say that V has permutable congruences iff $\varphi \cdot \psi = \psi \cdot \varphi$ for all congruences on any $A \in V$.) B. Jónsson proved [221] that all congruence lattices of algebras in V obey the distributive law iff there exist ternary terms $p_i(x,y,z)$ $(0 \leqslant i \leqslant n)$ such that the following equations hold identically in V:

$$\begin{array}{ll} p_i(x,y,x) = x & (0 \leqslant i \leqslant n) \\ p_0(x,y,z) = x \quad p_n(x,y,z) = z & \\ p_i(x,x,y) = p_{i+1}(x,x,y) & (\text{i even}) \\ p_i(x,y,y) = p_{i+1}(x,y,y) & (\text{i odd}). \end{array}$$

(But cf. 12.10 above.) And Day [98] proved a similar result for modularity of the congruence lattice. Properties of varieties definable in this way by the existence of terms have come to be known as *Malcev-definable* (see [420], [333] or [26] for a precise definition). The number of properties known to be Malcev-definable has been growing rapidly - see [420] for a summary of those known up to 1973 and [73] for a partial updating; also see [21], [44]. They include the following:

$|\mathbf{A}|$ divides $|\mathbf{B}|$ whenever $\mathbf{A} \subseteq \mathbf{B} \in V$, $\mathbf{B}$ finite;

V has no topological algebras with noncommutative homotopy (cf. §16);

no $\mathbf{A} \in V$ is a union of two proper subalgebras [73];

V has no nontrivial finite algebras (cf. 14.1).

The first three of these hold for all groups. The last one has the distinction that there is no way to recursively enumerate a Malcev condition for it, as was observed by J. Malitz - see [420, page 383]. See [420], [333], or [26] for a necessary and sufficient condition for a property of varieties to be Malcev-definable which easily entails all the above examples except the second (and many more).

Permutability and modularity of congruences have been important from earliest times in universal algebra (cf. 14.10 above and for recent examples, see [173] [428] and recent work of J. Smith). But the condition which has been most important recently is distributivity; this importance stems from Jónsson's theorem [221] that if X is any subset of a congruence-distributive variety then

$$\text{S.I.} \cap \mathbf{HSP}(X) \subseteq \mathbf{HSU}(X)$$

(here S.I. denotes the class of all subdirectly irreducible algebras (§4) and U(X) is the class of all ultraproducts of families of members of X). (Also see Baker [18].) In many cases this yields a *very* good representation theory in the sense of §4, e.g. if X is a finite set of finite algebras, in which case U(X) = X. (For the finite case, see also Quackenbush [373] for a somewhat simpler proof; a similar argument had earlier been known to A. F. Pixley.) Among many uses of this result has been the investigation of congruence lattices (see especially 13.2 and 13.11 above), and the "internal" model theory of many individual varieties whose algebras have the operations of lattice theory among their operations. See e.g. Davey [97], Berman [45], and references given there. A very important kind of algebra generating a congruence-distributive variety is a quasi-primal algebra, i.e., within equivalence, an algebra $\mathbf{A} = \langle A,T,F_1,F_2,\ldots\rangle$ where A is a finite set and

$$T(x,y,z) = \begin{cases} x \text{ if } x \neq y \\ z \text{ if } x = y \end{cases}$$

Every finite algebra in **HSP A** is uniquely a product of subalgebras of **A**. Many of the equivalence-invariants of these varieties (e.g. the fine spectrum, CEP, AP, $\omega_n(V)$ - see §14) are relatively easy to evaluate. See Pixley [362] and Quackenbush [369] for details and further references - the notion goes back essentially to Pixley, building on work of Foster and Rosenbloom. For infinite analogs of primal algebras, see Tulipani [433] and Iwanik [202]. For congruence-distributivity cf. also 9.11 and 14.7.

Clark and Krauss [89] [90] have given a remarkable theory of *para primal* algebras, a kind of non-distributive generalization of quasiprimal algebras, combining ideas of quasiprimality and linear algebra. Also see [371] [300] [173]; cf. 9.13 above.

Pixley and Wille gave an algorithm ([363] [442]; also see [420, Theorem 5.1]) to convert every identity on the congruence lattice (in $\wedge$, $\vee$, and $\circ$) into a Malcev condition. Which of these conditions are "new" remains an open question. For instance, Nation proved [322] that for certain lattice laws λ which do not imply the modular law, the following holds: if all congruence lattices of algebras in a variety V obey λ, then they all obey the modular law.

Very recently S. V. Polin has proved that Nation's result fails for some non-trivial lattice law λ. Non-modular "congruence varieties" are extensively investigated in a forthcoming paper of A. Day and R. Freese. Also see [103], [224] or [298] for further discussion of the state of affair just prior to Polin's result. Also see B. Jónsson's appendix to the forthcoming new edition of [163].

Another Malcev-definable property of varieties V which has received wide attention is that $\mathbf{F}_V(n) \cong \mathbf{F}_V(m)$. (See e.g. Marczewski [282], references given there, and various other articles in the same volume of Colloquium Mathematicum.) For fixed n_0, the set of numbers

$$\{n \in \omega : \mathbf{F}_V(n) \cong \mathbf{F}_V(n_0)\}$$

is always an arithmetic progression, and any progression can occur (Świerczkowski, et al.). If $\mathbf{F}_V(n) \cong \mathbf{F}_V(m)$ with $m \neq n$, then V has no non-trivial finite algebras (Jónsson and Tarski [231]) (cf. 14.1). (Also see Clark [87].)

See Csákány [95] for a collection of properties of varieties resembling, but more general than, Malcev conditions. A nice special example is in Klukovits [247].

16. Connections with topology. If $\mathrm{A} = (\mathrm{A},\mathrm{F}_t)_{t\in\mathrm{T}}$ is any algebra and $\mathcal{T}$ is any topology on A such that every $\mathrm{F}_t\colon \mathrm{A}^{n_t} \to \mathrm{A}$ is continuous (in the n_t-fold topological product of $\mathcal{T}$), then we say that $A = (\mathrm{A},\mathcal{T},\mathrm{F}_t)_{t\in\mathrm{T}}$ is a *topological algebra*. The space $(\mathrm{A},\mathcal{T})$ is not at all independent from the equational theory of A, but rather the two seem to influence each other quite strongly. This influence is poorly understood, but many interesting examples are known.

16.1. No sphere except S^0, S^1, S^3, S^7 can be an "H-space," i.e., can obey

$$ex = x = xe$$

for binary multiplication and constant e (Adams [2]), and S^7 cannot also obey the associative law (James [204]).

16.2. The space of a topological group must be homogeneous, with Abelian fundamental group, and if compact and uncountable, of power $\geqslant 2^{\aleph_0}$. (All of these facts are essentially well known.)

16.3. The space of a topological Boolean algebra, if compact, must be a power 2^n for 2 a 2-element space. (Kaplansky [237].)

16.4. The space of topological semilattice has zero homotopy in each of its components in each dimension (Taylor [429]), and if compact, connected and finite-dimensional, cannot be homogeneous (Lawson and Madison [253]).

16.5. If V is defined by the equations (*) of 14.3 above, then it is not hard to check that the topological algebras in V have "square" universe (as in Evans [118]), i.e. each is homeomorphic to the square of some other space.

16.6. If V is the product $U \otimes W$ of two varieties (see [424], §0]), then a topological algebra in V with product-indecomposable space must be either in U or W (this can be seen fairly easily from the methods outlined in [420, pages 357-358] or [424, pages 265-267]).

16.7. If a compact connected topological algebra obeys the law

$$(xy)(yz) = xz,$$

then it also obeys the law $xy = uv$. (Bednarek and Wallace [38].) It is an old problem of A. D. Wallace (see [378]) whether the "skew-associative" law

$$x(yz) = (zx)y$$

can hold on the unit interval without the associative law holding as well.

For some more examples consult [429]; also cf. 14.1 above. In [429] there is a fairly complete analysis of the influence of the laws obeyed by a topological algebra on the laws which must be obeyed by its homotopy groups. Świerczkowski's method [407] of topologizing free algebras was essential to this work. But we are still a long way from understanding the general interaction between the space of A and Eq **A**. There is of course no *a priori* reason for believing that Eq A is especially important here (rather than say the full first order theory of **A**), but experience has often yielded examples which involved identities. (For e.g. connected fields obeying $px = 0$, see [438], [320].)

Very closely connected is the study of *functional equations* - see Aczél's book [1] - a vast subject in itself. It proceeds like the above, also allowing certain "constants," i.e. function symbols whose meaning is prescribed in advance, such as ordinary addition + of real numbers - a typical early result being Cauchy's theorem that the continuous solutions of

$$f(x + y) = f(x) + f(y)$$

on the real numbers are the linear functions $f(x) = ax$.

17. **Miscellaneous.** Here we list a few topics that are concerned with equations in one way or another, but do not fit precisely into any of the earlier sections.

17.1. Algebraically closed algebras are defined analogously to algebraically closed fields. See e.g. Simmons [399], Bacsich [12], Forrest [139], Sabbagh [381] and Schupp [386] and references given there. All algebraically closed algebras in a variety V are simple iff every algebra in V can be embedded in a simple algebra of V. (B. H. Neumann had earlier proved that every algebraically closed group is simple.) (Cf. the final result of Evans in §12 - *Word Problems*, and that of McKenzie and Shelah in 14.8.) Of course the satisfiability of equations (in algebraically closed fields) is historically where the study of equations arose. In a recursively axiomatized variety V, if a finitely presented $\mathbf{A} \in V$ is embeddable in every algebraically closed $\mathbf{B} \in V$, then **A** has solvable word problem (Macintyre [271]; see also [329], [381] and [386]).

17.2. Satisfiability of equations, especially their unique satisfiability, figures heavily in the work of Sauer and Stone characterizing concrete endomorphism monoids. (See [383].)

17.3. See e.g. [193], [363] and references given there for "local" varieties.

17.4. There is a rapidly growing theory of combinatorial designs as varieties. Bruck first observed that if a binary operation obeys the laws

$$xx = x$$
$$xy = yx$$
$$x(xy) = y$$

then one has a Steiner triple system

$$\{\{x,y,xy\} : x \neq y\},$$

and conversely, all Steiner triple systems arise in this way. This idea can be greatly extended; consult Evans [126], Ganter and Werner [148], [149], Quackenbush [370], Banaschewski and Nelson [35], and Ganter [147].

17.5. Goodstein [159] propounded an equational axiom system for the natural numbers, equally as strong as Peano's. It differs from the systems described here in that it had another rule of proof, corresponding to uniqueness for inductive definitions.

17.6. An equation calculus for the recursive definition of functions was developed by Gödel and Herbrand - it is described in [246, §54]. But it differs widely from the logic of equations described here - for example, it lacks the symmetric law - an equation $f(\cdots) = g(\cdots)$ is an instruction meaning roughly, "if the RHS has been calculated, then regard this as a way of calculating the LHS," and this process cannot be reversed. Of course, this calculus greatly resembles the use of equations in e.g. Fortran, although there are obvious differences.

17.7. Henkin has proposed a theory of types which can be viewed as an equational theory. (See [182].)

17.8. A theory of "heterogeneous" varieties (objects of more than one type) was developed by G. Birkhoff and J. D. Lipson, A. I. Malcev, and P. J. Higgins. It was useful in Taylor [420] (q.v. for full references, pages 358-359). Boardman [58] used essentially the same theory in algebraic topology (his "colors").

17.9. One can study "products" of varieties in a sense which originated in group theory (see H. Neumann's book [330]), and was later extended by A. I. Malcev (see [279, page 422]) to some other varieties. For recent developments, see [249], [258].

17.10. For "identical inclusions" see e.g. [261]. An identical inclusion is a (non-first-order) condition of the form

$$\tau(x_1,x_2,\ldots) \in \{\sigma_1(x_1,x_2,\ldots),\ \sigma_2(x_1,x_2,\ldots),\ldots\},$$

where $\{\ldots\}$ denotes the subalgebra generated by ... .

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The Department of Mathematics and the University of Houston are greatly indebted to Mr. Roy H. Cullen for a major grant which made it possible to establish this publication.