Special Polynomials

Patterns
Certain polynomials can be factored by finding a pattern. This section deals with four special patterns for factoring polynomials: difference of squares, difference of cubes, sum of cubes, and perfect squares

Difference of Squares
The difference of squares pattern can be identified by looking at the polynomial. It must be a binomial, the first term must be a variable to the second power (a.k.a. squared) and a constant term must be subtracted from it. There is no first-order variable term in a difference-of-squares polynomial.

The formula is: $\boxed{a^2 - b^2 = (a-b)(a+b)}$

Example: Factor $x^2 - 25$.

This binomial has its first term is x^2, a second-order monomial. The only other term is 25, just a constant. This means $x^2 - 25$ can be factored using the difference of squares pattern, so $x^2 - 25 = (x)^2 - (5)^2 = (x - 5)(x + 5)$.

To check, we can multiply the factored form back together using the FOIL method:
$(x - 5)(x + 5) = x^2 + \cancel{5x} - \cancel{5x} - 25 = x^2 - 25$.

Example: Factor $9x^2 - 25$.
This binomial's highest order monomial is $9x^2$; the other monomial is the constant 25, so we can factor $9x^2 - 25$ using the difference of squares method:
$9x^2 - 25 = (3x)^2 - (5)^2 = (3x - 5)(3x + 5)$.

Note: There is **no** sum of squares factorization; that is, we can't factor $a^2 + b^2$.

Example 1: Factor $x^2 - 100$. $= (x)^2 - (10)^2 = (x-10)(x+10)$

Example 2: Factor $y^2 - 81$ $= (y)^2 - (9)^2 = (y-9)(y+9)$

Example 3: Factor $16y^2 - 9$ $= (4y)^2 - (3)^2 = (4y-3)(4y+3)$

Difference of Cubes
To find the pattern for the difference of cubes, the polynomial to factor must be a binomial, the first term must be a variable to the third power (a.k.a. cubed) and a constant term must be subtracted from it. There cannot be any first- or second-order variable terms in a difference-of-cubes polynomial.

The formula is: $\boxed{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$

Example: Factor the binomial $x^3 - 1$.
This binomial has x^3, a third-degree term, as its highest-powered monomial and only a constant, 1, is subtracted from it. We can factor by using the difference of cubes formula:
$x^3 - 1 = (x)^3 - (1)^3 = (x-1)(x^2 + (x)(1) + (1)^2) = (x-1)(x^2 + x + 1)$

Check: $(x-1)(x^2 + x + 1) = x\,.x^2 + x\cdot x + x\cdot 1 - 1\cdot x^2 - 1\cdot x - 1\cdot 1 = x^3 - 1$

Example: Factor the polynomial $b^3 - 64$.
The highest degree in this binomial is 3, on the b3 term. The only other monomial in the binomial is 64, so the difference of cubes form is useful for factoring:
$b^3 - 64 = (b)^3 - (4)^3 = (b-4)(b^2 + 4b + 16)$

Check: $(b-4)(b^2 + 4b + 16) = b^3 + 4b^2 + 16b - 16b - 64 = b^3 - 64$.

Example 4: Factor $a^3 - 8$. $= (a)^3 - (2)^3 = (a-2)(a^2+2a+4)$

Example 5: Factor $27x^3 - 1$. $= (3x)^3 - (1)^3 = (3x-1)(9x^2 + 3x + 1)$

Sum of Cubes
A sum of cubes polynomial is similar to the difference of cubes polynomial, except that the constant term is added to the third-order monomial instead of subtracted.

The formula is: $\boxed{a^3 + b^3 = (a+b)(a^2 - ab + b^2)\,.}$

Example: Factor the binomial $a^3 + 125$.
This binomial is a cubic monomial (a^3) plus a constant (125) so we can use the formula:
$a^3 + 125 = (a)^3 + (5)^3 = (a+5)(a^2 - 5a + 25)$.
Check: $(a+5)(a^2 - 5a + 25) = a^3 - 5a^2 + 25a + 5a^2 - 25a^2 + 125 = a^3 + 125$.

Example: Factor $x^3 + 27y^3$.
This binomial is a cubic monomial plus another cubic monomial, so we can use the formula:
$x^3 + 27y = (x)^3 + (3y)^3 = (x+3y)(x^2 - 3xy + 9y)$.
Check: $(x+3y)(x^2 - 3xy + 9y^2) = x^3 - 3x^2y + 9xy^2 + 3x^2y - 9xy + 27y^3 = x^3 + 27y^3$.

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

Example 6: Factor x^3+8. $=(\overset{a}{x})^3+(\overset{b}{2})^3=(\overset{a}{x}+\overset{b}{2})(\overset{a^2}{x^2}\overset{-ab}{-2x}+4)$

Example 7: Factor $8x^3+1$. $=(\overset{a}{2x})^3+(\overset{b}{1})^3=(2x+1)(4x^2-2x+1)$

Perfect Square Trinomials

There are two patterns for the perfect square trinomial:

$\boxed{x^2+2xy+y^2=(x+y)^2}$ and $\boxed{x^2-2xy+y^2=(x-y)^2}$.

This trinomial has three terms: two of the terms (usually the first and last) are monomials that we can easily take the square root of, and a third term is twice the roots of the other two terms multiplied together. Note that only the sign of the middle term can be positive or negative, and it matches the sign in the middle of the factored form. All of the other coefficients must be positive.

$2^2=4$

Example: Factor x^2-4x+4

$2\cdot 2\cdot x$

This is an example of a perfect square trinomial: it is easy to take the square root of the first and last terms (x^2 and 4), and the middle term is twice the roots of the end terms. We can use the formula. Since the middle term is negative, we will subtract the two terms in the formula:

$$x^2-4x+4=(\sqrt{x^2}-\sqrt{4})^2=(x-2)^2.$$

$x^2-4x+4=(x-2)^2$

Check: $(x-2)^2=(x-2)(x-2)=x^2-2x-2x+4$. $=x^2-4x+4$

Example: Factor the trinomial $25x^2+80x+64$

$(5x+8)$

$$25x^2+80x+64=(\sqrt{25x^2}+\sqrt{64})^2=(5x+8)^2.$$

$2\cdot(5x)(8)=80x$

Check: $(5x+8)(5x+8)=25x^2+40x+40x+64=25x^2+80x+64$.

Example 8: Factor x^2+6x+9. $=(\sqrt{x^2}+\sqrt{9})^2=(x+3)^2$ $\quad 2x(3)$ $\quad 6x$

Example 9: Factor $x^2-10x+25$. $=(\sqrt{x^2}-\sqrt{25})^2=(x-5)^2$

$2\cdot x(-5)=-10x$

Example 10: Factor $x^2-18x+81$.

$=(\sqrt{x^2}-\sqrt{81})^2$

$=(x-9)^2$ $\quad 2x(-9)=-18x$

Examples:

$a^2 - b^2 = (a-b)(a+b)$

1. Factor $4x^2 - 25$. $= (2x)^2 - (5)^2 = (2x-5)(2x+5)$

2. Factor $81a^2 - 16b^2$. $= (9a)^2 - (4b)^2 = (9a-4b)(9a+4b)$

3. Factor $\frac{1}{81}x^2 - y^2$. $= \left(\frac{1}{9}x\right)^2 - (y)^2 = \left(\frac{1}{9}x - y\right)\left(\frac{1}{9}x + y\right)$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

4. Factor $b^3 - 8a^3$. $= (b)^3 - (2a)^3 = (b-2a)(b^2+2ab+4a^2)$

$a^3 + b^3 = (a+b)(a^2-ab+b^2)$

5. Factor $64x^3 + 27$. $= (4x)^3 + (3)^3 = (4x+3)(16x^2-12x+9)$

6. Factor $a^3 - 27b^3$. $= (a)^3 - (3b)^3 = (a-3b)(a^2+3ab+9b^2)$

7. Factor $125y^3 - 64x^3$ $= (5y)^3 - (4x)^3 = (5y-4x)(25y^2+20xy+16x^2)$

8. Factor $125z^3 + 1$ $= (\underset{a}{5z})^3 + (\underset{b}{1})^3 = (5z+1)(\underset{a^2}{25z^2} \underset{-ab}{-5z} \underset{+b^2}{+1})$

9. Factor $5p^3 - 5q^3$ $= 5(p^3-q^3) = 5(p-q)(p^2+pq+q^2)$

10. Factor $x^2 + 2x + 1$ $= (x+1)^2$ ✓ $2\cdot x\cdot 1 = 2x$

11. Factor $4x^2 + \underline{36x} + 81$ $= \left(\sqrt{4x^2} + \sqrt{81}\right)^2 = (2x+9)^2$ ✓ $2\cdot(2x)(9) = 36x$

12. Factor $x^2 + 64$. Cannot factor!

13. Factor $x^4 - y^4$. $= (\underset{a}{x^2})^2 - (\underset{b}{y^2})^2 = (\underset{a-b}{x^2-y^2})(\underset{a+b}{x^2+y^2})$

$a^2 - b^2$

$= \boxed{(x-y)(x+y)(x^2+y^2)}$