Math 1311
Section 4.1
Exponential Growth and Decay

A function that grows or decays by a constant percentage change over each fixed change in input is called an exponential function.

a^n ← exponent
a ← base

Exponents – A quick review

1. $a^0 = 1$ $5^0 = 1$ $7^0 = 1$
2. $a^{-n} = \frac{1}{a^n}$ $5^{-1} = \frac{1}{5}$ $5^{-3} = \frac{1}{5^3}$
3. $a^{\frac{1}{n}} = \sqrt[n]{a}$ $5^{1/2} = \sqrt{5}$ $5^{1/3} = \sqrt[3]{5}$
4. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ $5^{3/4} = \sqrt[4]{5^3}$
5. $a^m a^n = a^{m+n}$ $5^2 \cdot 5^3 = 5^5$
6. $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$ $\frac{5^2}{5^3} = 5^{2-3} = 5^{-1} = \frac{1}{5}$
7. $(a^m)^n = a^{mn}$ $(5^2)^3 = 5^6$
8. $(ab)^n = a^n b^n$ $(3 \cdot 5)^2 = 3^2 \cdot 5^2$
9. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$ $\left(\frac{3}{5}\right)^2 = \frac{3^2}{5^2}$
10. For $b \neq 1$, $b^x = b^y$ means $x = y$. $3^2 = 3^x$ $x = 2$

Example 1: Simplify. Write your final answer without negative exponents.

1. $((a^2)^3)^4$

$= a^{2 \cdot 3 \cdot 4} = a^{24}$

2. $\frac{a^3 b^2}{a^2 b^3}$ $= a^{3-2} b^{2-3}$

$= a b^{-1} = \frac{a}{b}$

3. $a^3 b^2 a^4 b^{-1}$

$a^{3+4} b^{2-1} = a^7 b$ $\frac{a^6 b^2}{a^4 b^3}$

4. $\frac{a^{-4} b^{-3}}{a^{-6} b^{-2}}$ $= a^{-4-(-6)} b^{-3-(-2)}$

$= a^2 b^{-1} = \frac{a^2}{b}$

Exponential Growth
Example 2: A petri dish contains 500 bacteria at the start of an experiment. The number of bacteria double each hour. We can calculate the number of bacteria in the dish as a function of the number of hours since the experiment started. Here is the beginning of the chart:

Time, in hours	0	1	2	3	4	5
Number of bacteria	500	1000	2000	4000	8000	16000

Let $N = N(t)$ be the number of bacteria in the dish t hours after the experiment started.

Let's use the pattern in the chart above to develop a formula for $N(t)$.

$N(0) = 500$

$N(1) = 2 \cdot 500$

$N(2) = 2(2 \cdot 500) = 2^2 \cdot 500$

$N(3) = 2(2^2 \cdot 500) = 2^3 \cdot 500$

$N(t) = \underbrace{500}_{\text{Initial value}} \cdot \underbrace{2}_{\text{Growth factor}}{}^t$

The growth factor is: 2

The Exponential Growth Formula: $\boldsymbol{N(t) = Pa^t,\ a > 1.}$

P is the initial value, t is the time and a is the growth factor for each unit of time.

Exponential Decay

Example 3: Suppose we change the experiment in Example 2 by introducing an antibiotic into the petri dish. Now, the number of bacteria in the dish is cut in half each hour.

Time, in hours	0	1	2	3	4	5
Number of bacteria	500	250	125	62.5	31.25	15.625

Let $N(t)$ be the number of bacteria in the petri dish t hours after the antibiotic was introduced. The pattern in the above chart suggests the following formula for $N(t)$:

$N(0) = 500$

$N(1) = \frac{1}{2}(500)$

$N(2) = \frac{1}{2}\left(\frac{1}{2} \cdot 500\right) = \left(\frac{1}{2}\right)^2 \cdot 500$

$N(t) = \left(\frac{1}{2}\right)^t \cdot 500$

$N(t) = 500\left(\frac{1}{2}\right)^t$

The decay factor is: $\frac{1}{2}$

The Exponential Decay Formula: $\boldsymbol{N(t) = Pa^t,\ a < 1.}$

Example 4: **Radioactive Decay**

If there is 1 gram of heavy hydrogen in a container, then as a result of radioactive decay there will be .783 grams of heavy hydrogen in the container one year later. Suppose a container starts with 25 grams of hydrogen.

a. Find the formula for the number of grams of hydrogen in the container as a function of the time t in years.
b. How much heavy hydrogen is left after 5 years?
c. Plot the graph of the function.
d. Find the time t when $1/2$ of the hydrogen is left in the container.

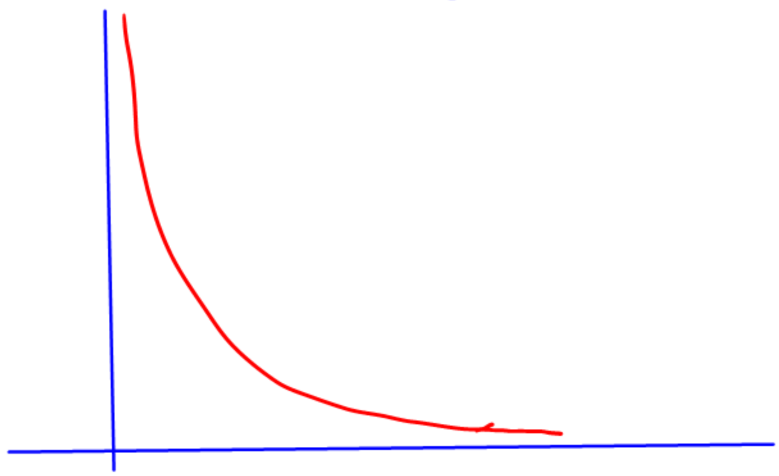
½ of original amount = 12.5 grams

(a) Decay factor = .783

$A(t) = 25 \cdot (.783)^t$

(b) $A(5) = 25(.783)^5 = 7.36$ grams

(c)



(d) $\underbrace{12.5}_{Y_1} = \underbrace{25(.783)^t}_{Y_2}$

$t = 2.83$

$\boxed{\text{after } 2.83 \text{ years}}$

Constant Proportional Change

A function is exponential if it shows constant percentage (or proportional) growth or decay.
Growth: For an exponential function with discrete (yearly, monthly, etc.) percentage growth rate r as a decimal, the growth factor $\boldsymbol{a = 1 + r}$.
Decay: For an exponential function with discrete (yearly, monthly, etc.) percentage decay rate r as a decimal, the decay factor $\boldsymbol{a = 1 - r}$.

Example 5: A certain phenomenon has an initial value of 23 and grows at a rate of 6% per year. Give an exponential function which describes this phenomenon.

$N(t) = P a^t$

$6\% \to 0.06$

$r = 0.06$

$\boxed{N(t) = 23(1.06)^t}$

$a = 1 + r$

$= 1 + 0.06 = 1.06$

$N(5) = 23(1.06)^5 = 30.78$

Example 6: Suppose the amount of pollution in a tank starts at 100 pounds and decreases by 16% per hour. Find the decay constant and the formula for the amount of pollutant in the tank in pounds as a function of time in hours. How much is left in the tank after 10 hours? 15 hours?

$N(t) = P a^t$

$16\% \to .16$

$N(t) = 100 \cdot a^t$

$a = 1 - r$

$= 1 - .16 = .84$

$N(t) = 100(.84)^t$

$N(10) = 100(.84)^{10} = 17.49$ pounds

$N(15) = 100(.84)^{15} = 7.31$ pounds

Growth or Decay Factor Unit Conversion
If the growth or decay factor for one period of time is a, then the growth or decay factor for k periods of time is given by $A = a^k$.

Here is the conversion diagram:

GROWTH FACTOR FOR ONE PERIOD OF TIME	Raise to k^{th} Power $\longrightarrow$ $\longleftarrow$ Raise to $(1/k)^{th}$ Power	GROWTH FACTOR FOR k PERIODS OF TIME

What does this mean in practical terms?
Example 7: Census data is collected every 10 years. Suppose the census data shows that the population increases by 23% per decade. What is the yearly growth factor?

% increase for 10 years (0.23) → growth factor for 10 years ($1+0.23 = 1.23$) →

→ growth factor for 1 year ($(1.23)^{1/10} = 1.02$) → % increase for 1 year ($1.02-1 = .02 \to 2\%$)

$\boxed{1.02}$

Example 8: Terry deposits $10,000 in an account with a 0.75% monthly interest rate. What is the yearly growth factor?

$.75\% \to .0075 = r$

$a = 1+r = 1+.0075 = 1.0075$ monthly growth factor

$(1.0075)^{12} = 1.094$ yearly growth factor

Find the amount in the account after 10 years.

$N(t) = 10000\,(1.094)^t$

$N(10) = 10000\,(1.094)^{10} = \$24,556.88$