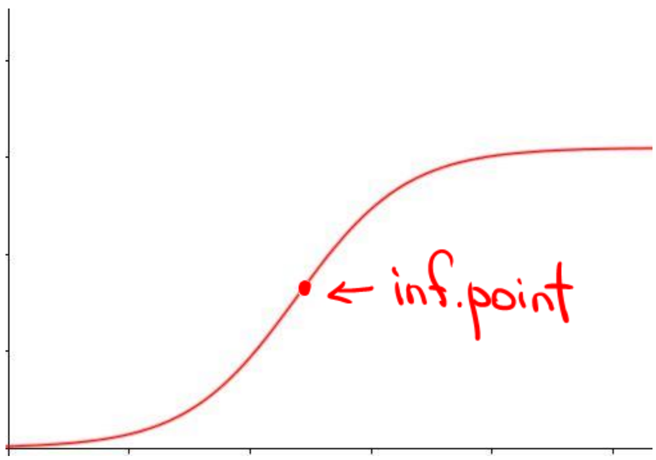
Math 1311
Section 5.1
Logistic Functions

We can assume in many situations that growth is exponential. For population growth, an exponential model is a consequence of the assumption that percentage change (birth rate minus death rate) is constant. In reality a population cannot undergo such rapid growth indefinitely. The species will begin to exhaust local resources, and we expect that instead of remaining constant, the birth rate will begin to decrease and death rate to increase.
A model that takes into account the limited potential for growth is the logistic growth model.

Logistic Growth Curve



The logistic growth curve has the following properties:

- Initially the growth is rapid, nearly exponential.
- The inflection point represents the location of most rapid growth.
- After the inflection point, the growth rate declines. The function has a limiting value, known as the carrying capacity.
- The point of inflection occurs at an output of half of the carrying capacity. This is the level of maximum growth. This level is often called the optimum yield level.

Logistic Model Formula

$$N = \frac{K}{1 + be^{-rt}}$$

- The constant K is the carrying capacity. It is the limiting value of N.
- The point of inflection, or optimum yield level, occurs at an output of $\frac{K}{2}$

- The constant b is determined by the formula $b = \frac{K}{N(0)} - 1$.

- The r-value is the intrinsic exponential growth rate. In the absence of limiting factors, growth would be exponential according to the formula $N = N(0)e^{rt}$. The corresponding growth factor would be $a = e^r$.

- Note that we can solve $a = e^r$ for r and get $r = \ln a$

Example 1: The Pacific Sardine

The function

$$N = \frac{2.4}{1 + 239e^{-0.338t}}$$

$N = \frac{K}{1+be^{-rt}}$

models the Pacific sardine population (measure in million tons of fish) t years from now.

a. What is r for the Pacific sardine?

$r = .338$

b. What would be the annual percentage growth rate for the Pacific sardine in the absence of limiting factors?

$a = e^r = e^{.338} = 1.4$

$1.4 - 1 = .4$

40%

c. What is the carrying capacity?

2.4 million ton of fish

d. What is the optimum yield level?

$\frac{2.4}{2} = 1.2$ million ton of fish

e. Make a graph of N versus t.

f. At what time should the population be harvested?

$\underbrace{N(t)}_{Y_1} = \underbrace{1.2}_{Y_2}$ t - ?

≈ 16.2 years

0 to 1.2 $\quad 1.2 \to \infty$

g. What portion of the graph is concave up? Concave down?

Example 2: Constructing a logistic function
We begin selling a new magazine in a small town. Initial sales are 250 magazines per month. We believe that in the absence of limiting factors, our sales will increase by 6% per month, but the size of the town limits our total sales to 1000 magazines per month.

a. Construct a logistic model for our magazine sales under these conditions.

N = sales per month after t months.

$$N(t) = \frac{k}{1+be^{-rt}}$$

$N(0) = 250$

6% → .06

$a = 1+.06 = 1.06$

✓ $r = \ln 1.06 = .058$

✓ $k = 1000$

✓ $b = \frac{k}{N(0)} - 1 = \frac{1000}{250} - 1 = 4-1 = 3$

$$b = \frac{k}{N(0)} - 1$$

$$r = \ln a$$

b. When can we expect sales to reach 750 magazines per month?

$$N(t) = \underbrace{\frac{1000}{1+3e^{-.058t}}}_{Y_1} = \underbrace{750}_{Y_2}$$

$\boxed{\text{38 months}}$

Logistic Regression
When a scatterplot of data shows the behavior of logistic growth (initially exponential looking but then leveling off after an inflection point), then the data may be best modeled with a logistic regression. Following the same steps we have done in the TI for other regressions, we can get a logistic regression model.

Example 3: A company is developing a new computer chip. Each month a collection of prototypes is tested and percentage *P* of chips that operate successfully is recorded. Here are the results.

m = month	1	2	3	4	5	6	7
P = % successful	22	29	37	45	54	63	71

a. Find a logistic model for *P* versus m.

$$P = \frac{102.55}{1+5.13e^{-.35m}}$$

b. Reliability is a percentage of success. The chip is ready for production when reliability reaches 95%. When does the logistic model predict the chip will be ready for production? Round your answer to the nearest month.

$P = 95 \quad x = 12 \rightarrow$ 1 year