Math 1311
Section 5.5
Polynomials and Rational Functions

In addition to linear, exponential, logarithmic, and power functions, many other types of functions occur in mathematics and its applications. In this section we look at polynomials (quadratic and higher degree) and rational functions.

Definition: A polynomial is a function whose formula can be written as a sum of power functions where each of the powers is a non-negative whole number.

The degree of a polynomial is the highest exponent of x.

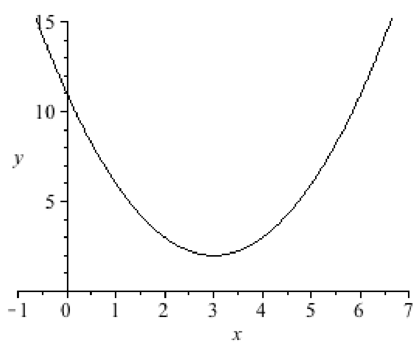
Example 1:
Polynomials: $3x^5 - 7x^2 + \frac{1}{2}x - \sqrt{2}$, $x^2 + 6x + 9$
Not polynomials: $\sqrt{x}, \frac{1}{x}, 2^x, e^x, \ln x$

$\sqrt{x} = x^{1/2}$ $\quad \frac{1}{x} = x^{-1}$

Definition: A quadratic function is a function that is written in the form $f(x) = ax^2 + bx + c$, where $a, b,$ and c are real numbers, and a is not zero.

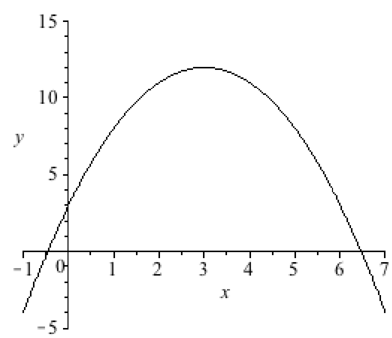
Quadratic functions are a specific type of polynomial functions.

The graph of a quadratic function is a parabola. If a is positive, the parabola opens upward. If a is negative, it opens downward.



$5x^2 - \sqrt{2}$

$5 > 0$ ∪

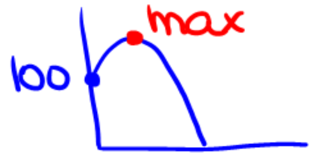


$-\frac{1}{2}x^2 + 3x - 7$

$-\frac{1}{2} < 0$ ∩

Example 2: The height of a rocket is modeled by the function $h(t) = -16t^2 + 64t + 100$ where t is given in seconds after launch and $h(t)$ is given in feet.

a. Graph the function.



Graph

b. Find $h(0)$. What does this represent?

$h(0) = 100$ ft initial height

c. At what time does the rocket reach the highest point in its flight?

$\text{max} = (\overset{x}{2}, \overset{y}{164})$

at 2 seconds

d. What is the rocket's maximum height?

164 ft

e. At what time(s) is the rocket 75 feet above the ground?

$\underbrace{-16t^2 + 64t + 100}_{Y_1} = \underbrace{75}_{Y_2}$

(4.36, 75)

at 4.36 sec.

f. When does the rocket hit the ground?

$Y_1 = \underbrace{0}_{Y_2}$

after 5.2 sec.

Quadratic Regression

Data that is "almost quadratic" can be modeled using quadratic regression.

Example 3: Use quadratic regression to find a model for the following data set:

x	-2.4	-1.1	5.7	16.4	18.3
f(x)	17.3	4.1	93.6	743.6	864.5

$f(x) = 2.47x^2 + 2.93x + 5.77$

a. Use the model to approximate the values with $x = 10$.

$f(10) = 281.62$

b. For what value of x is the function equal to 200.

$\underbrace{f(x)}_{Y_1} = \underbrace{200}_{Y_2}$

$\boxed{\begin{array}{l} x = 8.3 \\ x = -9.49 \end{array}}$

$y = 200$

$y = 200$

Example 4: The following table shows the rate R of vehicular involvement in traffic accidents (per 100,000,000 vehicle-miles) as a function of vehicular speed, s, in miles per hour, for commercial vehicles driving at night on urban streets.

Speed s	20	25	30	35	40	45	50
Accident rate R	1600	700	250	300	700	1300	2480

a. Use regression to find a quadratic model for the data.

$R(s) = 7.79s^2 - 514.36s + 8733.57$

b. Calculate $R(50)$ and explain what your answer means.

$R(50) = 2480$

c. At what speed is vehicular involvement in traffic accidents at a minimum?

33 miles / hour

Popper 29 5 As

The Quadratic Formula

There is a well-known formula, the quadratic formula, which makes possible the solution of quadratic equations by hand calculation.

A quadratic equation $ax^2 + bx + c = 0$ has two solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 5: Use the quadratic formula to solve $-2x^2 + 2x + 5 = 0$

$a = -2 \quad b = 2 \quad c = 5$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(-2)(5)}}{2(-2)} = \frac{-2 \pm \sqrt{4+40}}{-4} = \frac{-2 \pm \sqrt{44}}{-4}$$

$$= \frac{-2 \pm 2\sqrt{11}}{-4}$$

$$\frac{-2 + 2\sqrt{11}}{-4} = \boxed{\frac{1}{2} - \frac{1}{2}\sqrt{11}}$$

$$\frac{-2 - 2\sqrt{11}}{-4} = \boxed{\frac{1}{2} + \frac{1}{2}\sqrt{11}}$$

$$\sqrt{44} = \sqrt{4 \cdot 11} = \sqrt{4} \cdot \sqrt{11} = 2\sqrt{11}$$

Higher Degree Polynomials

Now we will consider a couple of other polynomial functions, cubic functions and quartic functions. We rarely see data that exactly follow cubic or quartic patterns, so we will look only at regression in this section.

The graphs of these functions are interesting and useful as models, because we can use them to find maximum and minimum values.

Cubic and Quartic Functions

A cubic function is a function whose highest power of the variable is 3; a quartic function is a function whose highest power of the variable is 4.

Example 6: This table gives the average price of natural gas for home use at certain times, *t*. Prices are given in dollars per thousand cubic feet, and time is given in years since 1982.

	1982	1983							
Time	0	1	2	3	4	5	6	7	8
Price	5.17	6.06	6.12	6.12	5.83	5.54	5.47	5.65	5.77

a. Find a cubic function that models this data.

$$f(x) = .021x^3 - .273x^2 + .915x + 5.248$$

b. Graph the cubic function along with the data points.

Graph

c. Now find a quartic function that models this data.

$$g(x) = -.003x^4 + .069x^3 - .513x^2 + 1.291x + 5.176$$

d. Graph the quartic function along with the data points.

Graph

e. Which model seems to fit the data better, the cubic function or the quartic function?

Rational functions

Definition: rational functions are functions that can be written as a ratio~~n~~ of polynomial functions.

Example 7: $\frac{1}{x}$, $\frac{x^{12}-x}{0.7x^6+3}$

Definition: A function has a pole at $x = a$ if the function is not defined at $x = a$ and the values of the function become larger and larger in size as x gets near a.

Definition: If a function has a pole at $x = a$, then we say that the line $x = a$ is a vertical asymptote of the graph of the function and the graph gets closer and closer to this line as x tends toward a.

Definition: A horizontal asymptote is a line to which a graph gets closer and closer to.
The idea of a horizontal asymptote is a geometric version of a limiting value.

Example 7: Find the poles of $\frac{x}{x^2-3x+2}$.

$x^2-3x+2=0$ $\qquad$ $(x-2)(x-1)=0$

Mult. to 2
Add up to -3] -2, -1

$x-2=0$ $\qquad$ $x-1=0$

$\boxed{x=2 \qquad x=1}$

Example 8: Holling's functional response curve describes the feeding habits of a predator in terms of the density of the prey. An example of such a curve is given by a rational function $y = \frac{16x}{1+2x}$. Here x is the density of the prey – that is the number of prey per unit area, and y is the number of prey eaten per day by a certain predator.

a. Make a graph of y as a function of x.

Graph

b. In terms of a predator and its prey, why is it reasonable for the graph to be increasing and concave down?

c. Does the function have a limiting value? What is the equation of the horizontal asymptote?

$y=8$

d. Explain the significance of the horizontal asymptote in terms of the predator and prey.

Large density of prey => each predator eats 8 a day