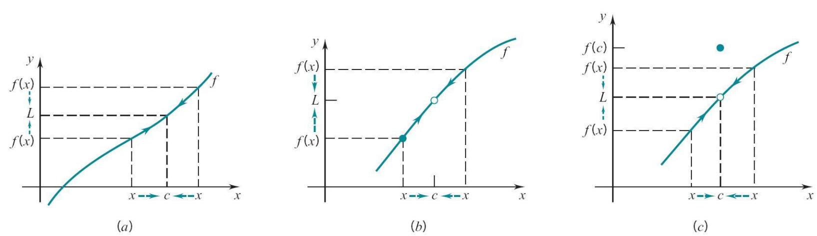
Lecture 2 Section 2.2 Definition of Limit Section 2.3 Some Limit Theorems

Jiwen He

1 Review

1.1 The Ideal of Limit

Graphical Introduction to Limit

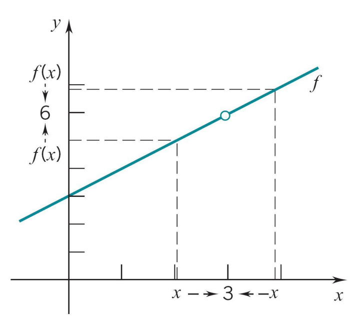


$$\lim_{x \to c} f(x) = L$$

- In taking the limit of a function f as x approaches c, it does not matter whether f is defined at c and, if so, how it is defined there.
- The only thing that matters is the values that f takes on when x is near c.

1.2 Examples

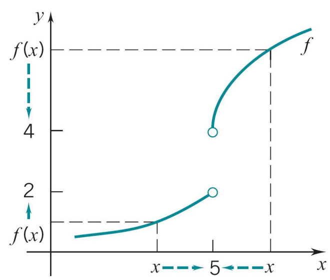
Example: $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$



Let $f(x) = \frac{x^2-9}{x-3}$. Use the graph of f to find

(a) $f(3)$ (b) $\lim_{x \to 3^-} f(x)$ (c) $\lim_{x \to 3^+} f(x)$ (d) $\lim_{x \to 3} f(x)$

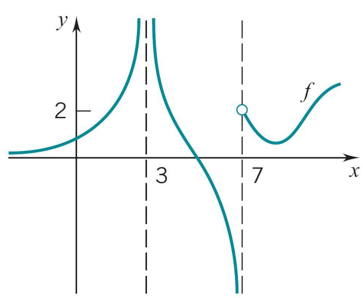
Example



Use the graph of f to find

(a) $f(5)$ (b) $\lim_{x \to 5^-} f(x)$ (c) $\lim_{x \to 5^+} f(x)$ (d) $\lim_{x \to 5} f(x)$

Example

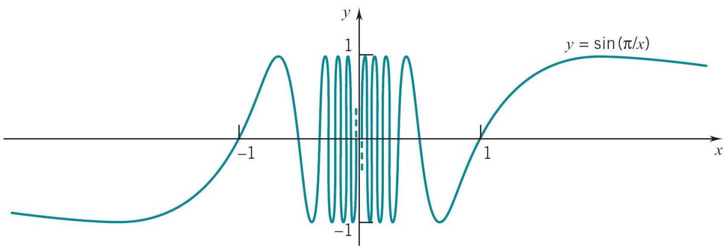


Use the graph of f to find

(a) $f(3)$ (b) $\lim_{x \to 3^-} f(x)$ (c) $\lim_{x \to 3^+} f(x)$ (d) $\lim_{x \to 3} f(x)$

(e) $f(7)$ (f) $\lim_{x \to 7^-} f(x)$ (g) $\lim_{x \to 7^+} f(x)$ (h) $\lim_{x \to 7} f(x)$

Example: $\lim_{x\to 0} \sin\dfrac{\pi}{x}$



Let $f(x) = \sin\frac{\pi}{x}$. Use the graph of f to find

(a) $f(0)$ (b) $\lim_{x\to 0^-} f(x)$ (c) $\lim_{x\to 0^+} f(x)$ (d) $\lim_{x\to 0} f(x)$

1.3 Theorem

An Important Theorem

Theorem 1.

$$\lim_{x\to c} f(x) = L$$

if and only if both

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L$$

2 Section 2.2 Definition of Limit

2.1 Definition of Limit

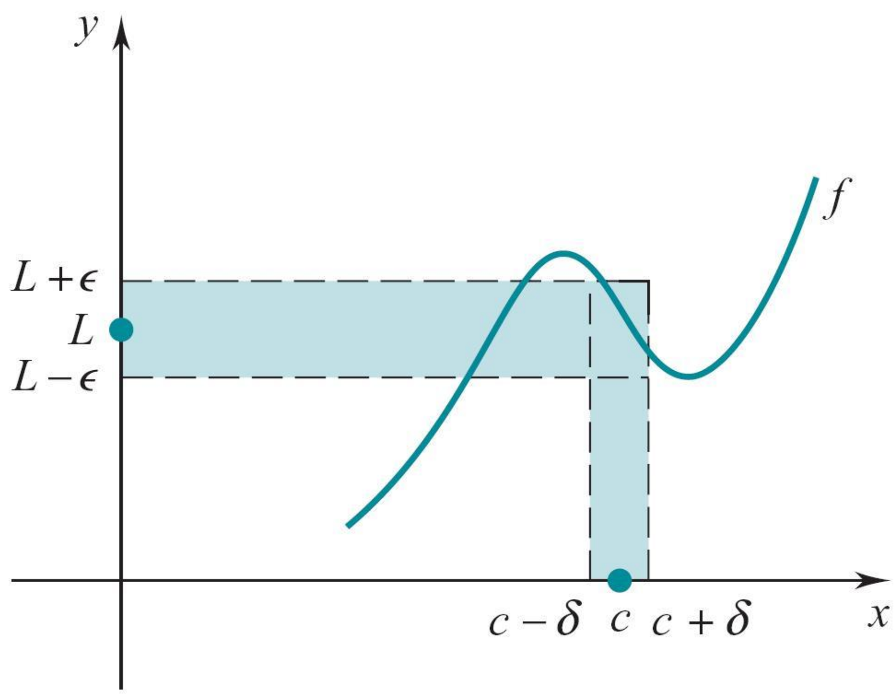
Definition of Limit: ϵ, δ statement

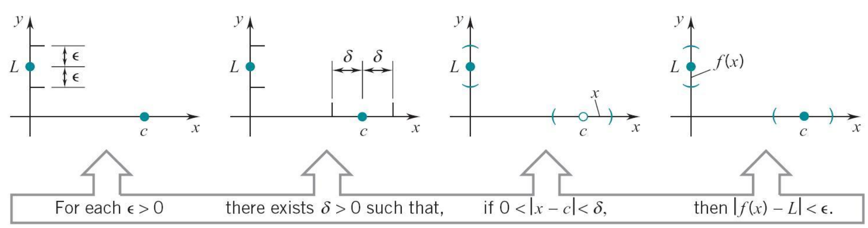
We say that

$$\lim_{x\to c} f(x) = L$$

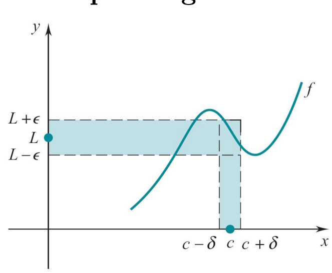
if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

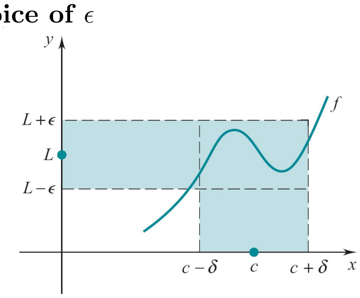
$$\text{if } 0 < |x - c| < \delta, \text{ then } |f(x) - L| < \epsilon$$

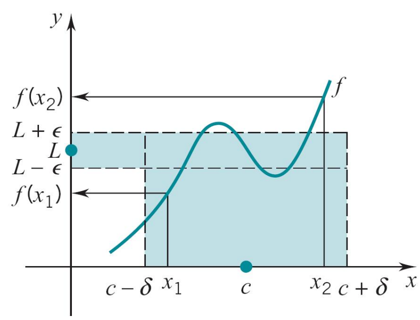




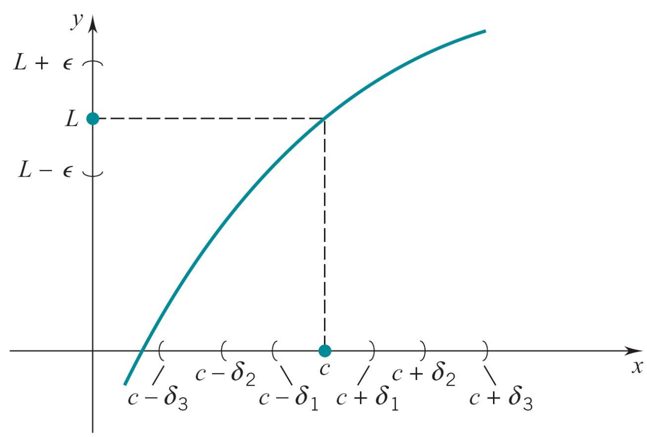
Choice of δ Depending on the Choice of ϵ







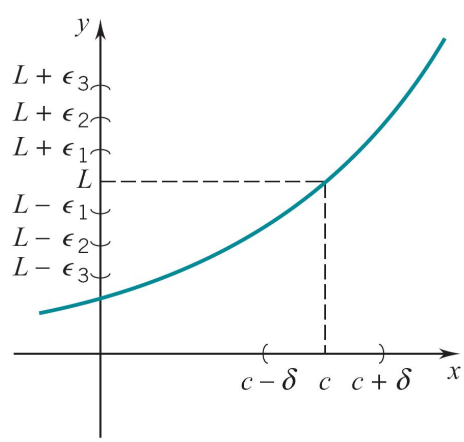
Example



Which of the δ's "works" for the given ϵ?

(a) δ_1 (b) δ_2 (c) δ_3

Example

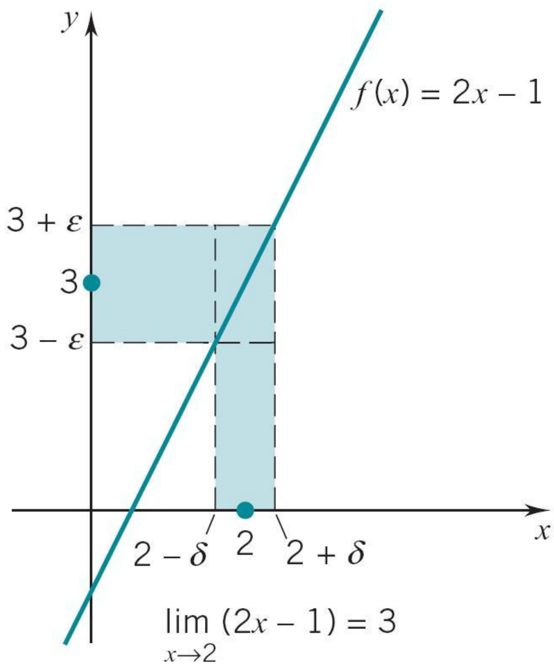


For which of the ϵ's given does the specified δ "work"?

$$\text{(a) } \epsilon_1) \qquad \text{(b) } \epsilon_2 \qquad \text{(c) } \epsilon_3$$

2.2 Examples

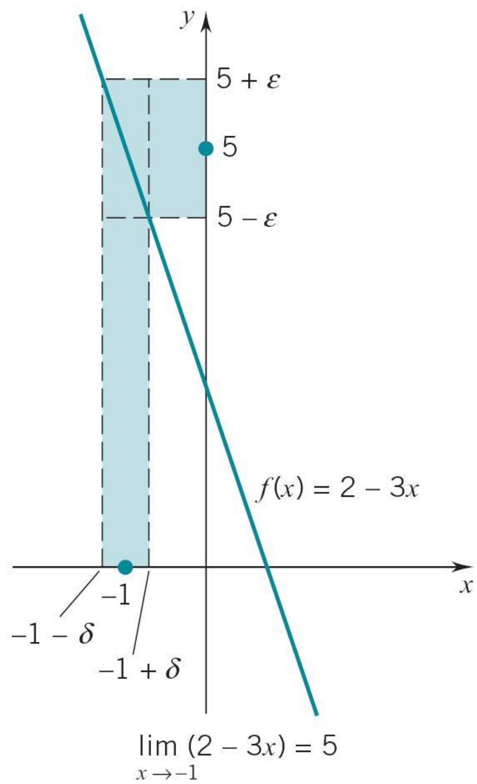
Example: $\lim_{x \to 2}(2x-1) = 3$
Show that $\lim_{x \to 2}(2x-1) = 3$.



Let $\epsilon > 0$. We choose $\delta = \frac{1}{2}\epsilon$ such that

$$\text{if } \; 0 < |x-2| < \delta, \qquad \text{then } \; |(2x-1)-3| < \epsilon$$

Example: $\lim_{x \to -1}(2-3x) = 5$
Show that $\lim_{x \to -1}(2-3x) = 5$.

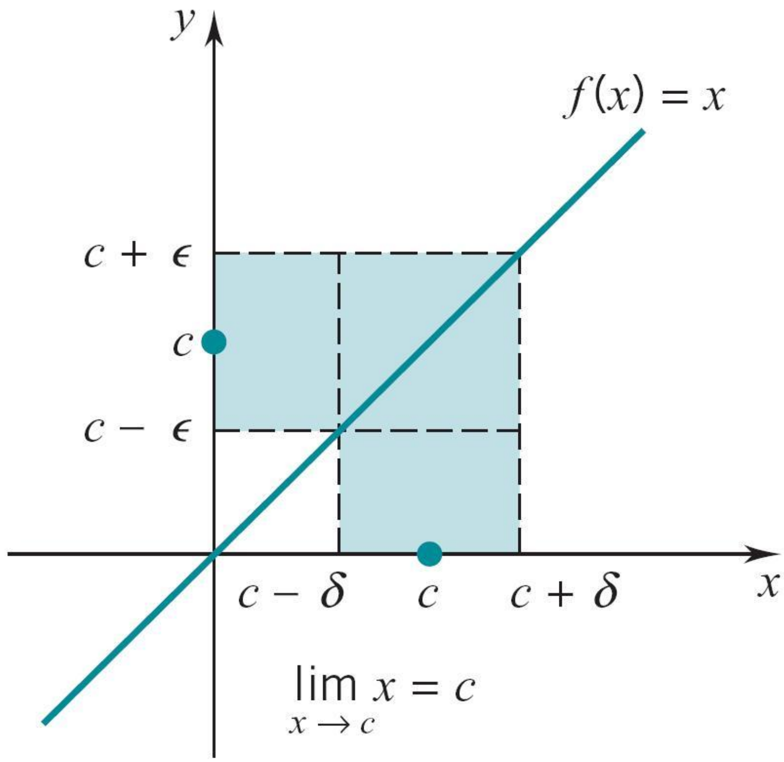


Let $\epsilon > 0$. We choose $\delta = \frac{1}{3}\epsilon$ such that

$$\text{if } \; 0 < |x - (-1)| < \delta, \qquad \text{then } \; |(2 - 3x) - 5| < \epsilon$$

2.3 Four Basic Limits

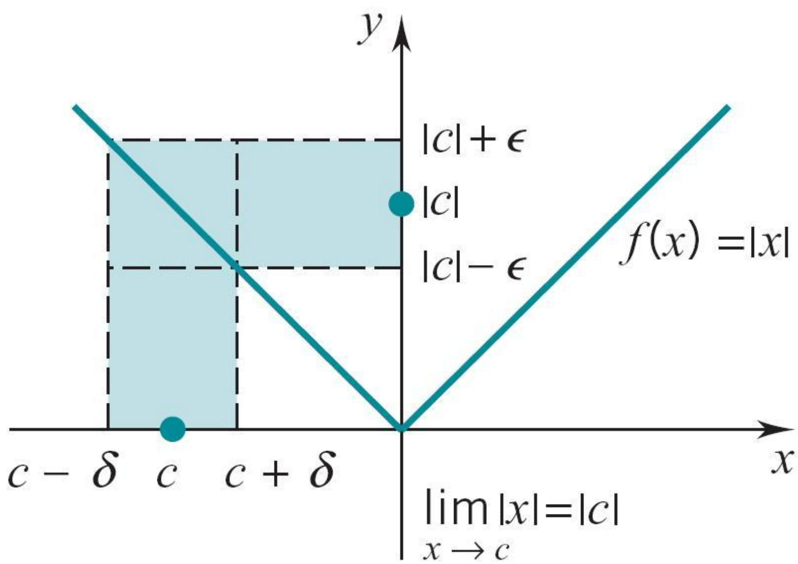
Basic Limit $\lim x = c$
Show that $\lim_{x \to c} x = c$.



Let $\epsilon > 0$. We choose $\delta = \epsilon$ such that

$$\text{if } \; 0 < |x - c| < \delta, \qquad \text{then } \; |x - c| < \epsilon$$

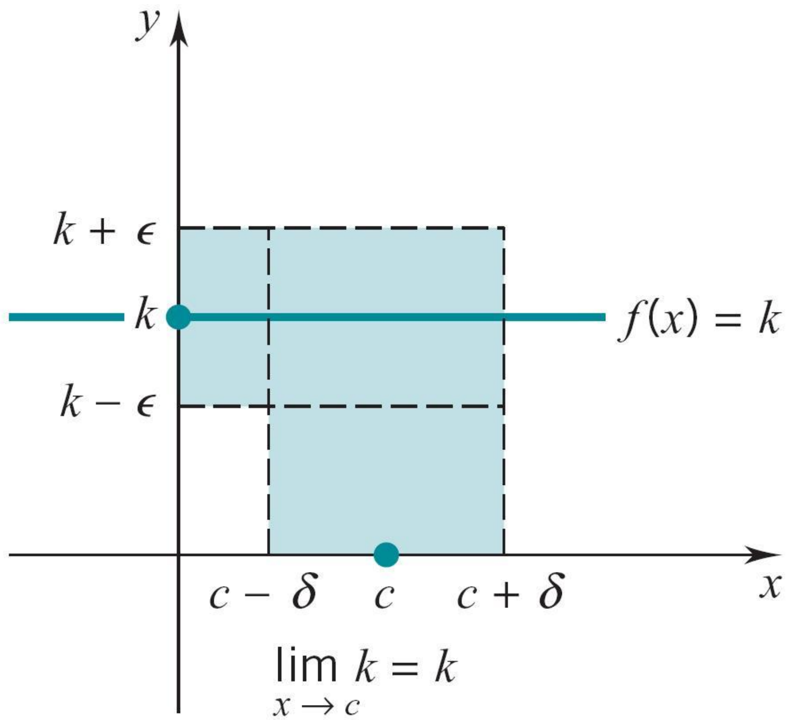
Basic Limit $\lim_{x \to c} |x| = |c|$
Show that $\lim_{x \to c} |x| = |c|$.



Let $\epsilon > 0$. We choose $\delta = \epsilon$ such that

$$\text{if } \ 0 < |x - c| < \delta, \qquad \text{then } \ ||x| - |c|| < \epsilon$$

Basic Limit $\lim_{x \to c} k = k$
Show that $\lim_{x \to c} k = k$.

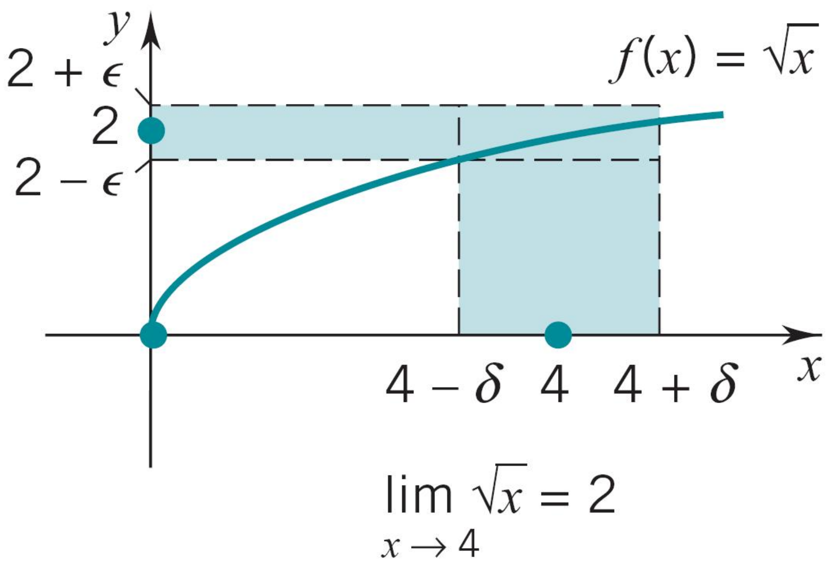


Let $\epsilon > 0$. We can choose any number $\delta > 0$ such that

$$\text{if } \ 0 < |x - c| < \delta, \qquad \text{then } \ |k - k| < \epsilon$$

Basic Limit $\lim_{x \to c} \sqrt{x} = \sqrt{c}$
Show that for $c > 0$,

$$\lim_{x \to c} \sqrt{x} = \sqrt{c}$$



Let $\epsilon > 0$. We choose $\delta = \min\{c, \epsilon\}$ such that

$$\text{if } \; 0 < |x - c| < \delta, \qquad \text{then } \; |\sqrt{x} - \sqrt{c}| < \epsilon$$

3 Section 2.3 Some Limit Theorems

3.1 Some Limit Theorems

Some Limit Theorems - Sum and Product

Theorem 2. *If* $\lim_{x \to c} f(x)$ *and* $\lim_{x \to c} g(x)$ *each exists, then*

1. $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
2. $\lim_{x \to c} (\alpha\, f(x)) = \alpha \, \lim_{x \to c} f(x)$
3. $\lim_{x \to c} (f(x)\, g(x)) = \lim_{x \to c} f(x) \, \lim_{x \to c} g(x)$

Some Limit Theorems - Quotient

Theorem 3. *If* $\lim_{x \to c} f(x)$ *and* $\lim_{x \to c} g(x)$ *each exists, then*

1. *if* $\lim_{x \to c} g(x) \neq 0$, *then* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$

2. *if* $\lim_{x\to c} g(x) = 0$ *while* $\lim_{x\to c} f(x) \neq 0$, *then* $\lim_{x\to c} \frac{f(x)}{g(x)}$ *does not exist.*

3. *if* $\lim_{x\to c} g(x) = 0$ *and* $\lim_{x\to c} f(x) = 0$, *then* $\lim_{x\to c} \frac{f(x)}{g(x)}$ *may or may not exist.*

Limit of a Polynomial

Theorem 4. *Let* $P(x) = a_n x^n + \cdots + a_1 x + a_0$ *be a polynomial and* c *be any number. Then*

$$\lim_{x\to c} P(x) = P(c)$$

Examples 5.

$$\lim_{x\to -1}\left(2x^3 + x^2 - 2x - 3\right) = 2(-1)^3 + (-1)^2 - 2(-1) - 3 = -2$$

$$\lim_{x\to 0}\left(14x^5 - 7x^2 + 2x + 8\right) = 14(0)^5 - 7(0)^2 + 2(0) + 8 = 8$$

Limit of a Rational Function

Theorem 6. *Let* $R(x) = \frac{P(x)}{Q(x)}$ *be a rational function (a quotient of two polynomials) and let* c *be a number. Then*

1. *if* $Q(c) \neq 0$, *then* $\lim_{x\to c} R(x) = \lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)} = R(c)$

2. *if* $Q(c) = 0$ *while* $P(c) \neq 0$, *then* $\lim_{x\to c} R(x) = \lim_{x\to c} \frac{P(x)}{Q(x)}$ *does not exist.*

3. *if* $Q(c) = 0$ *while* $P(c) = 0$, *then* $\lim_{x\to c} R(x) = \lim_{x\to c} \frac{P(x)}{Q(x)}$ *may or may not exist.*

Examples

Examples 7.

$$\lim_{x\to 3} \frac{x^3 - 3x^2}{1 - x^2} = \frac{(3)^3 - 3(3)^2}{1 - (3)^2} = \frac{27 - 27}{1 - 9} = 0.$$

$$\lim_{x\to 1} \frac{x^2}{x-1} \text{ does not exist, } \lim_{x\to 1^-} \frac{x^2}{x-1} = -\infty, \lim_{x\to 1^+} \frac{x^2}{x-1} = \infty,$$

$$\lim_{x\to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3} = \lim_{x\to 3}(x+2) = (3) + 2 = 5$$

$$\lim_{x\to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$$