Lecture 3

Section 2.4 Continuity

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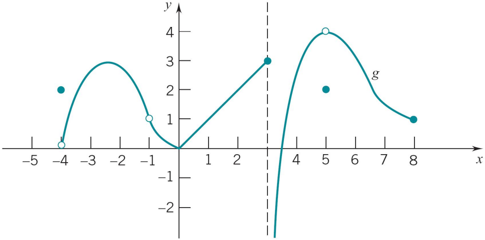


Homework and Quizzes

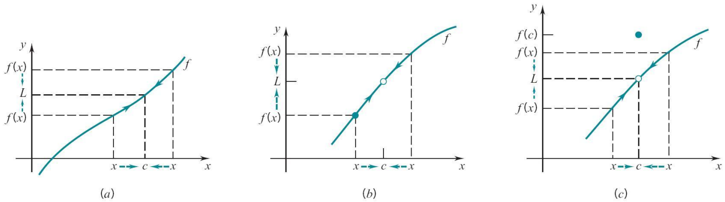
Homework 1 & 2

- Homework 1 is due September 4th in lab.
- Homework 2 is due September 9th in lab.

Quizzes 1 & 2

- Quizzes 1 and 2 are available on CourseWare!

The Ideal of Limits



Theorem

$$\lim_{x \longrightarrow c} f(x) = L$$

if and only if both

$$\lim_{x \longrightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \longrightarrow c^+} f(x) = L$$

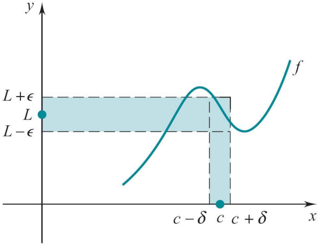
Definition of Limit: ϵ, δ statement

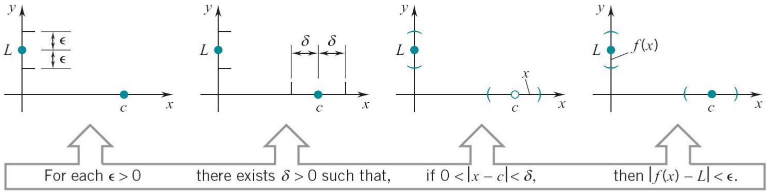
We say that

$$\lim_{x \to c} f(x) = L$$

if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

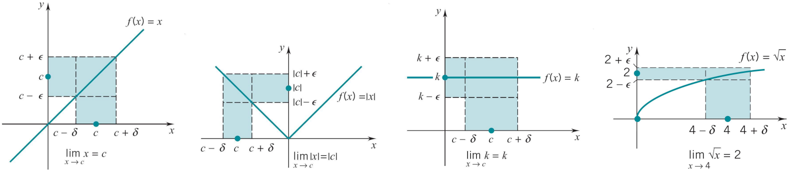
if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$





Four Basic Limits

1. $\lim_{x \to c} x = c.$
2. $\lim_{x \to c} |x| = |c|.$
3. $\lim_{x \to c} k = k.$
4. $\lim_{x \to c} \sqrt{x} = \sqrt{c}$, for $c > 0$.



Limits of Sums and Products - Polynomials

Theorem

If $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *each exists, then*

1. $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. $\lim_{x\to c}(\alpha f(x)) = \alpha \lim_{x\to c} f(x)$
3. $\lim_{x\to c}(f(x)\,g(x)) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$

Theorem

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ *be a polynomial and* c *be any number. Then*

$$\lim_{x\to c} P(x) = P(c)$$

Limits of Quotients

Theorem

If $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *each exists, then*

1. *if* $\lim_{x\to c} g(x) \neq 0$, *then* $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$
2. *if* $\lim_{x\to c} g(x) = 0$ *while* $\lim_{x\to c} f(x) \neq 0$, *then* $\lim_{x\to c} \frac{f(x)}{g(x)}$ *does not exist.*
3. *if* $\lim_{x\to c} g(x) = 0$ *and* $\lim_{x\to c} f(x) = 0$, *then* $\lim_{x\to c} \frac{f(x)}{g(x)}$ *may or may not exist.*

Limits of Rational Functions

Theorem

Let $R(x) = \dfrac{P(x)}{Q(x)}$ *be a rational function (a quotient of two polynomials) and let* c *be a number. Then*

1. *if* $Q(c) \neq 0$, *then* $\lim_{x \to c} R(x) = \lim_{x \to c} \dfrac{P(x)}{Q(x)} = \dfrac{P(c)}{Q(c)} = R(c)$
2. *if* $Q(c) = 0$ *while* $P(c) \neq 0$,
   then $\lim_{x \to c} R(x) = \lim_{x \to c} \dfrac{P(x)}{Q(x)}$ *does not exist.*
3. *if* $Q(c) = 0$ *while* $P(c) = 0$,
   then $\lim_{x \to c} R(x) = \lim_{x \to c} \dfrac{P(x)}{Q(x)}$ *may or may not exist.*

Definition of Continuity at a Point

Definition

Let f be a function defined on an open interval centered at c. We say that f is continuous at c if

$$\lim_{x \longrightarrow c} f(x) = f(c).$$

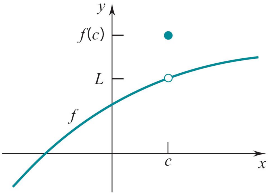
Remark

f is continuous at c if

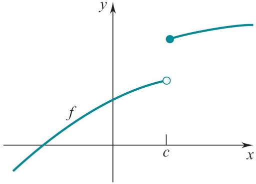
1. f is defined at c,
2. $\lim_{x \longrightarrow c} f(x)$ exists, and
3. $\lim_{x \longrightarrow c} f(x) = f(c)$.

Three Types of "Simple" Discontinuity

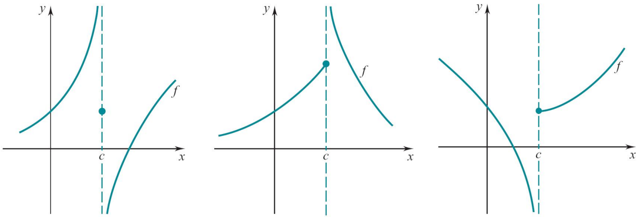
"Removable" Discontinuity



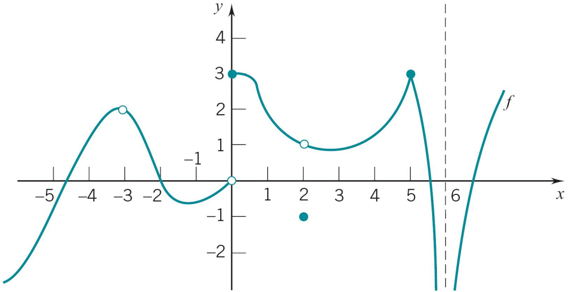
"Jump" Discontinuity



"Infinite" Discontinuity

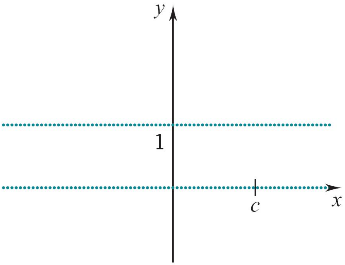


Example



At which points is f discontinuous? And what type of discontinuity does f have?

Dirichlet Function (Discontinuous Everywhere)



Let $f(x) = \begin{cases} 1, & x \text{ rational}, \\ 0, & x \text{ irrational}. \end{cases}$

At no point c does f have a limit, thus f is everywhere discontinuous.

Continuity Properties of Sums, Products and Quotients

Theorem

If f and g are continuous at c, then

1. $f + g$ *and* $f - g$ *are continuous at* c.
2. kf *is continuous at* c *for each real* k.
3. $f \cdot g$ *is continuous at* c.
4. f/g *is continuous at* c *provided* $g(c) \neq 0$.

Theorem

- *The absolute function* $|x|$ *is continuous everywhere.*
- *The square root function* $\sqrt{x}$ *is continuous at any positive number.*
- *Polynomials are continuous everywhere.*
- *Rational Functions are continuous everywhere they are defined.*

Continuity Properties of Compositions

Theorem

If g is continuous at c and f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

Examples

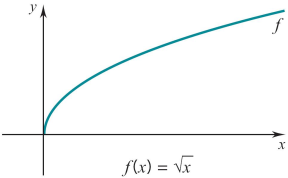
- $F(x) = \sqrt{\dfrac{x^2+1}{x-3}}$ is continuous wherever it is defined, i.e., at any number $c > 3$. Note that $F = f \circ g$ where $f(x) = \sqrt{x}$ and $g(x) = \frac{x^2+1}{x-3}$.
- $F(x) = \dfrac{1}{5 - \sqrt{x^2+16}}$ is continuous wherever it is defined, i.e., at any number $c \neq \pm 3$. Note that $F = f \circ g \circ h$ where $f(x) = \frac{1}{5-x}$, $g(x) = \sqrt{x}$ and $h(x) = x^2 + 16$.

One Sided Continuity

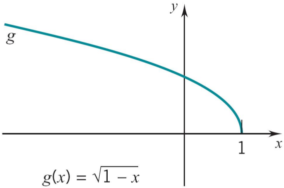
Definition

- f is left continuous at c if $\lim_{x \to c^-} f(x) = f(c)$.
- f is right continuous at c if $\lim_{x \to c^+} f(x) = f(c)$.

$f(x) = \sqrt{x}$ is right-continuous at 0.



$f(x) = \sqrt{1-x}$ is left-continuous at 1.

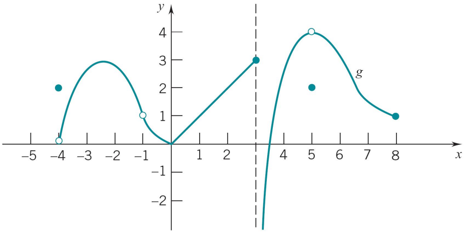


Continuity on Intervals

Definition

Let I be an interval of form: (a,b), $[a,b]$, $[a,b)$, $(a,b]$, (a,∞), $[a,\infty)$, $(-\infty,b)$, $(-\infty,b]$, or $(-\infty,\infty)$. The f is said to be continuous on I if for every number c in I,

- f is continuous at c if c is not an endpoint of I,
- f is left continuous at c if c is a right-endpoint of I,
- f is right continuous at c if c is a left-endpoint of I.



continuous on $(-4,1)\cup(-1,3]\cup(-3,5)\cup(5,8]$.