Lecture 4

Section 2.5 The Pinching Theorem
Section 2.6 Two Basic Properties of Continuity

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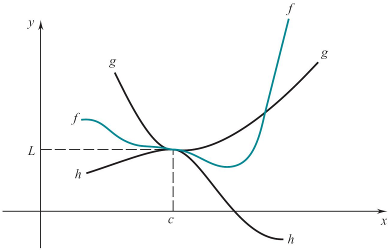


Homework and Quizzes

Homework 1 & 2

- Homework 1 is due September 4th in lab.
- Homework 2 is due September 9th in lab.

Quizzes 1, 2 & 3

- Quizzes 1, 2 and 3 are available on CourseWare!
- Quizzes 1 and 2 are due on this Friday!

Daily Grades

- Daily grades will be given in lecture beginning next Tuesday.
- The daily grades form is posted on the course homepage. You must print out this form and BRING it to class every day.
- Questions will be asked in lecture at random times.
- You will mark your answers on the daily grades form and drop the form in a box at the end of class.

Weekly Written Quizzes in Lab

- Quizzes will be given every week on Thursday in lab beginning the next week.
- The weekly written quizzes form is posted on the course homepage. You must print out this form and BRING it to class every Thursday.

Definition of Continuity at a Point

Definition

Let f be a function defined on an open interval centered at c. We say that f is continuous at c if

$$\lim_{x \to c} f(x) = f(c).$$

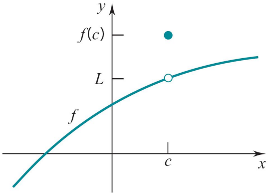
Remark

f is continuous at c if

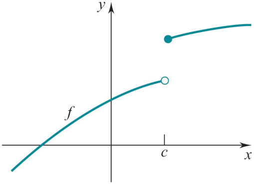
1. f is defined at c,
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

Three Types of "Simple" Discontinuity

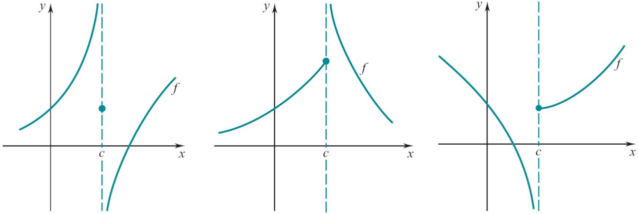
"Removable" Discontinuity



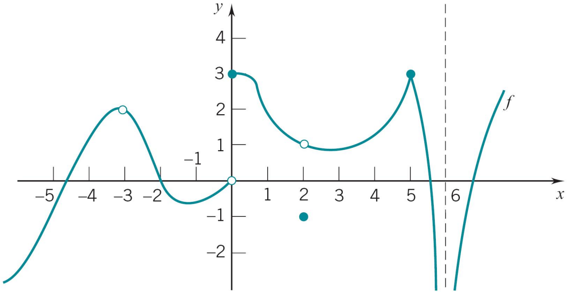
"Jump" Discontinuity



"Infinite" Discontinuity



Example



At which points is f discontinuous? And what type of discontinuity does f have?

Continuity Properties of Elementary Functions

Theorem

- *The absolute function* $f(x) = |x|$ *is continuous everywhere.*
- *The square root function* $f(x) = \sqrt{x}$ *is continuous at any positive number.*
- *Polynomials are continuous everywhere.*
- *Rational Functions are continuous everywhere defined.*

Continuity Properties of Sums, Products and Quotients

Theorem

If f and g are continuous at c, then

1. $f + g$ *and* $f - g$ *are continuous at* c.
2. kf *is continuous at* c *for each real* k.
3. $f \cdot g$ *is continuous at* c.
4. f/g *is continuous at* c *provided* $g(c) \neq 0$.

Continuity Properties of Compositions

Theorem

If g is continuous at c and f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

Examples

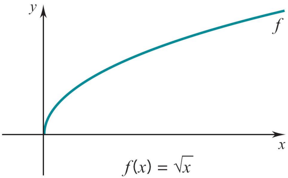
- $F(x) = \sqrt{\dfrac{x^2+1}{x-3}}$ is continuous wherever it is defined, i.e., at any number $c > 3$. Note that $F = f \circ g$ where $f(x) = \sqrt{x}$ and $g(x) = \frac{x^2+1}{x-3}$.

One Sided Continuity

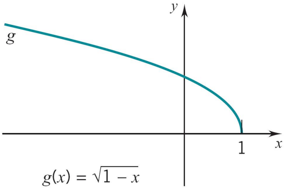
Definition

- f is left continuous at c if $\lim_{x \to c^-} f(x) = f(c)$.
- f is right continuous at c if $\lim_{x \to c^+} f(x) = f(c)$.

$f(x) = \sqrt{x}$ is right-continuous at 0.



$f(x) = \sqrt{1-x}$ is left-continuous at 1.

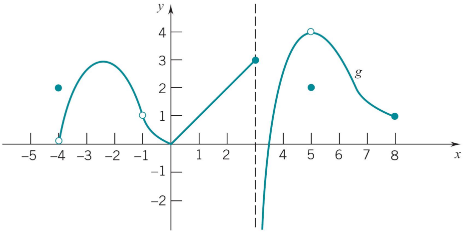


Continuity on Intervals

Definition

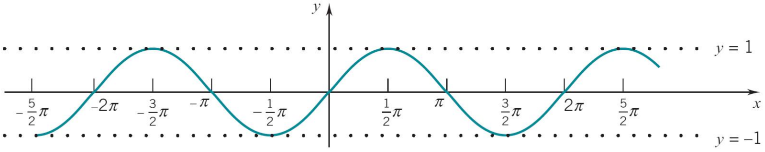
Let I be an interval of form: (a, b), $[a, b]$, $[a, b)$, $(a, b]$, (a, ∞), $[a, \infty)$, $(-\infty, b)$, $(-\infty, b]$, or $(-\infty, \infty)$. The f is said to be continuous on I if for every number c in I,

- f is continuous at c if c is not an endpoint of I,
- f is left continuous at c if c is a right-endpoint of I,
- f is right continuous at c if c is a left-endpoint of I.

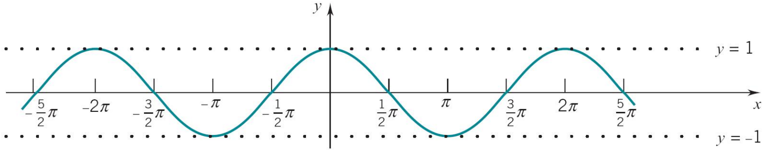


continuous on $(-4, 1) \cup (-1, 3] \cup (-3, 5) \cup (5, 8]$.

Trigonometric Functions: Sine and Cosine



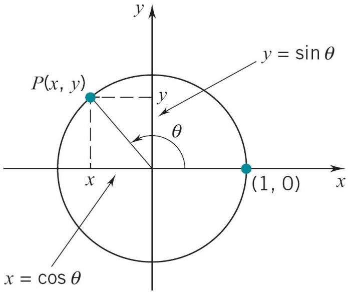
$y = \sin x$
period 2π



$y = \cos x$
period 2π

sin and cos are continuous everywhere.

Sine and Cosine: Important Identities



Unit Circle

$$\sin^2\theta + \cos^2\theta = 1.$$

Odd/Even Function

$$\sin(-\theta) = -\sin\theta,$$
$$\cos(-\theta) = \cos\theta.$$

Addition Formula

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Other Trigonometric Functions: Identities

Continuity

The remaining trigonometric functions

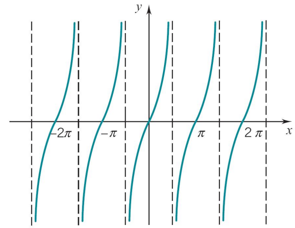
$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

are all continuous where defined.

Important Identities

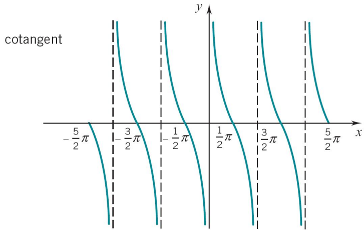
$$\tan^2 x + 1 = \sec^2 x, \quad \cot^2 x + 1 = \csc^2 x.$$

Other Trigonometric Functions: Graphs



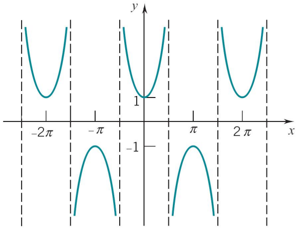
$y = \tan x$
period π

vertical asymptotes $x = (n + \frac{1}{2})\pi$, n an integer

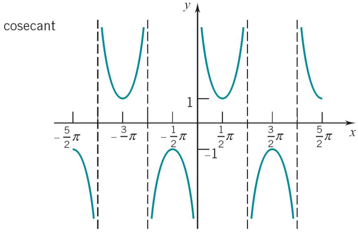


$y = \cot x$
period π

vertical asymptotes $x = n\pi$, n an integer

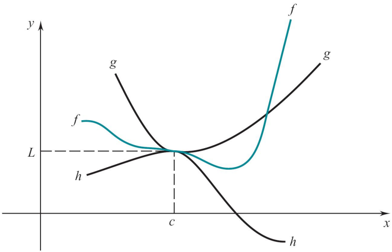


$y = \sec x$
period π



$y = \csc x$
period π

The Pinching Theorem



Theorem

Let $p > 0$. Suppose that, for all x such that $0 < |x - c| < p$

$$h(x) \leq f(x) \leq g(x).$$

If

$$\lim_{x \to c} h(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = L.$$

then

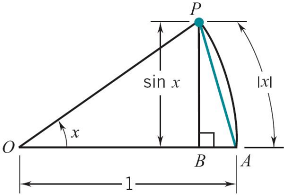
$$\lim_{x \to c} f(x) = L.$$

The Pinching Theorem: Continuity of Sine and Cosine

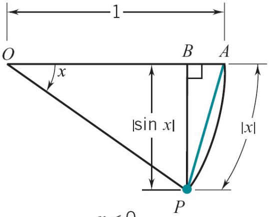
Theorem

$$\lim_{x\to 0} \sin x = 0, \quad \lim_{x\to 0} \cos x = 1,$$
$$\lim_{x\to c} \sin x = \sin c, \quad \lim_{x\to c} \cos x = \cos c.$$

$$0 < |\sin x| < |x|.$$



$x > 0$

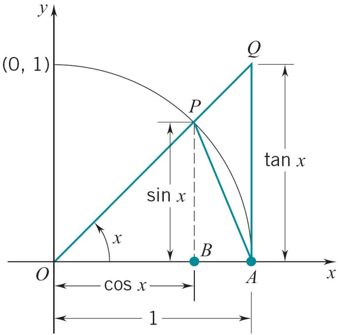


$x < 0$

The Pinching Theorem: Trigonometric Limits

Theorem

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$



Proof.

Use Geometric argument to get

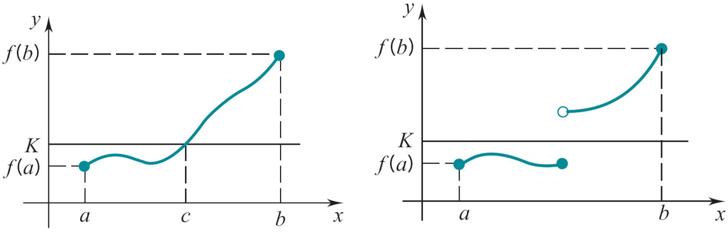
$$\cos x < \frac{\sin x}{x} < 1,$$

then apply the pinching theorem.

The Intermediate-Value Theorem

Theorem

If f is continuous on $[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there is at least one number c in the interval (a, b) such that $f(c) = K$.



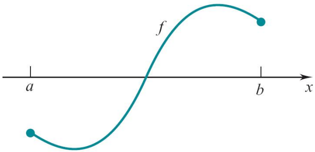
The Intermediate-Value Theorem: Roots of Equation

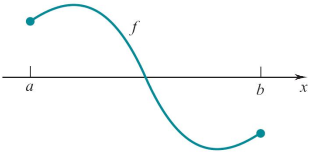
Theorem

If f is continuous on $[a, b]$ and

$$f(a) < 0 < f(b), \quad \text{or} \quad f(b) < 0 < f(a),$$

then the equation $f(x) = 0$ has at least a root in (a, b).

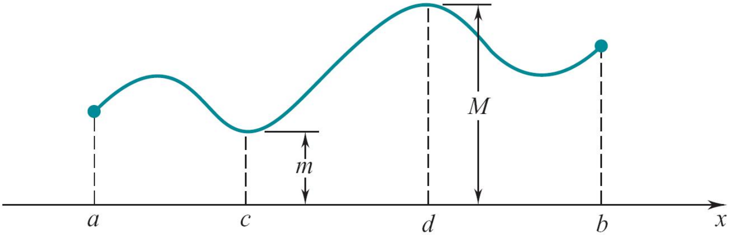




The Extreme-Value Theorem

Theorem

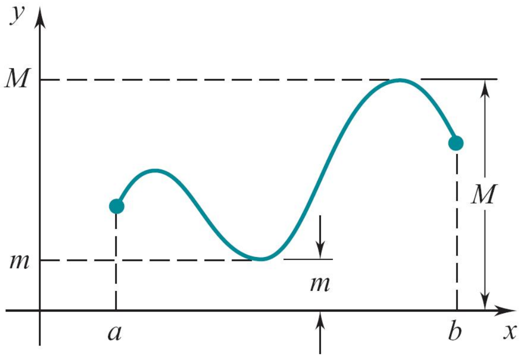
A function f continuous on a bounded closed $[a, b]$ takes on both a maximum value M and a minimum value m.



The Extreme-Value Theorem: Bounded Closed Intervals

Theorem

Continuous functions map bounded closed intervals $[a, b]$ onto bounded closed intervals $[m, M]$.



$f: [a, b] \to [m, M]$