Lecture 4 Section 2.5 The Pinching Theorem Section 2.6 Two Basic Properties of Continuity

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1 Review

1.1 Continuity

Homework and Quizzes

Homework 1 & 2

- Homework 1 is due September 4th in lab.
- Homework 2 is due September 9th in lab.

Quizzes 1, 2 & 3

- Quizzes 1, 2 and 3 are available on CourseWare!
- Quizzes 1 and 2 are due on this Friday!

Daily Grades

- Daily grades will be given in lecture beginning next Tuesday.
- The daily grades form is posted on the course homepage. You must print out this form and *BRING it to class every day.*
- Questions will be asked in lecture at random times.
- You will mark your answers on the daily grades form and drop the form in a box at the end of class.

Weekly Written Quizzes in Lab

- Quizzes will be given every week on Thursday in lab beginning the next week.
- The weekly written quizzes form is posted on the course homepage. You must print out this form and *BRING it to class every Thursday.*

Definition of Continuity at a Point

Definition 1. Let f be a function defined on an open interval centered at c. We say that f is continuous at c if

$$\lim_{x \to c} f(x) = f(c).$$

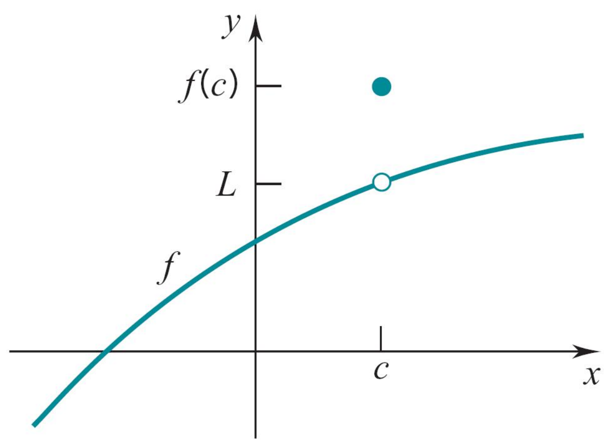
Remark

f is continuous at c if

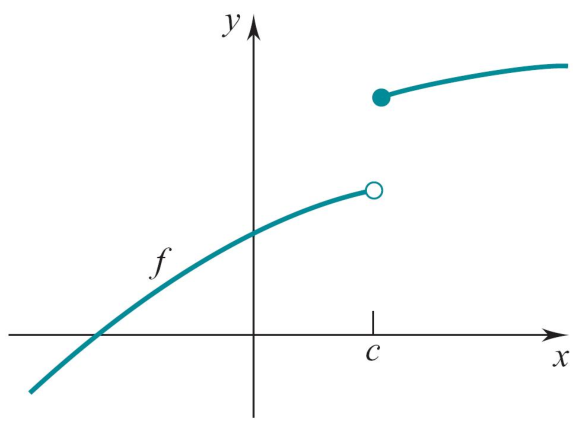
1. f is defined at c,
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

Three Types of "Simple" Discontinuity

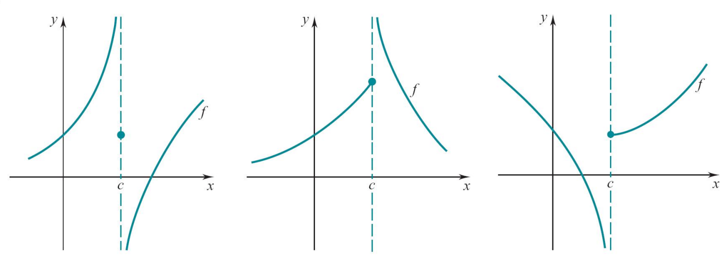
"Removable" Discontinuity



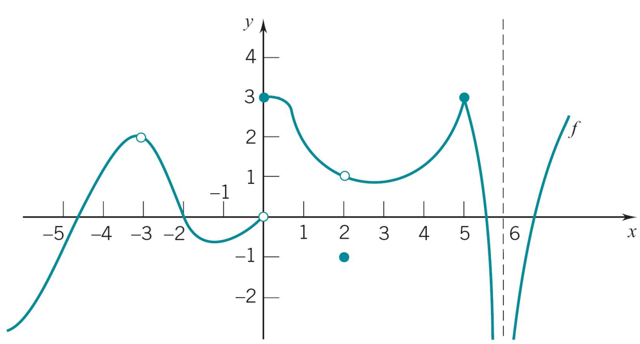
"Jump" Discontinuity



“Infinite” Discontinuity



Example



At which points is f discontinuous? And what type of discontinuity does f have?

Continuity Properties of Elementary Functions

Theorem 2. • *The absolute function $f(x) = |x|$ is continuous everywhere.*

- *The square root function* $f(x) = \sqrt{x}$ *is continuous at any positive number.*
- *Polynomials are continuous everywhere.*
- *Rational Functions are continuous everywhere defined.*

Continuity Properties of Sums, Products and Quotients

Theorem 3. *If* f *and* g *are continuous at* c*, then*

1. $f + g$ *and* $f - g$ *are continuous at* c.
2. $k\,f$ *is continuous at* c *for each real* k.
3. $f \cdot g$ *is continuous at* c.
4. f/g *is continuous at* c *provided* $g(c) \neq 0$.

Continuity Properties of Compositions

Theorem 4. *If* g *is continuous at* c *and* f *is continuous at* $g(c)$*, then the composition* $f \circ g$ *is continuous at* c.

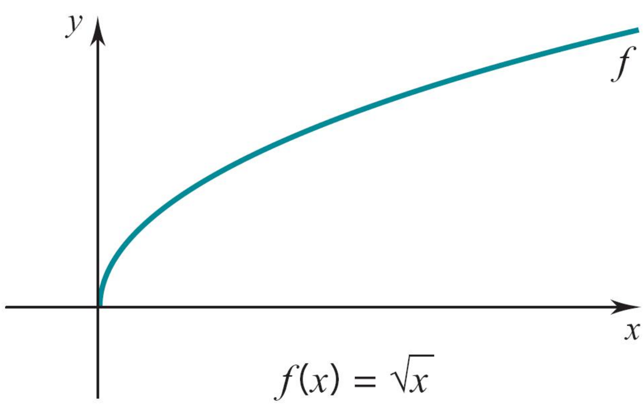
Examples 5. • $F(x) = \sqrt{\dfrac{x^2+1}{x-3}}$ is continuous wherever it is defined, i.e., at any number $c > 3$. Note that $F = f \circ g$ where $f(x) = \sqrt{x}$ and $g(x) = \frac{x^2+1}{x-3}$.

One Sided Continuity

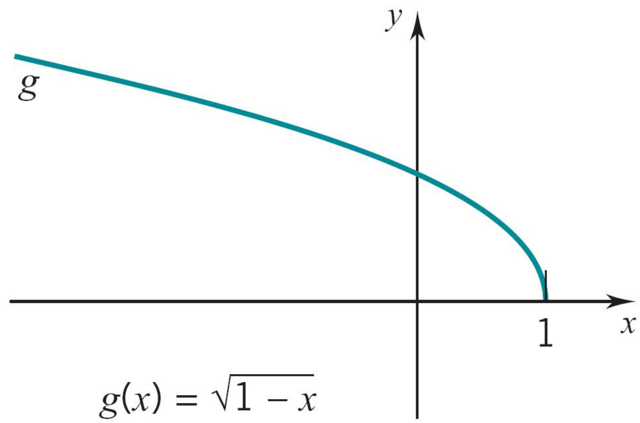
Definition 6. • f is left continuous at c if $\lim_{x \to c^-} f(x) = f(c)$.

- f is right continuous at c if $\lim_{x \to c^+} f(x) = f(c)$.

$f(x) = \sqrt{x}$ is right-continuous at 0.



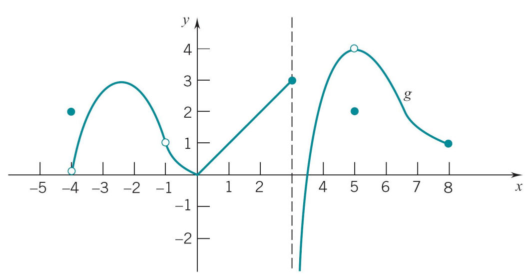
$f(x) = \sqrt{1-x}$ is left-continuous at 1.



Continuity on Intervals

Definition 7. Let I be an interval of form: (a,b), $[a,b]$, $[a,b)$, $(a,b]$, (a,∞), $[a,\infty)$, $(-\infty,b)$, $(-\infty,b]$, or $(-\infty,\infty)$. The f is said to be continuous on I if for every number c in I,

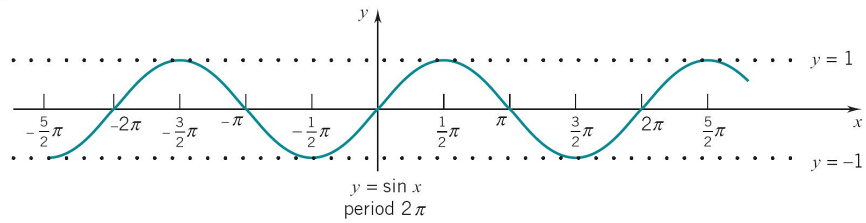
- f is continuous at c if c is not an endpoint of I,
- f is left continuous at c if c is a right-endpoint of I,
- f is right continuous at c if c is a left-endpoint of I.

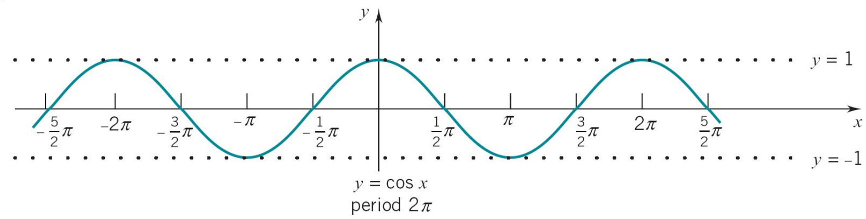


continuous on $(-4,1) \cup (-1,3] \cup (-3,5) \cup (5,8]$.

1.2 Trigonometric Functions

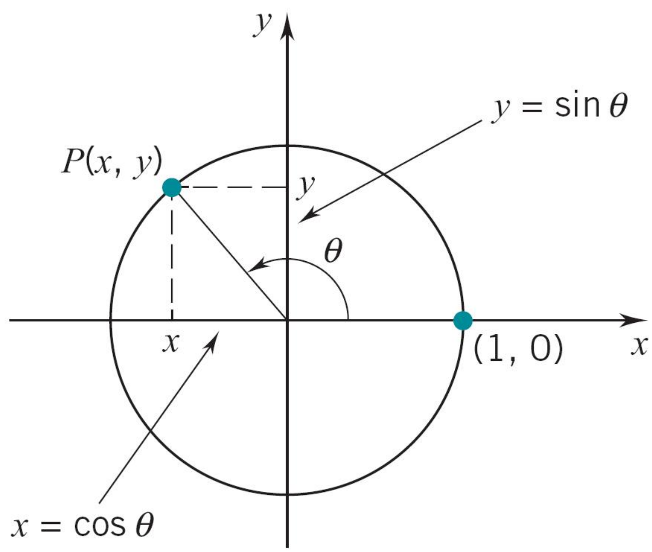
Trigonometric Functions: Sine and Cosine





sin and cos are continuous everywhere.

Sine and Cosine: Important Identities



Unit Circle

$$\sin^2\theta + \cos^2\theta = 1.$$

Odd/Even Function

$$\sin(-\theta) = -\sin\theta,$$
$$\cos(-\theta) = \cos\theta.$$

Addition Formula

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Other Trigonometric Functions: Identities

Continuity

The remaining trigonometric functions

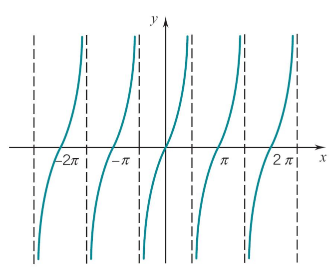
$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

are all continuous where defined.

Important Identities

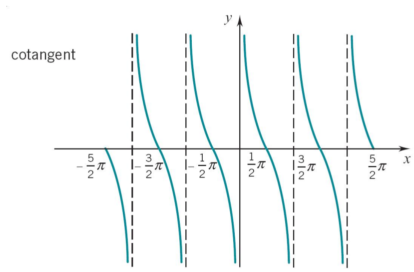
$$\tan^2 x + 1 = \sec^2 x, \quad \cot^2 x + 1 = \csc^2 x.$$

Other Trigonometric Functions: Graphs



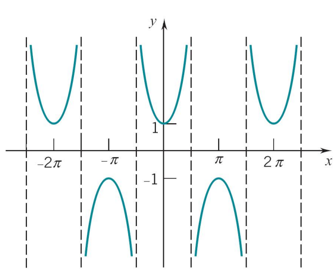
$y = \tan x$
period π

vertical asymptotes $x = (n + \frac{1}{2})\pi$, n an integer



$y = \cot x$
period π

vertical asymptotes $x = n\pi$, n an integer



$y = \sec x$
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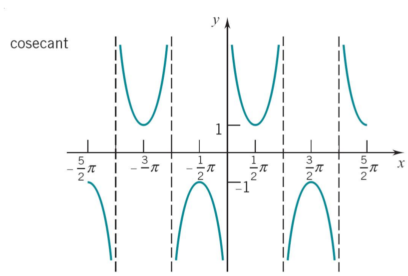


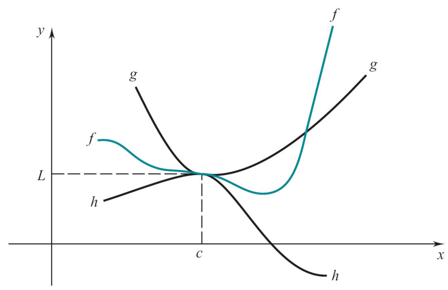
$y = \csc x$
period π

vertical asymptotes $x = n\pi$, n an integer

2 Section 2.5 The Pinching Theorem; Trigonometric Limits

2.1 The Pinching Theorem

The Pinching Theorem



Theorem 8. *Let $p > 0$. Suppose that, for all x such that $0 < |x - c| < p$*

$$h(x) \le f(x) \le g(x).$$

If

$$\lim_{x \to c} h(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = L.$$

then

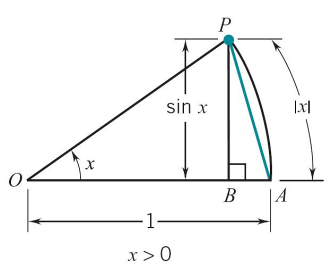
$$\lim_{x \to c} f(x) = L.$$

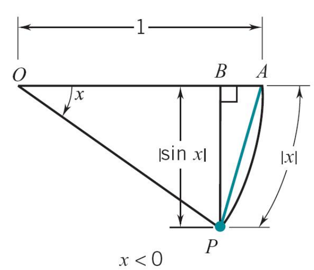
The Pinching Theorem: Continuity of Sine and Cosine

Theorem 9.

$$\lim_{x \to 0} \sin x = 0, \quad \lim_{x \to 0} \cos x = 1,$$
$$\lim_{x \to c} \sin x = \sin c, \quad \lim_{x \to c} \cos x = \cos c.$$

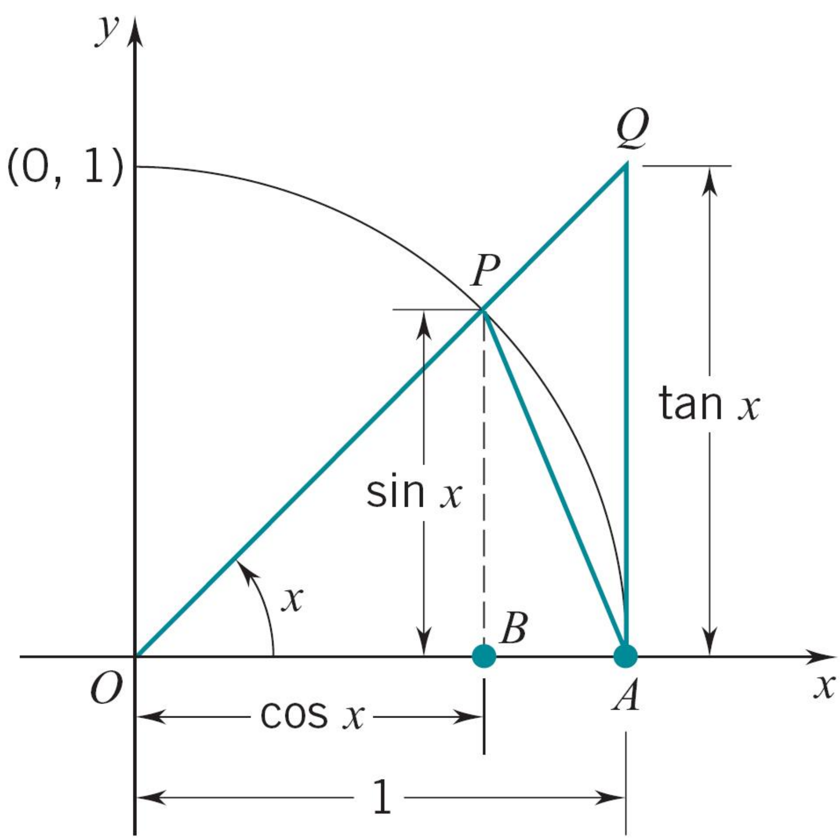
$$0 < |\sin x| < |x|.$$





The Pinching Theorem: Trigonometric Limits
Theorem 10.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$



Proof.
Use Geometric argument to get

$$\cos x < \frac{\sin x}{x} < 1,$$

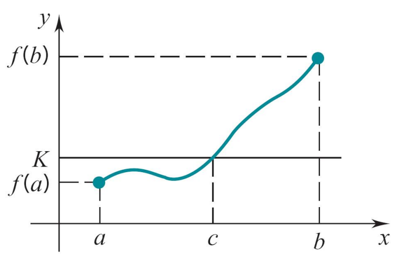
then apply the pinching theorem.

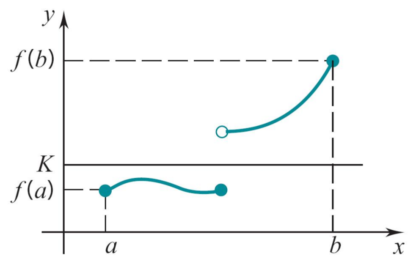
3 Section 2.6 Two Basic Properties of Continuous Functions

3.1 The Intermediate-Value Theorem

The Intermediate-Value Theorem

Theorem 11. *If f is continuous on $[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there is at least one number c in the interval (a, b) such that $f(c) = K$.*



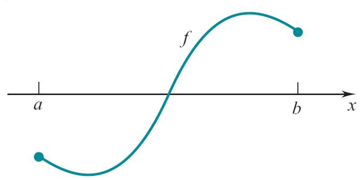


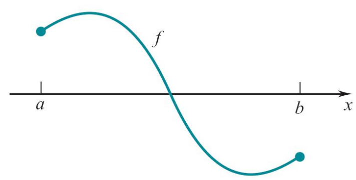
The Intermediate-Value Theorem: Roots of Equation

Theorem 12. *If f is continuous on $[a, b]$ and*

$$f(a) < 0 < f(b), \quad \textit{or} \quad f(b) < 0 < f(a),$$

then the equation $f(x) = 0$ has at least a root in (a, b).

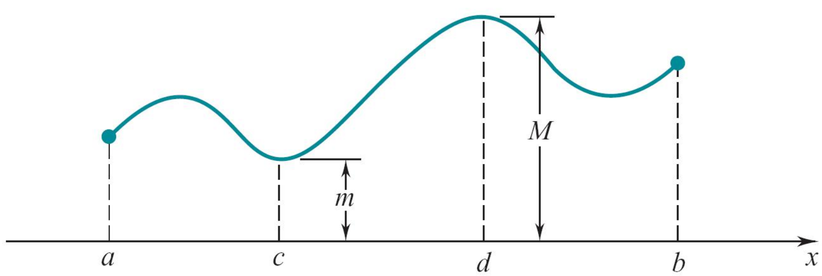




3.2 The Extreme-Value Theorem

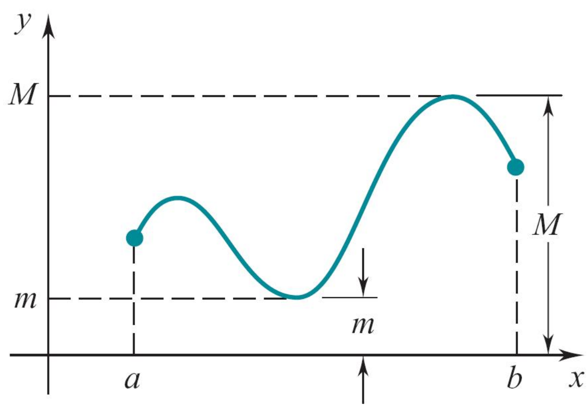
The Extreme-Value Theorem

Theorem 13. *A function f continuous on a bounded closed $[a, b]$ takes on both a maximum value M and a minimum value m.*



The Extreme-Value Theorem: Bounded Closed Intervals

Theorem 14. *Continuous functions map bounded closed intervals* $[a, b]$ *onto bounded closed intervals* $[m, M]$.



$f: [a, b] \to [m, M]$