# Lecture 5 Section 3.1 The Derivative

**Jiwen He**

## 1 Review

### 1.1 Info

**Homework and Quizzes**

**Homework 2 & 3**

- Homework 2 is due today in lab.
- Homework 3 is due September 16th in lab.

**Online Quizzes**

- Quizzes 1 and 2 have expired!
- Quiz 3 is posted and due on this Friday before 11:59 PM!
- Quiz 4 is posted and due 9/19!

**Weekly Written Quizzes in Lab**

- Quizzes will be given every week on Thursday in lab beginning THIS WEEK.
- The weekly written quizzes form is posted on the course homepage. You must print out this form and *BRING it to class every Thursday.*

**Daily Grades**

- Daily grades start Today.
- The daily grades form is posted on the course homepage. You must print out this form and *BRING it to class every day.*

**Quiz 1**

**Quiz 1**

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} =$$

a. $1$

b. $2$

c. $-1$

d. None of these

**Quiz 2**

**Quiz 2**

Classify the discontinuity at $x = 1$ for

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

a. Jump

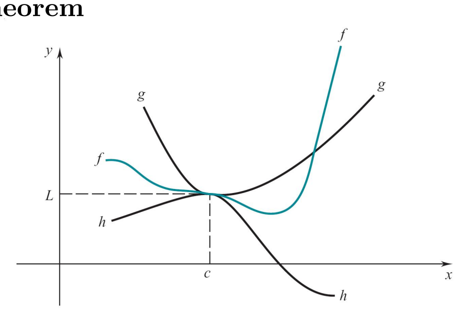
b. Infinite

c. Removable

d. None of these

## 1.2 The Pinching Theorem

**The Pinching Theorem**




**Theorem 1.** *Let $p > 0$. Suppose that, for all $x$ such that $0 < |x - c| < p$*

$$h(x) \le f(x) \le g(x).$$

*If*

$$\lim_{x \to c} h(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = L.$$

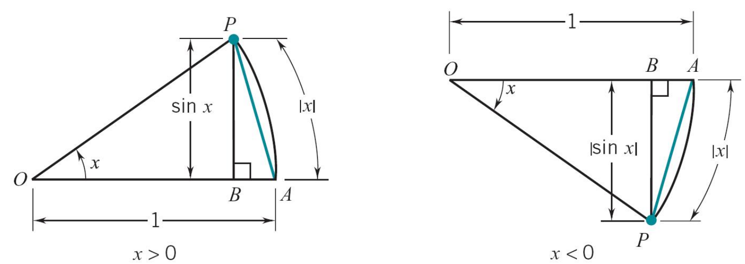
*then*

$$\lim_{x \to c} f(x) = L.$$

**The Pinching Theorem: Continuity of Sine and Cosine**
**Theorem 2.**

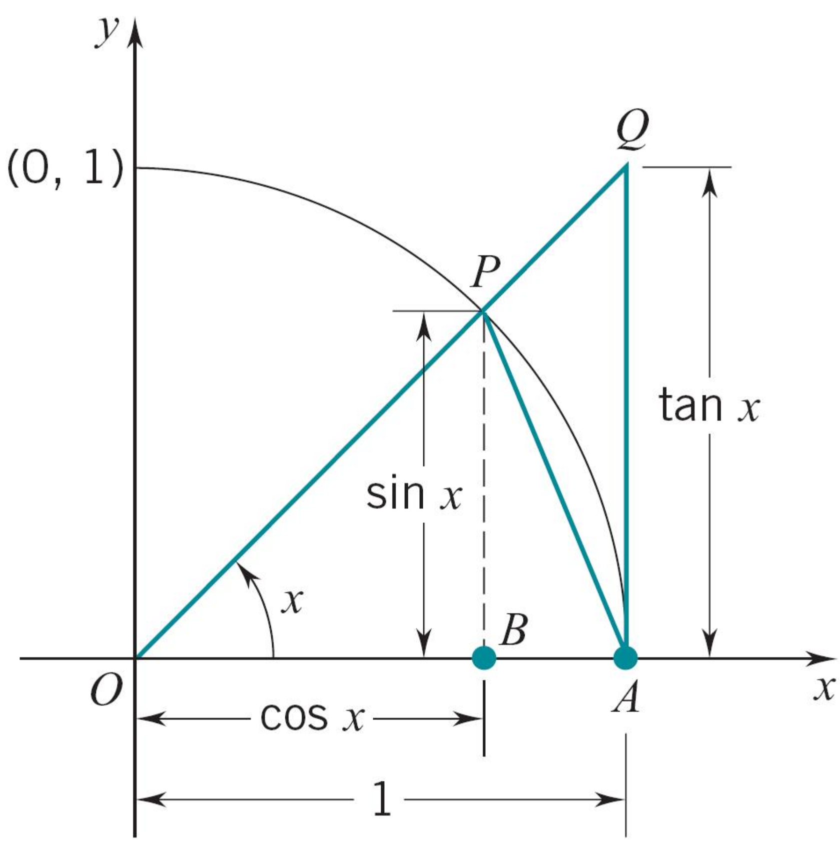
$$\lim_{x\to 0} \sin x = 0, \quad \lim_{x\to 0} \cos x = 1,$$
$$\lim_{x\to c} \sin x = \sin c, \quad \lim_{x\to c} \cos x = \cos c.$$

$$0 < |\sin x| < |x|.$$




**The Pinching Theorem: Trigonometric Limits**
**Theorem 3.**

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$




**Proof.**
Use Geometric argument to get

$$\cos x < \frac{\sin x}{x} < 1,$$

then apply the pinching theorem.

**More Trigonometric Limits**

**Theorem 4.**

$$\lim_{x\to 0} \frac{x}{\sin x} = 1 \quad \Rightarrow \quad \sin x \approx x \ \textit{for } x \textit{ near } 0.$$

**Theorem 5.**

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \quad \Rightarrow \quad \cos x \approx 1 - \frac{1}{2}x^2 \ \textit{for } x \textit{ near } 0.$$

**Theorem 6.** *For any number* $\alpha \neq 0$,

$$\lim_{x\to 0} \frac{\sin \alpha x}{\alpha x} = 1, \quad \lim_{x\to 0} \frac{1-\cos \alpha x}{\alpha x} = 0,$$
$$\lim_{x\to 0} \frac{\alpha x}{\sin \alpha x} = 1, \quad \lim_{x\to 0} \frac{1-\cos \alpha x}{(\alpha x)^2} = \frac{1}{2}.$$

**Quiz 3**

**Quiz 3**

What day is today?

a. Monday

b. Wednesday

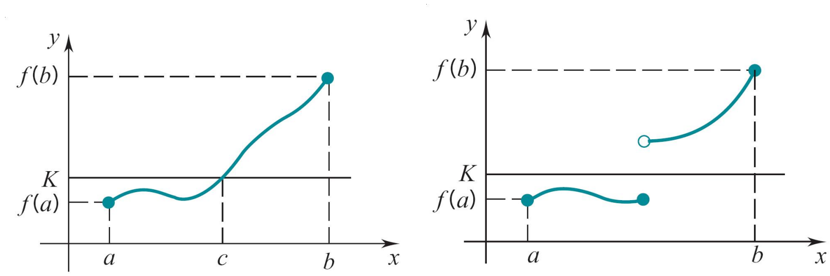
c. Friday

d. None of these

## 1.3 The Intermediate-Value Theorem

**The Intermediate-Value Theorem**

**Theorem 7.** *If $f$ is continuous on $[a, b]$ and $K$ is any number between $f(a)$ and $f(b)$, then there is at least one number $c$ in the interval $(a, b)$ such that $f(c) = K$.*

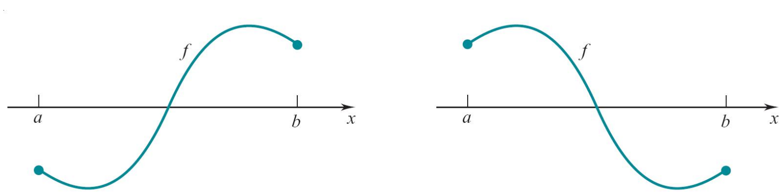



**The Intermediate-Value Theorem: Roots of Equation**

**Theorem 8.** *If $f$ is continuous on $[a, b]$ and*

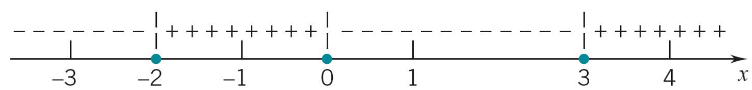
$$f(a) < 0 < f(b), \quad \text{or} \quad f(b) < 0 < f(a),$$

*then the equation $f(x) = 0$ has at least a root in $(a, b)$.*



**The Intermediate-Value Theorem: Solution of Inequality**
Solve the inequality

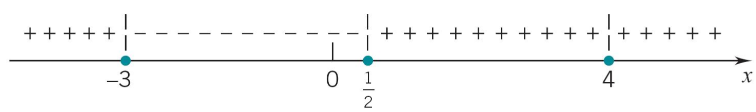
$$x^3 - x^2 - 6x > 0$$




Solution: $(-2, 0) \cup (3, \infty)$.

**Solution of Inequality**
Solve the inequality

$$(x+3)^3(2x-1)(x-4)^2 \leq 0$$

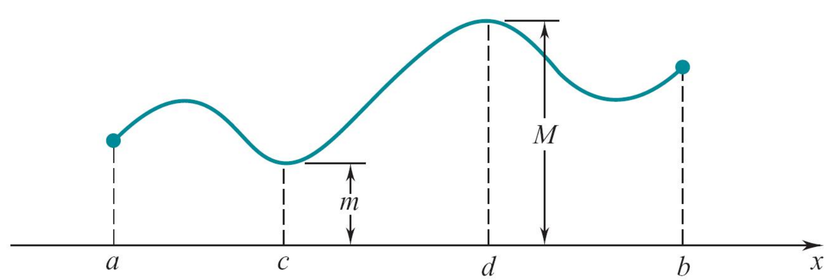



Solution: $[-3, 1/2]$.

## 1.4 The Extreme-Value Theorem

**The Extreme-Value Theorem**

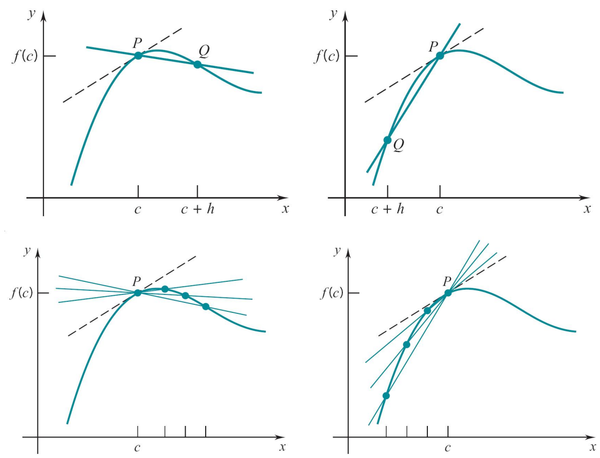
**Theorem 9.** *A function $f$ continuous on a bounded closed $[a, b]$ takes on both a maximum value $M$ and a minimum value $m$.*



# 2 Section 3.1 The Derivative

## 2.1 The Derivative

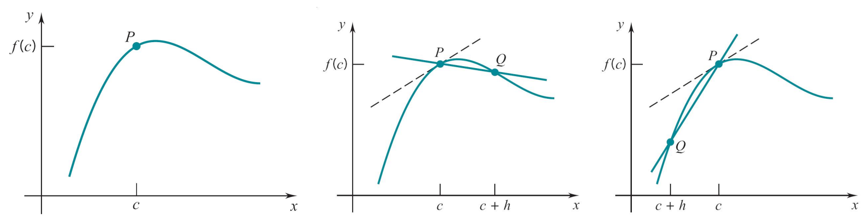
### Secant Lines vs. Tangent Lines




**Definition 10.** The slope of the graph at the point $(c, f(c))$ is given by

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}, \qquad \text{provided the limit exists}$$

### Derivative and Differentiation




**Definition 11.** A function $f$ is *differentiable* at $c$ if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \qquad \text{exists.}$$

If this limit exists, it is called the *derivative* of $f$ at $c$, and is denoted by $f'(c)$.

## 2.2 Derivative as Function

**Derivative as Function**

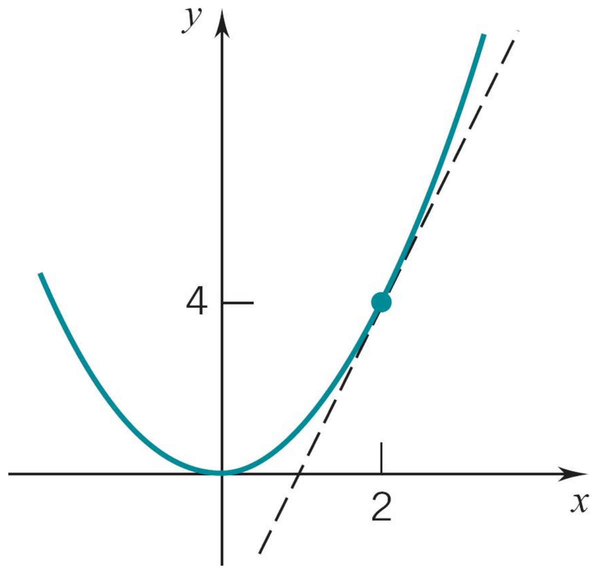
**Definition 12.** The *derivative* of a function $f$ is the function $f'$ with value at $x$ given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \qquad \text{provided the limit exists.}$$

To *differentiate* a function $f$ is to find its derivative.

*Examples* 13.

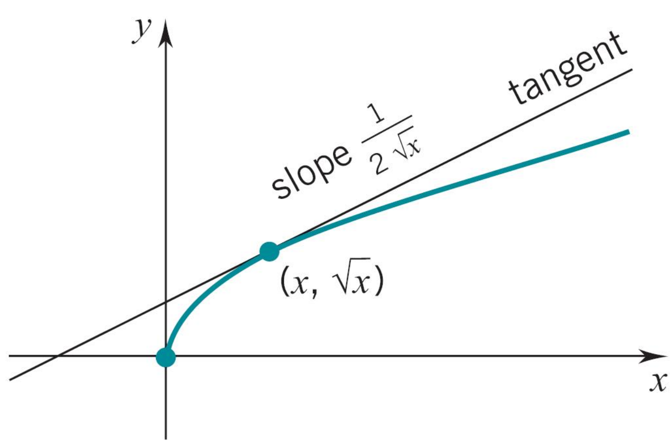
$$f(x) = x^2, \quad f'(x) = 2x.$$




**Derivative as Function: More Examples**

*Examples* 14.

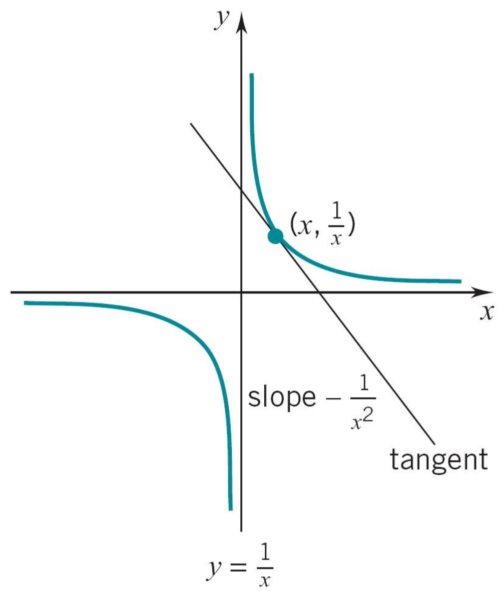
$$f(x) = \sqrt{x}, \quad f'(x) = \frac{1}{2\sqrt{x}}.$$




square root function

*Examples* 15.

$$f(x) = \frac{1}{x}, \quad f'(x) = -\frac{1}{x^2}.$$

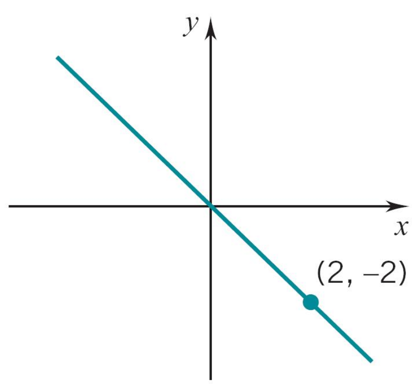



**Line Functions**

For a line function $f(x) = mx + b$, the derivative $f'(x) = m$.

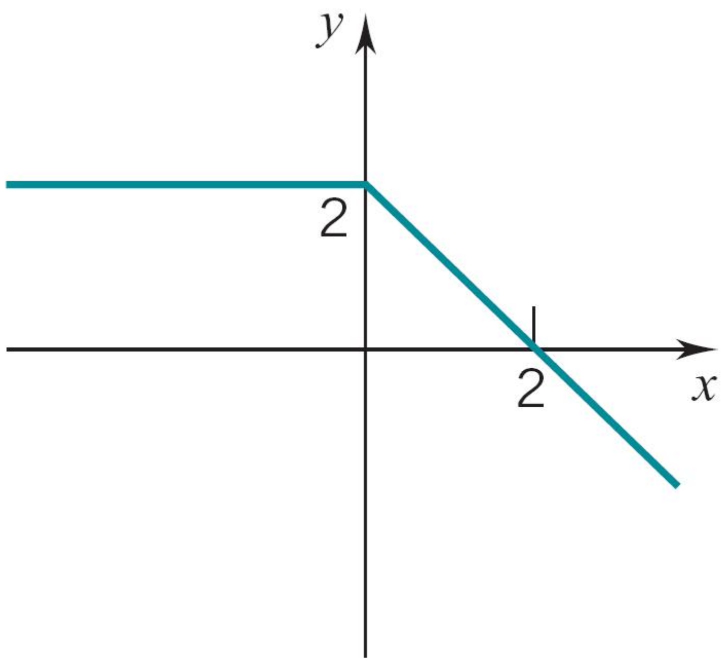
*Examples* 16.

$$f(x) = -x, \quad f'(x) = -1.$$




*Examples* 17.

$$f(x) = \begin{cases} 2, & x \leq 0, \\ -x+2, & x > 0. \end{cases} \quad f'(x) = \begin{cases} 0, & x < 0, \\ -1, & x > 0. \end{cases}$$

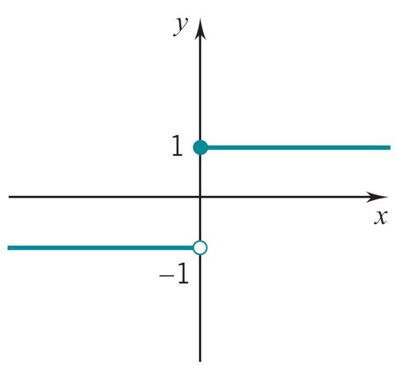



## 2.3 Nondifferentiability

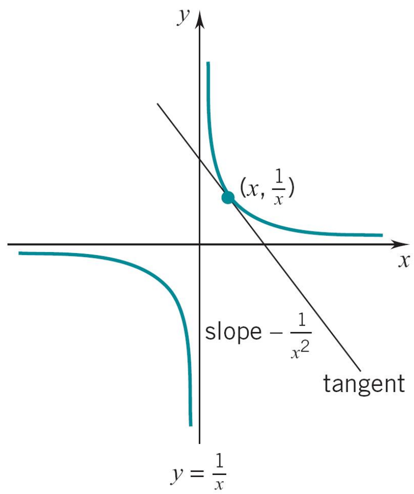
**Nondifferentiability: Discontinuity**
A function $f$ is *not differentiable* at $c$ if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \quad \text{does not exist.}$$

Jump discontinuity at 0 as $x \to 0^-$, $f'(x) \to \infty$; as $x \to 0^+$, $f'(x) \to 0$;



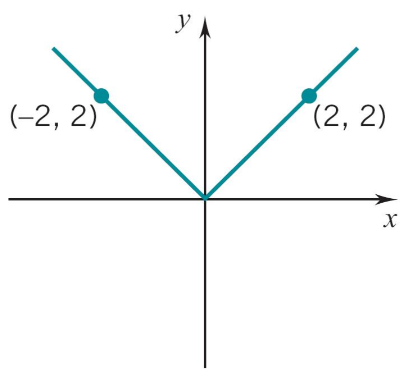
Infinite discontinuity at 0  as $x \to 0^-$, $f'(x) \to \infty$;  as $x \to 0^+$, $f'(x) \to \infty$;



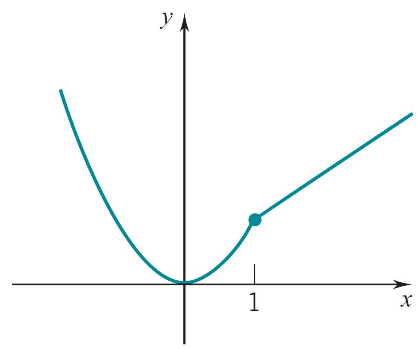

**Nondifferentiability: Corner Points**
A function $f$ is *not differentiable* at $c$ if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \quad \text{does not exist.}$$

Corner point at 0  as $x \to 0^-$, $f'(x) \to -1$;  as $x \to 0^+$, $f'(x) \to 1$;




Corner point at 1  as $x \to 1^-$, $f'(x) \to 2$;  as $x \to 1^+$, $f'(x) \to \frac{1}{2}$;

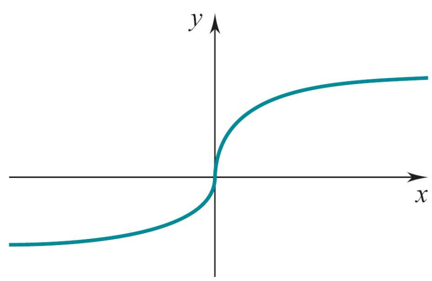



no derivative at 1

**Nondifferentiability: Vertical tangents**
A function $f$ is *not differentiable* at $c$ if

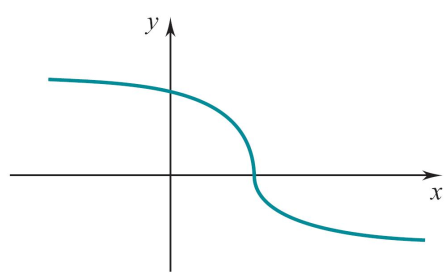
$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \quad \text{does not exist.}$$

vertical tangent at 0 as $x \to 0,\ f'(x) \to \infty$.




$f(x) = x^{1/3}$

vertical tangent at 2 as $x \to 2,\ f'(x) \to -\infty$.

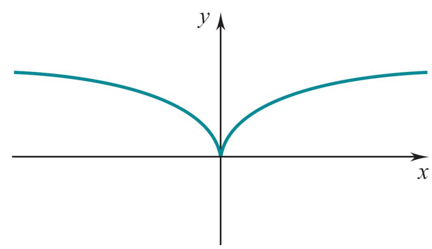



$g(x) = (2 - x)^{1/5}$

**Nondifferentiability: Vertical Cusps**
A function $f$ is *not differentiable* at $c$ if

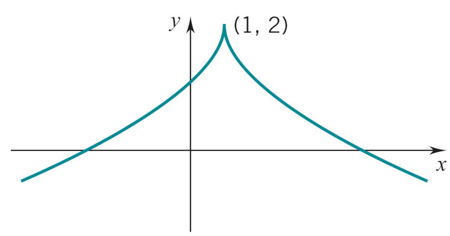
$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \quad \text{does not exist.}$$

vertical cusp at 0 as $x \to 0^-$, $f'(x) \to -\infty$; as $x \to 0^+$, $f'(x) \to \infty$.



$f(x) = x^{2/3}$

vertical cusp at 1 as $x \to 1^-$, $f'(x) \to \infty$; as $x \to 1^+$, $f'(x) \to +\infty$.



$g(x) = 2 - (x - 1)^{2/5}$