Lecture 17

Sections 5.1-5.4 Integration

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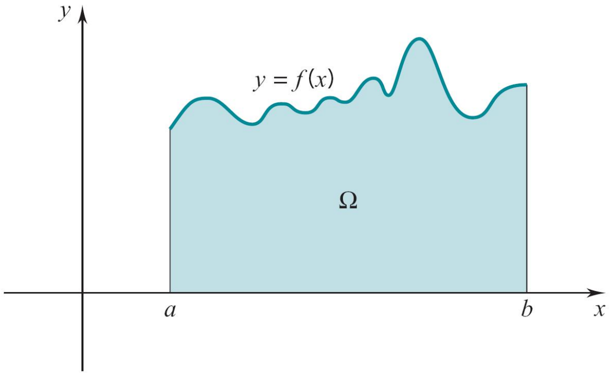
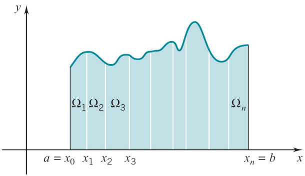
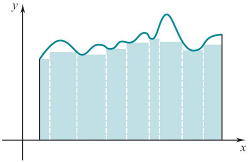
Area and Definite Integral



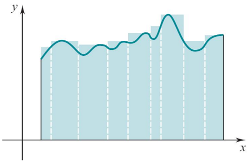
$$\text{Area of } \Omega = \int_a^b f(x)\,dx$$

Definite Integral and Lower/Upper Sums





area of shaded region is a lower sum for f



area of shaded region is an upper sum for f

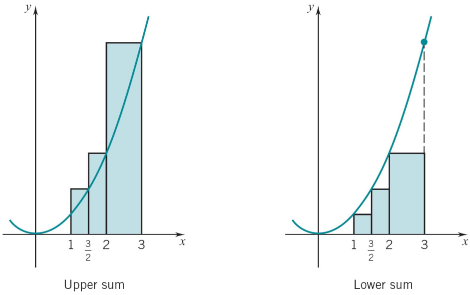
$$\text{Area of } \Omega = \text{Area of } \Omega_1 + \text{Area of } \Omega_2 + \cdots + \text{Area of } \Omega_n,$$

$$L_f(P) = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n$$

$$U_f(P) = M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n$$

$$L_f(P) \le \int_a^b f(x)\,dx \le U_f(P), \qquad \text{for all partitions } P \text{ of } [a,b]$$

Example: $\int_1^3 x^2\,dx$

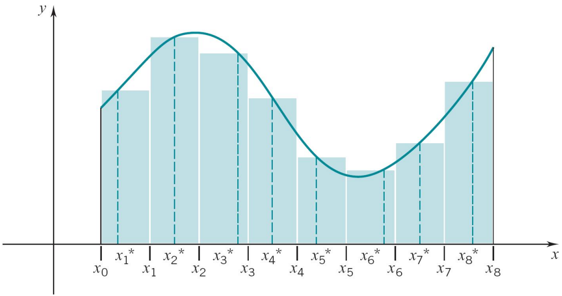


$$L_f(P) = 1\left(\frac{1}{2}\right) + \frac{9}{4}\left(\frac{1}{2}\right) + 4(1) = \frac{45}{8} \approx 5.625$$

$$U_f(P) = \frac{9}{4}\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) + 9(1) = \frac{97}{8} \approx 12.125$$

$$L_f(P) \le \int_1^3 x^2\,dx = \frac{1}{3}(3^3 - 1^3) = \frac{26}{3} \approx 8.667 \le U_f(P).$$

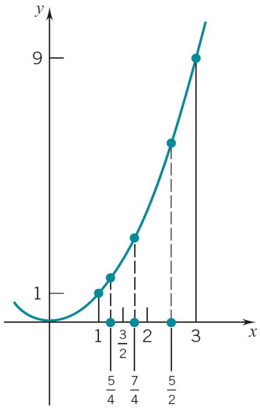
Lower/Upper Sums and Riemann Sums



$$S^*(P) = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

$$L_f(P) \leq S^*(P) \leq U_f(P), \qquad \text{for all partitions } P \text{ of } [a, b]$$

Example: $\int_1^3 x^2\,dx$



$$S^*(P) = f\left(\frac{5}{4}\right)\left(\frac{1}{2}\right) + f\left(\frac{7}{4}\right)\left(\frac{1}{2}\right) + f\left(\frac{5}{2}\right)(1)$$
$$= \frac{137}{16} \approx 8.563$$

$$L_f(P) \approx 5.625, \quad U_f(P) \approx 12.125,$$

$$\int_1^3 x^2\,dx = \frac{1}{3}\left(3^3 - 1^3\right) = \frac{26}{3} \approx 8.667.$$

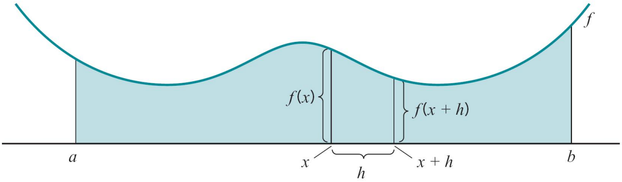
$$L_f(P) \le S^*(P) \approx \int_a^b f(x)\,dx \le U_f(P).$$

Definite Integral as the Limit of Riemann Sums

Theorem

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\Big[f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n\Big].$$

Definite Integral and Antiderivative



$F(x)$ = area from a to x and $F(x+h)$ = area from a to $x+h$.
Therefore $F(x+h) - F(x)$ = area from x to $x+h \cong f(x)\,h$ if h is small and

$$\frac{F(x+h)-F(x)}{h} \cong \frac{f(x)\,h}{h} = f(x).$$

Theorem

Let

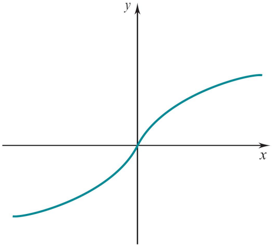
$$F(x) = \int_a^x f(t)\,dt.$$

Then

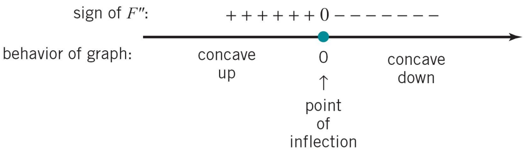
$$F'(x) = f(x) \quad \text{for all } x \text{ in } (a, b).$$

Example: $F(x) = \int_0^x \frac{1}{1+t^2}\,dt$

Sketch the graph of $F(x) = \displaystyle\int_0^x \frac{1}{1+t^2}\,dt$.



$$F'(x) = \frac{1}{1+x^2} > 0, \quad F''(x) = \frac{-2x}{(1+x^2)^2}.$$



Note that $F(x) = \displaystyle\int_0^x \frac{1}{1+t^2}\,dt = \tan^{-1} x$.

Fundamental Theorem of Integral Calculus

Theorem

In general,

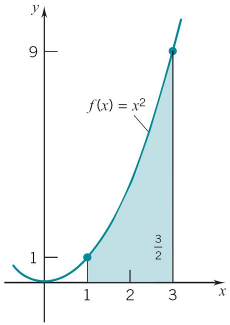
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

where $F(x)$ is an antiderivative of $f(x)$.

Function	Antiderivative
x^r	$\dfrac{x^{r+1}}{r+1}$ (r a rational number $\neq -1$)
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\csc^2 x$	$-\cot x$
$\csc x \cot x$	$-\csc x$

Example: $\int_1^3 x^2\,dx$

Evaluate $\displaystyle\int_1^3 x^2\,dx$.



Area of the shaded region: $\int_1^3 x^2\,dx = \frac{26}{3}$

Step 1. Get an antiderivative for $f(x) = x^2$:

$$F(x) = \frac{1}{3}x^3$$

Step 2.

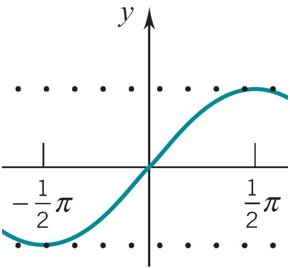
$$\int_1^3 x^2\,dx = F(3) - F(1)$$

$$= \frac{1}{3}(3)^3 - \frac{1}{3}(1)^3 = \frac{26}{3}.$$

$f(x) = x^2 > 0 \implies \displaystyle\int_1^3 x^2\,dx =$ area of the shaded region.

Example: $\int_{-\pi/2}^{\pi/2} \sin x \, dx$

Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x \, dx$.



$y = \sin x$

Step 1. Get an antiderivative for $f(x) = \sin x$:

$$F(x) = -\cos x$$

Step 2.

$$\int_{-\pi/2}^{\pi/2} \sin x \, dx = F(\pi/2) - F(-\pi/2)$$
$$= -\cos(\pi/2) - \left[-\cos(-\pi/2)\right] = 0.$$

$f(x) = \sin x \not\geq 0 \quad \Longrightarrow \quad \displaystyle\int_{-\pi/2}^{\pi/2} \sin x \, dx$ is not an area.

Further Properties of Integral Calculus

Theorem

1. $$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2. $$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$$

3. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

4. *If* $f \geq g$ *on* $[a, b]$, *then*
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Examples

Examples

Evaluate

1. $$\int_0^1 \left(2x - 6x^4 + 5\right)\,dx$$

2. $$\int_1^2 \frac{x^4+1}{x^2}\,dx$$

3. $$\int_0^1 \left(4-\sqrt{x}\right)^2\,dx$$

4. $$\int_0^{\pi/4} \sec x\left(2\tan x - 5\sec x\right)\,dx$$