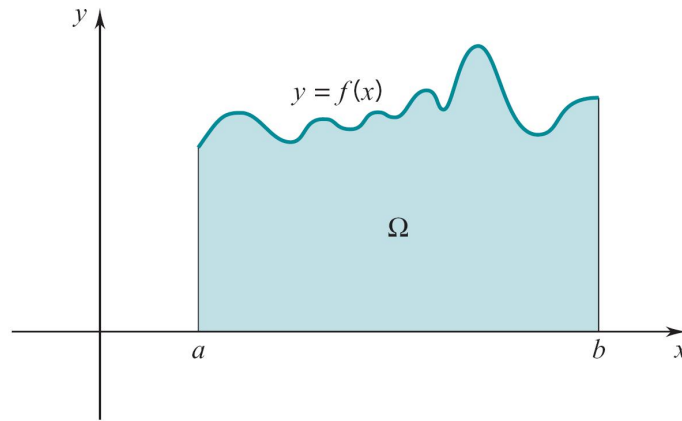


Lecture 17 Sections 5.1-5.4 Integration

Jiwen He

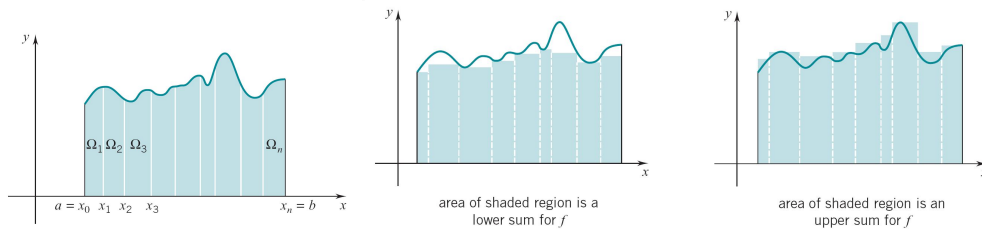
1 Sections 5.1 & 5.2 Area and Definite Integral

Area and Definite Integral



$$\text{Area of } \Omega = \int_a^b f(x) dx$$

Definite Integral and Lower/Upper Sums



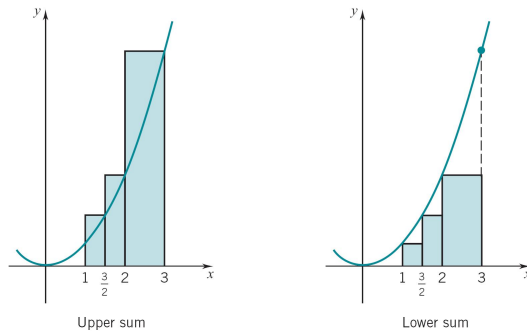
$$\text{Area of } \Omega = \text{Area of } \Omega_1 + \text{Area of } \Omega_2 + \dots + \text{Area of } \Omega_n,$$

$$L_f(P) = m_1 \Delta x_1 + m_2 \Delta x_2 + \dots + m_n \Delta x_n$$

$$U_f(P) = M_1 \Delta x_1 + M_2 \Delta x_2 + \dots + M_n \Delta x_n$$

$$L_f(P) \leq \int_a^b f(x) dx \leq U_f(P), \quad \text{for all partitions } P \text{ of } [a, b]$$

Example: $\int_1^3 x^2 dx$

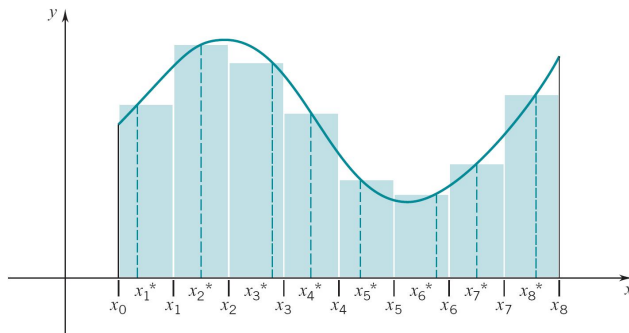


$$L_f(P) = 1\left(\frac{1}{2}\right) + \frac{9}{4}\left(\frac{1}{2}\right) + 4(1) = \frac{45}{8} \approx 5.625$$

$$U_f(P) = \frac{9}{4}\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) + 9(1) = \frac{97}{8} \approx 12.125$$

$$L_f(P) \leq \int_1^3 x^2 dx = \frac{1}{3}(3^3 - 1^3) = \frac{26}{3} \approx 8.667 \leq U_f(P).$$

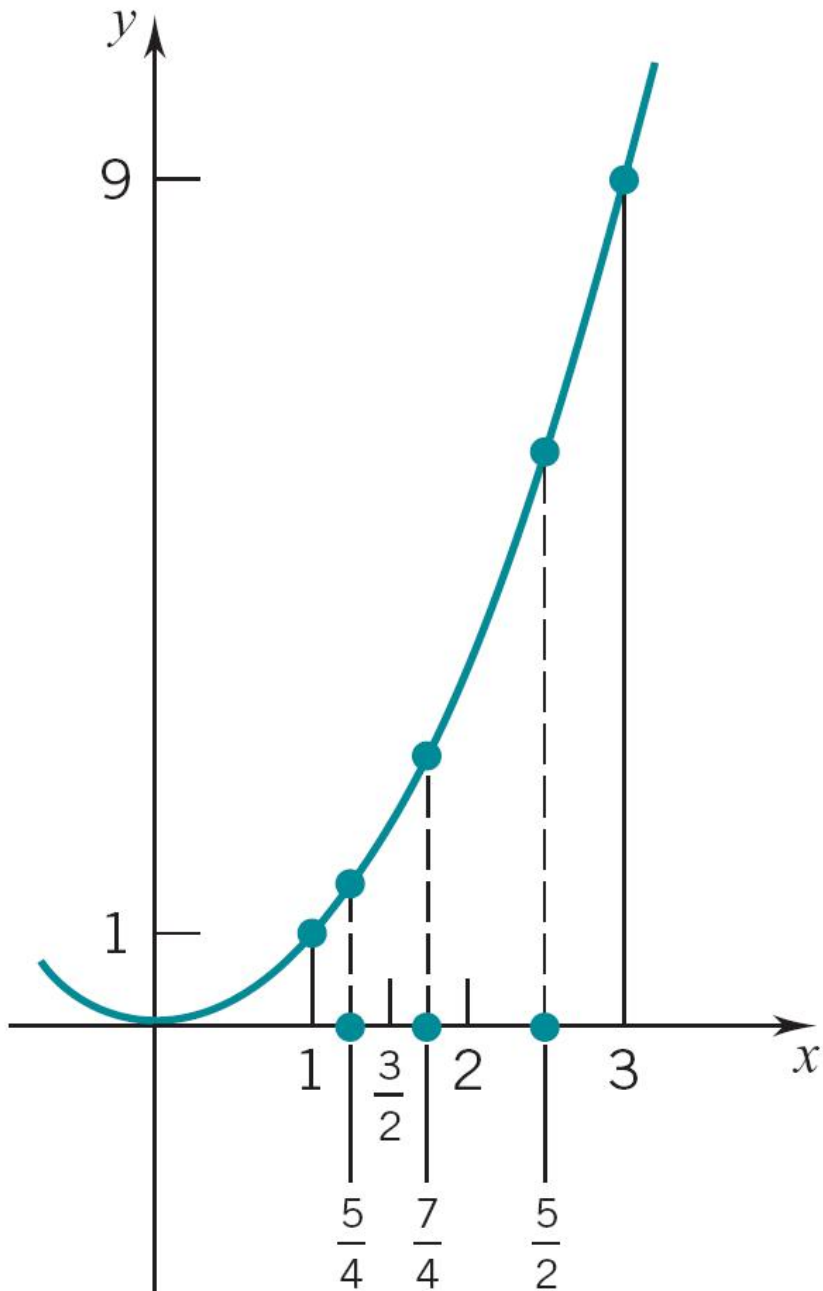
Lower/Upper Sums and Riemann Sums



$$S^*(P) = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

$$L_f(P) \leq S^*(P) \leq U_f(P), \quad \text{for all partitions } P \text{ of } [a, b]$$

Example: $\int_1^3 x^2 dx$



$$\begin{aligned}
 S^*(P) &= f\left(\frac{5}{4}\right)\left(\frac{1}{2}\right) + f\left(\frac{7}{4}\right)\left(\frac{1}{2}\right) + f\left(\frac{5}{2}\right)(1) \\
 &= \frac{137}{16} \approx 8.563
 \end{aligned}$$

$$L_f(P) \approx 5.625, \quad U_f(P) \approx 12.125,$$

$$\int_1^3 x^2 dx = \frac{1}{3}(3^3 - 1^3) = \frac{26}{3} \approx 8.667.$$

$$L_f(P) \leq S^*(P) \approx \int_a^b f(x) dx \leq U_f(P).$$

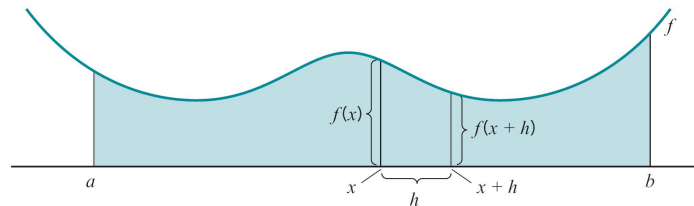
Definite Integral as the Limit of Riemann Sums

Theorem 1.

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \left[f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n \right].$$

2 Sections 5.3 & 5.4 Definite Integral and Antiderivative

Definite Integral and Antiderivative



$F(x)$ = area from a to x and $F(x+h)$ = area from a to $x+h$.
Therefore $F(x+h) - F(x)$ = area from x to $x+h \cong f(x)h$ if h is small and

$$\frac{F(x+h) - F(x)}{h} \cong \frac{f(x)h}{h} = f(x).$$

Theorem 2. *Let*

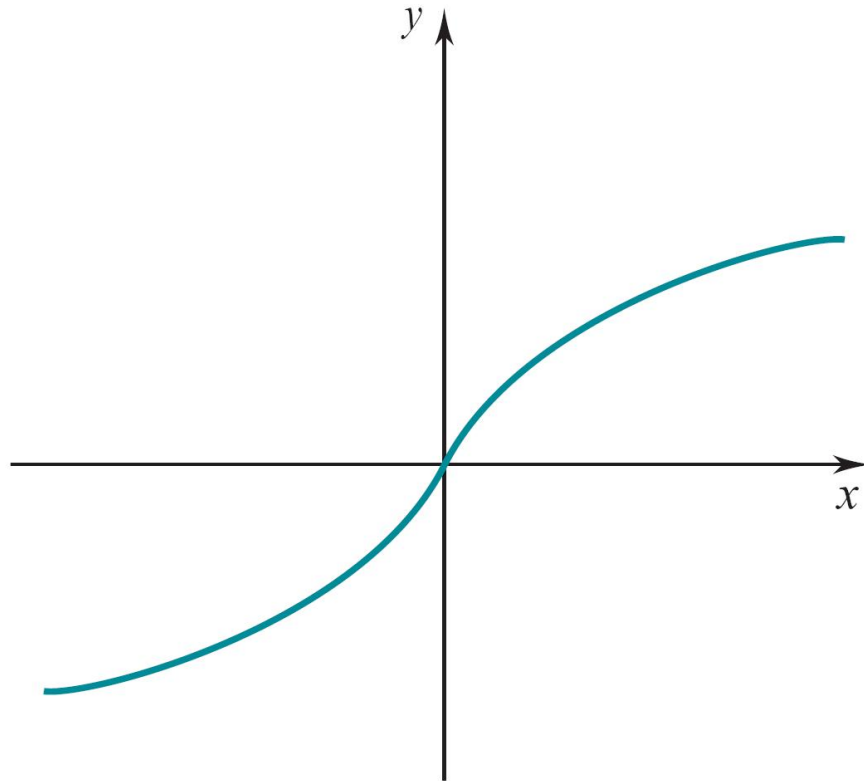
$$F(x) = \int_a^x f(t) dt.$$

Then

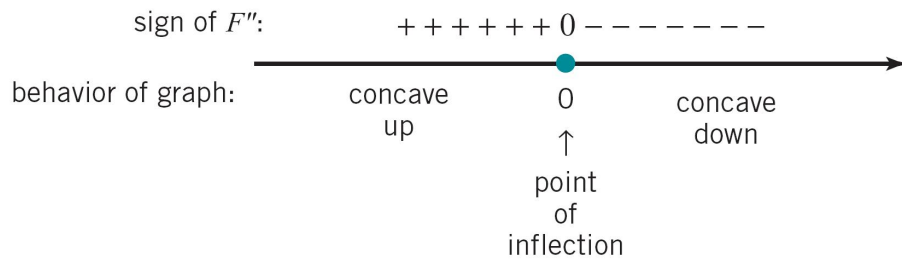
$$F'(x) = f(x) \quad \text{for all } x \text{ in } (a, b).$$

Example: $F(x) = \int_0^x \frac{1}{1+t^2} dt$

Sketch the graph of $F(x) = \int_0^x \frac{1}{1+t^2} dt$.



$$F'(x) = \frac{1}{1+x^2} > 0, \quad F''(x) = \frac{-2x}{(1+x^2)^2}.$$



Note that $F(x) = \int_0^x \frac{1}{1+t^2} dt = \tan^{-1} x$.

Fundamental Theorem of Integral Calculus

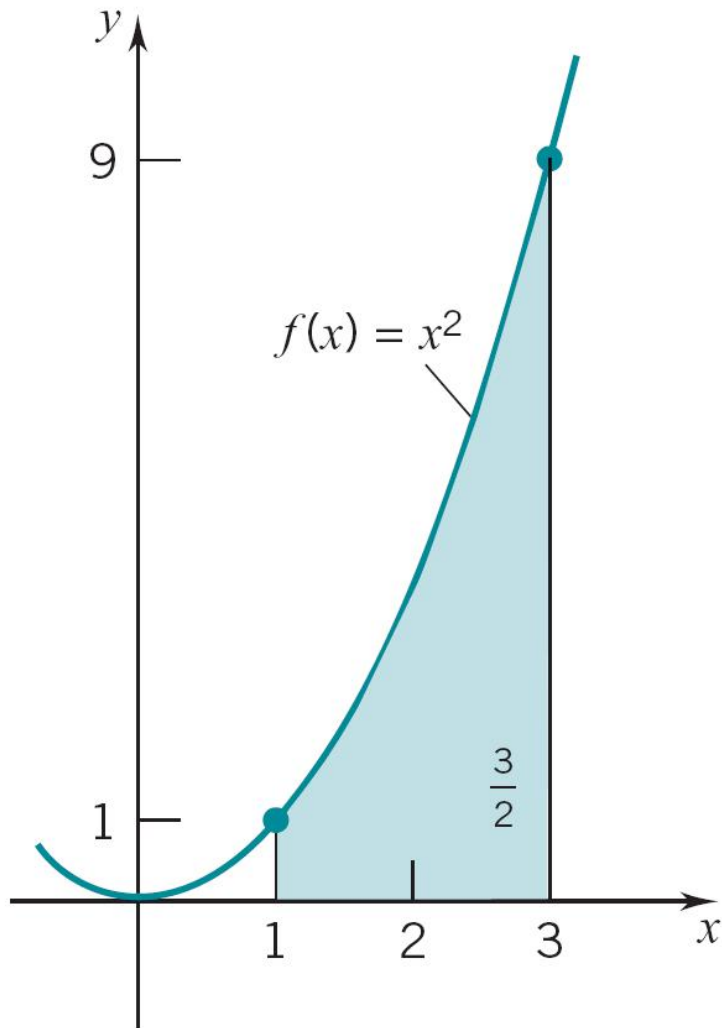
Theorem 3. *In general,*

$$\int_a^b f(x) dx = F(b) - F(a).$$

where $F(x)$ is an antiderivative of $f(x)$.

Function	Antiderivative
x^r	$\frac{x^{r+1}}{r+1}$ (r a rational number $\neq -1$)
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\csc^2 x$	$-\cot x$
$\csc x \cot x$	$-\csc x$

Example: $\int_1^3 x^2 dx$
Evaluate $\int_1^3 x^2 dx$.



Area of the shaded region: $\int_1^3 x^2 dx = \frac{26}{3}$

Step 1. Get an antiderivative for $f(x) = x^2$:

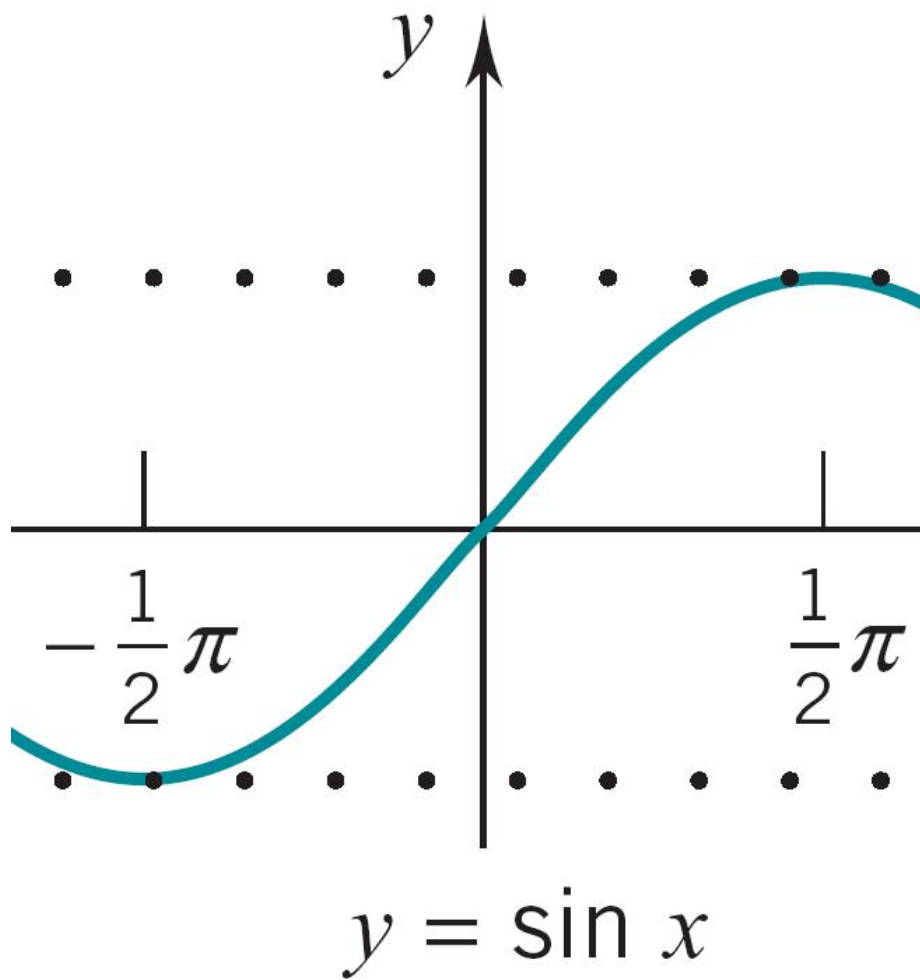
$$F(x) = \frac{1}{3}x^3$$

Step 2.

$$\begin{aligned}\int_1^3 x^2 dx &= F(3) - F(1) \\ &= \frac{1}{3}(3)^3 - \frac{1}{3}(1)^3 = \frac{26}{3}.\end{aligned}$$

$$f(x) = x^2 > 0 \implies \int_1^3 x^2 dx = \text{area of the shaded region.}$$

Example: $\int_{-\pi/2}^{\pi/2} \sin x dx$
Evaluate $\int_{-\pi/2}^{\pi/2} \sin x dx$.



Step 1. Get an antiderivative for $f(x) = \sin x$:

$$F(x) = -\cos x$$

Step 2.

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \sin x \, dx &= F(\pi/2) - F(-\pi/2) \\ &= -\cos(\pi/2) - [-\cos(-\pi/2)] = 0. \end{aligned}$$

$$f(x) = \sin x \not\geq 0 \implies \int_{-\pi/2}^{\pi/2} \sin x \, dx \text{ is not an area.}$$

Further Properties of Integral Calculus

Theorem 4.

1. $\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$
2. $\int_a^b \alpha f(x) \, dx = \alpha \int_a^b f(x) \, dx$
3. $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$
4. If $f \geq g$ on $[a, b]$, then
$$\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx$$

Examples

Examples 5. Evaluate

1. $\int_0^1 (2x - 6x^4 + 5) \, dx$
2. $\int_1^2 \frac{x^4 + 1}{x^2} \, dx$
3. $\int_0^1 (4 - \sqrt{x})^2 \, dx$
4. $\int_0^{\pi/4} \sec x (2 \tan x - 5 \sec x) \, dx$