Lecture 19

Section 5.6 Indefinite Integrals
Section 5.7 The u-Substitution

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Quiz 1

What is today?

a. Monday

b. Wednesday

c. Friday

d. None of these

Definite Integral

Fundamental Theorem of Integral Calculus

In general,

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b.$$

where $F(x)$ is an antiderivative of $f(x)$, i.e., $F'(x) = f(x)$.

Function	Antiderivative
x^r	$\dfrac{x^{r+1}}{r+1}$ (r a rational number $\neq -1$)
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\csc^2 x$	$-\cot x$
$\csc x \cot x$	$-\csc x$

Quiz 2

Give the value of $\int_{-1}^{2} (2x^2 - 3x - 5)\, dx$.

a. -12

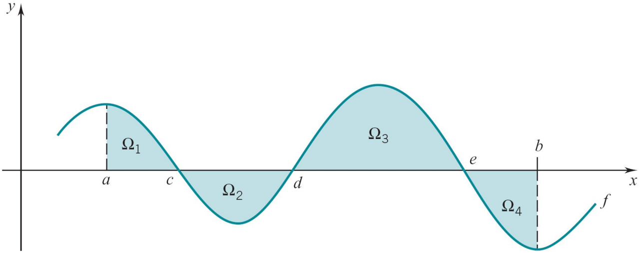
b. -13

c. $-\dfrac{27}{2}$

d. $-\dfrac{29}{2}$

e. None of these

Definite Integral as Signed Area



$$\int_a^b f(x)\,dx$$

$$= \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^e f(x)\,dx + \int_e^b f(x)\,dx$$

$$= \text{Area of } \Omega_1 + \left(-\text{Area of } \Omega_2\right) + \text{Area of } \Omega_3 + \left(-\text{Area of } \Omega_4\right)$$

$$= \left(\text{Area of } \Omega_1 + \text{Area of } \Omega_3\right) - \left(\text{Area of } \Omega_2 + \text{Area of } \Omega_4\right)$$

$$= \text{Area above the } x\text{-axis} - \text{Area below the } x\text{-axis}.$$

Quiz 4

The graph of $y = f(x)$ is shown below, where Ω_1 has area 2, Ω_2 has area 3, Ω_3 has area 8, and Ω_4 has area 4. Give $\int_a^b f(x)\,dx$.

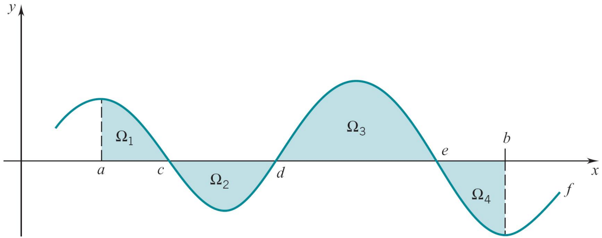
a. 1

b. 3

c. 12

d. 14

e. None of these



Quiz 5

The graph of $y = f(x)$ is shown below, where Ω_1 has area 2, Ω_2 has area 3, Ω_3 has area 8, and Ω_4 has area 4. Give $\int_a^e f(x)\,dx$.

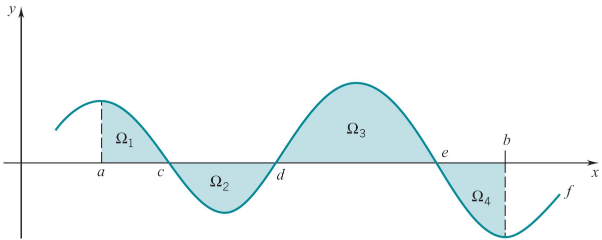
a. 1

b. 3

c. 12

d. 14

e. None of these



Quiz 6

The graph of $y = f(x)$ is shown below, where Ω_1 has area 2, Ω_2 has area 3, Ω_3 has area 8, and Ω_4 has area 4. Give the area bounded by the graph of f and the x-axis for $d \leq x \leq b$.

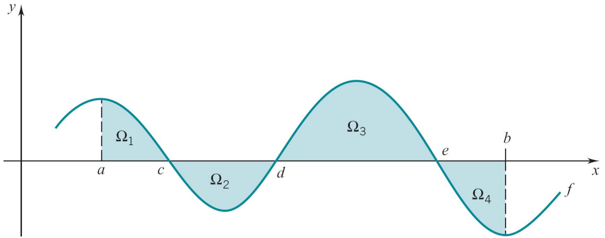
a. 1

b. 3

c. 12

d. 14

e. None of these



Indefinite Integral as General Antiderivative

The Indefinite Integral of f

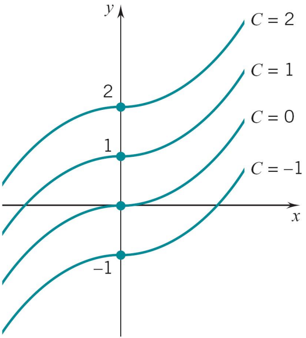
$$\int f(x)\,dx = \text{ The general anti-derivative of } f.$$

Example

$$\int x^2\,dx = \frac{1}{3}x^3 + C$$

where C is an arbitrary constant.

The indefinite integral f is a family of antiderivatives of f.



Indefinite Integral as General Antiderivative

The Indefinite Integral of f

In general,

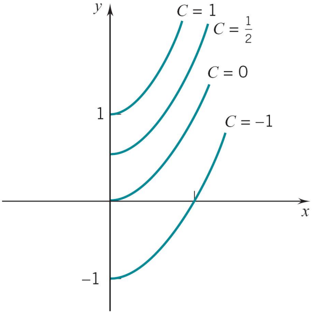
$$\int f(x)\,dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

Example

$$\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C.$$

The indefinite integral f is a family of antiderivatives of f.



Indefinite Integral as General Antiderivative

The Indefinite Integral of f

In general,

$$\int f(x)\,dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \qquad (r \text{ rational},\ r \neq -1)$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc^2 x\,dx = -\cot x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

Further Properties of Integral Calculus

Theorem

1. $$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

2. $$\int \alpha f(x)\,dx = \alpha \int f(x)\,dx, \qquad \alpha \text{ a constant}$$

In general,

$$\int (\alpha f(x) + \beta g(x))\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$$

where α and β are constants.

Example

Example

Calculate $\int \left(5x^{3/2} - 2\csc^2 x\right)\,dx$

Example

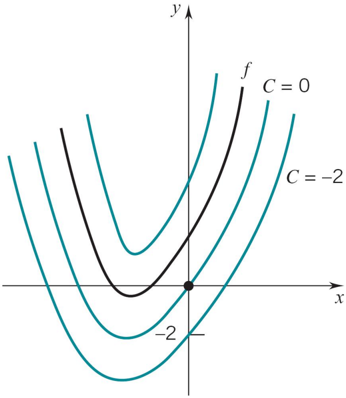
Example

Find f given that $f'(x) = x^3 + 2$ and $f(0) = 1$.

Note that, since f' is the derivative of f, f is an antiderivative for f':

$$f(x) = \int f'(x)\,dx$$

The constant of integration C can be evaluated using the fact that $f(0) = 1$.



Undoing the Chain Rule: The *u*-Substitution

Note: Most differentiation involves the chain rule, so we should expect that most antidifferentiation will involve

undoing the chain rule

(the *u*-Substitution)

If F is an antiderivative for f, then

$$\frac{d}{dx}\left[F(u(x))\right] = F'(u(x))\,u'(x) = f(u(x))\,u'(x)$$

$$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du = F(u) + C = F(u(x)) + C.$$

Undoing the Chain Rule

Undoing the Chain Rule

$F(x)$	$F'(x)$
$\frac{1}{n+1}\left[u(x)\right]^{n+1}$	$\left[u(x)\right]^{n} u'(x)$
$\sin\left[u(x)\right]$	$\cos\left[u(x)\right]\ u'(x)$
$\cos\left[u(x)\right]$	$-\sin\left[u(x)\right]\ u'(x)$
$\tan\left[u(x)\right]$	$\sec^2\left[u(x)\right]\ u'(x)$
$\cot\left[u(x)\right]$	$-\csc^2\left[u(x)\right]\ u'(x)$
$\sec\left[u(x)\right]$	$\sec\left[u(x)\right]\tan\left[u(x)\right]\ u'(x)$
$\csc\left[u(x)\right]$	$-\csc\left[u(x)\right]\cot\left[u(x)\right]\ u'(x)$

The u-Substitution

Original Integral	$u = g(x)$, $du = g'(x)dx$	New Integral
$\int [g(x)]^r g'(x)\, dx$	$\rightarrow$	$\int u^r du = \frac{u^{r+1}}{r+1} + C = \frac{[g(x)]^{r+1}}{r+1} + C \quad (r \neq -1)$
$\int \sin [g(x)]\, g'(x)\, dx$	$\rightarrow$	$\int \sin u\, du = -\cos u + C = -\cos [g(x)] + C$
$\int \cos [g(x)]\, g'(x)\, dx$	$\rightarrow$	$\int \cos u\, du = \sin u + C = \sin [g(x)] + C$
$\int \sec^2 [g(x)]\, g'(x)\, dx$	$\rightarrow$	$\int \sec^2 u\, du = \tan u + C = \tan [g(x)] + C$
$\int \sec [g(x)] \tan [g(x)] g'(x)\, dx$	$\rightarrow$	$\int \sec u\, \tan u\, du = \sec u + C = \sec [g(x)] + C$
$\int \csc^2 [g(x)]\, g'(x)\, dx$	$\rightarrow$	$\int \csc^2 u\, du = -\cot u + C = -\cot [g(x)] + C$
$\int \csc [g(x)] \cot [g(x)]\, g'(x)\, dx$	$\rightarrow$	$\int \csc u \cot u\, du = -\csc u + C = -\csc [g(x)] + C$

Examples

Examples

Calculate

1. $$\int (x^2 - 1)^4 2x \, dx$$
2. $$\int 3x^2 \cos(x^3 + 2) \, dx$$
3. $$\int \sin x \cos x \, dx$$
4. $$\int \frac{dx}{(3 + 5x)^2}$$

Examples

Examples

Calculate

5. $\displaystyle\int x^2\sqrt{4+x^3}\,dx$

6. $\displaystyle\int 2x^3\sec^2(x^4+1)\,dx$

7. $\displaystyle\int \sec^3 x\tan x\,dx$

8. $\displaystyle\int x(x-3)^5\,dx$

Substitution in Definite Integrals

The Change of Variables Formula

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

We change the limits of integration to reflect the substitution.

Examples

Examples

Evaluate

1. $$\int_0^2 (x^2 - 1)(x^3 - 3x + 2)^3\, dx$$

2. $$\int_0^{1/2} \cos^3 \pi x \sin \pi x\, dx$$

3. $$\int_0^{\sqrt{3}} x^5 \sqrt{x^2 + 1}\, dx$$