Lecture 19 Section 5.6 Indefinite Integrals Section 5.7 The u-Substitution

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Quiz 1
What is today?

a. Monday

b. Wednesday

c. Friday

d. None of these

Definite Integral

Fundamental Theorem of Integral Calculus
In general,

$$\int_a^b f(x)\,dx = \big[F(x)\big]_a^b.$$

where $F(x)$ is an antiderivative of $f(x)$, i.e., $F'(x) = f(x)$.

Function	Antiderivative
x^r	$\dfrac{x^{r+1}}{r+1}$ (r a rational number $\neq -1$)
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\csc^2 x$	$-\cot x$
$\csc x \cot x$	$-\csc x$

Quiz 2

Give the value of $\displaystyle\int_{-1}^{2} \left(2x^2 - 3x - 5\right) dx$.

a. -12

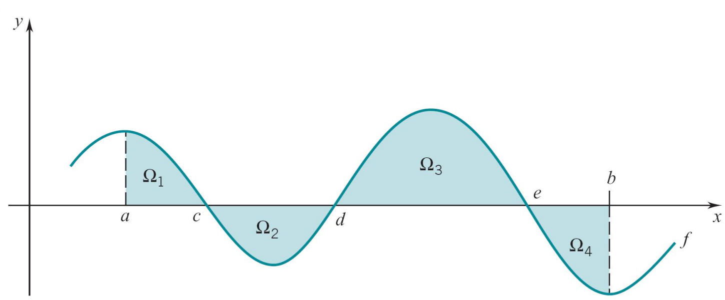
b. -13

c. $-\dfrac{27}{2}$

d. $-\dfrac{29}{2}$

e. None of these

Definite Integral as Signed Area



$$
\begin{aligned}
&\int_a^b f(x)\,dx \\
&= \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^e f(x)\,dx + \int_e^b f(x)\,dx \\
&= \text{Area of } \Omega_1 + \left(-\text{Area of } \Omega_2\right) + \text{Area of } \Omega_3 + \left(-\text{Area of } \Omega_4\right) \\
&= \left(\text{Area of } \Omega_1 + \text{Area of } \Omega_3\right) - \left(\text{Area of } \Omega_2 + \text{Area of } \Omega_4\right) \\
&= \text{Area above the } x\text{-axis} - \text{Area below the } x\text{-axis.}
\end{aligned}
$$

Quiz 4

The graph of $y = f(x)$ is shown below, where Ω_1 has area 2, Ω_2 has area 3, Ω_3 has area 8, and Ω_4 has area 4. Give $\displaystyle\int_a^b f(x)\,dx$.

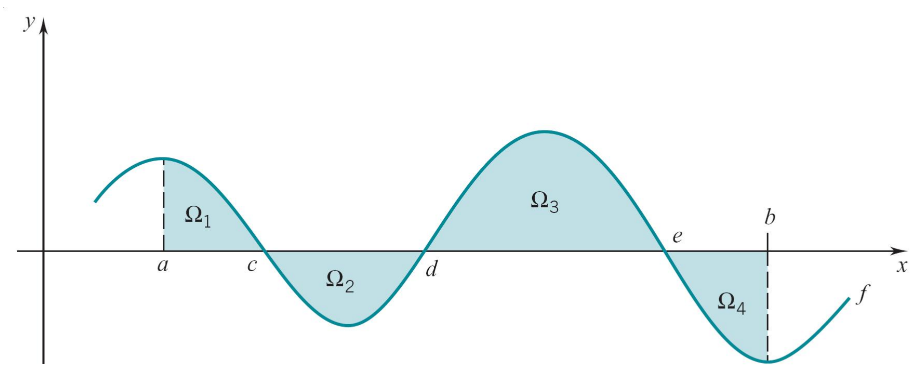
a. 1

b. 3

c. 12

d. 14

e. None of these



Quiz 5

The graph of $y = f(x)$ is shown below, where Ω_1 has area 2, Ω_2 has area 3, Ω_3 has area 8, and Ω_4 has area 4. Give $\int_a^e f(x)\,dx$.

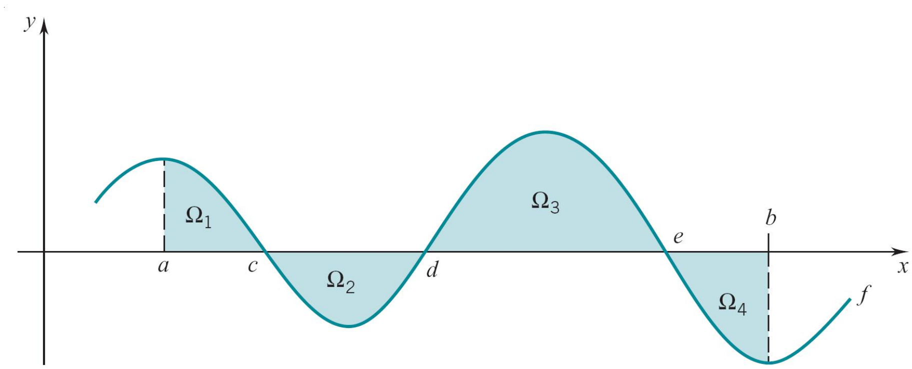
a. 1

b. 3

c. 12

d. 14

e. None of these



Quiz 6

The graph of $y = f(x)$ is shown below, where Ω_1 has area 2, Ω_2 has area 3, Ω_3 has area 8, and Ω_4 has area 4. Give the area bounded by the graph of f and the x-axis for $d \leq x \leq b$.

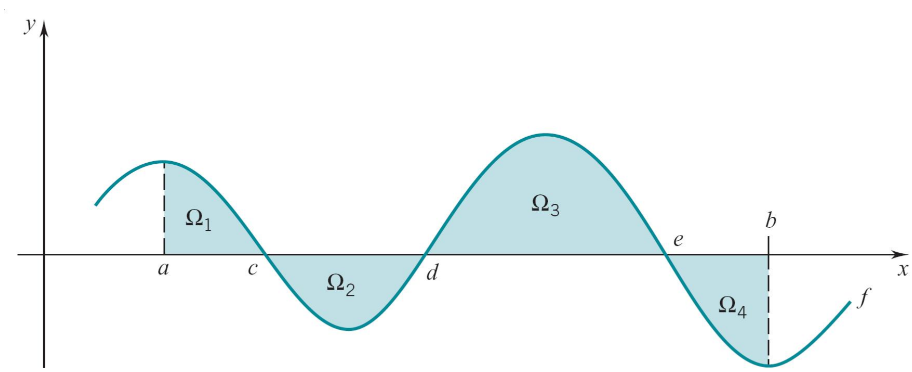
a. 1

b. 3

c. 12

d. 14

e. None of these



1 Section 5.6 Indefinite Integrals

Indefinite Integral as General Antiderivative

The Indefinite Integral of f

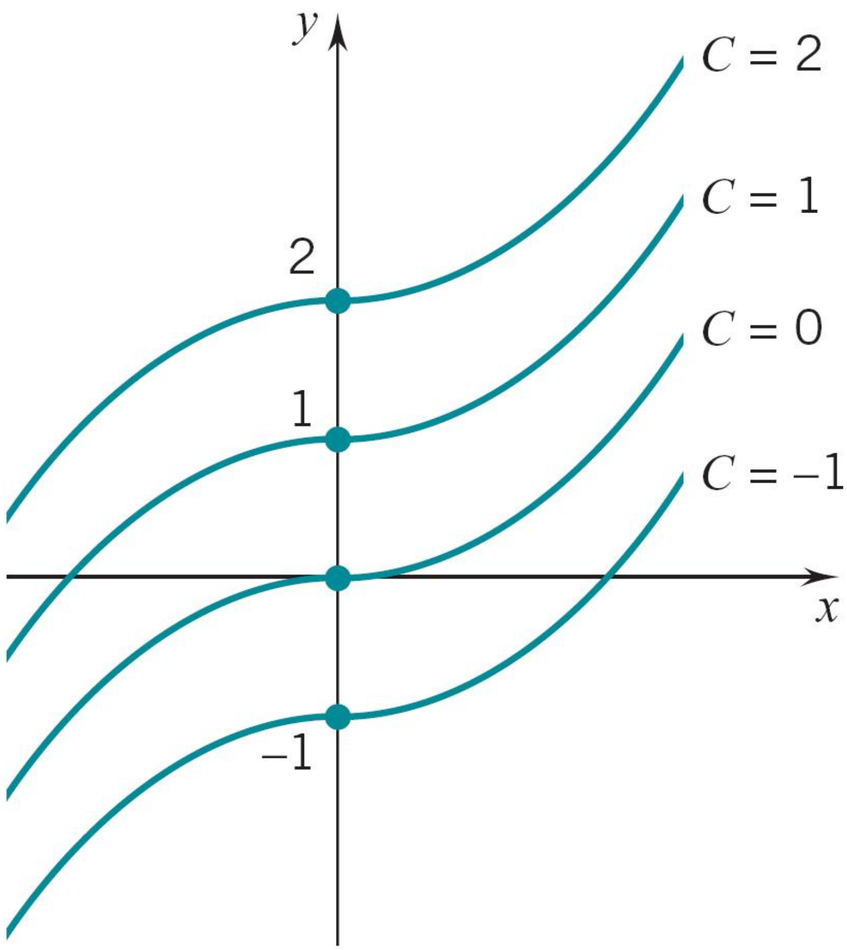
$$\int f(x)\,dx = \text{ The general anti-derivative of } f.$$

Example 1.

$$\int x^2\,dx = \frac{1}{3}x^3 + C$$

where C is an arbitrary constant.

The indefinite integral f is a family of antiderivatives of f.



Indefinite Integral as General Antiderivative

The Indefinite Integral of f
In general,

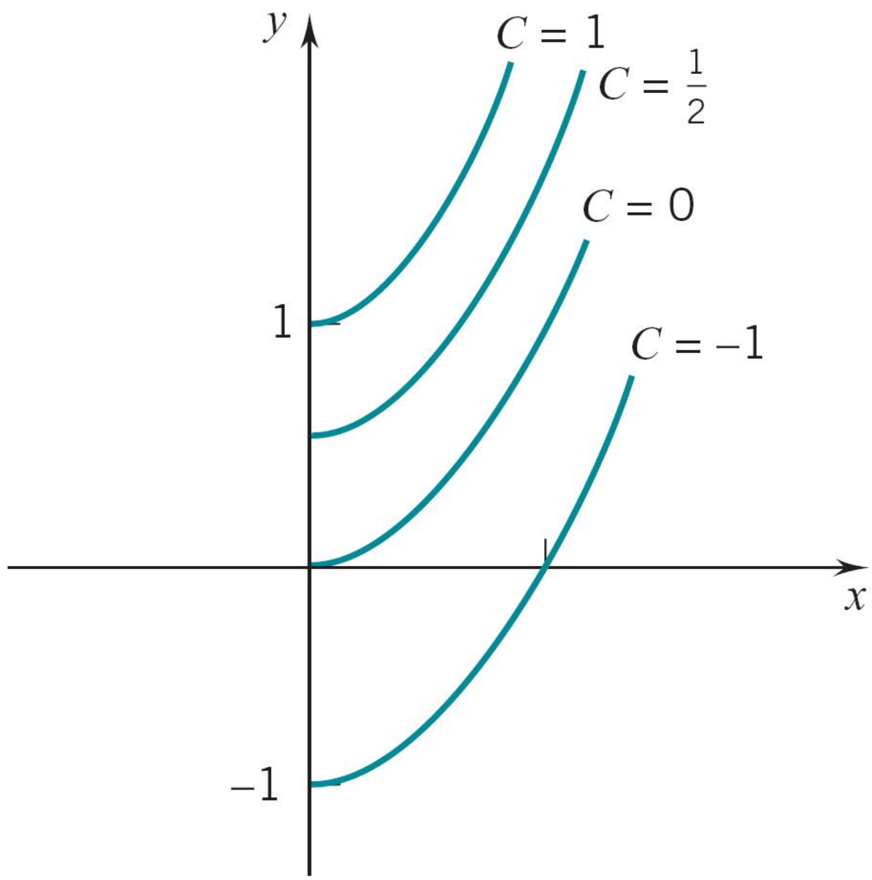
$$\int f(x)\,dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

Example 2.

$$\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C.$$

The indefinite integral f is a family of antiderivatives of f.



Indefinite Integral as General Antiderivative

The Indefinite Integral of f
In general,

$$\int f(x)\,dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \qquad (r \text{ rational},\ r \neq -1)$	
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \sec x \tan x\,dx = \sec x + C$
$\int \csc^2 x\,dx = -\cot x + C$	$\int \csc x \cot x\,dx = -\csc x + C$

Further Properties of Integral Calculus

Theorem 3.

1. $$\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

2. $$\int \alpha f(x)\,dx = \alpha \int f(x)\,dx, \qquad \alpha \textit{ a constant}$$

In general,

$$\int \big(\alpha f(x) + \beta g(x)\big)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$$

where α and β are constants.

Example

Example 4. Calculate $\int \big(5x^{3/2} - 2\csc^2 x\big)\,dx$

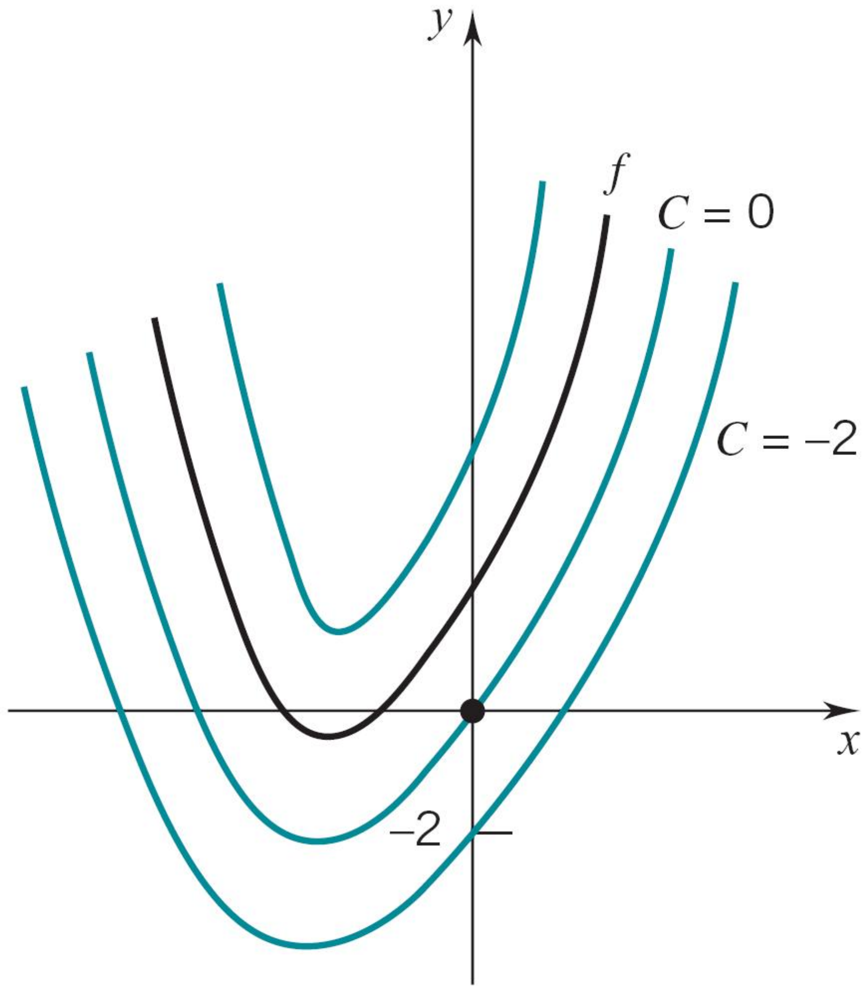
Example

Example 5. Find f given that $f'(x) = x^3 + 2$ and $f(0) = 1$.

Note that, since f' is the derivative of f, f is an antiderivative for f':

$$f(x) = \int f'(x)\,dx$$

The constant of integration C can be evaluated using the fact that $f(0) = 1$.



2 Section 5.7 The u-Substitution

Undoing the Chain Rule: The u-Substitution

Note: Most differentiation involves the chain rule, so we should expect that most antidifferentiation will involve

undoing the chain rule [1ex] (*the u-Substitution*)

If F is an antiderivative for f, then

$$\frac{d}{dx}\big[F(u(x))\big] = F'(u(x))\,u'(x) = f(u(x))\,u'(x)$$

$$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du = F(u) + C = F(u(x)) + C.$$

Undoing the Chain Rule
Undoing the Chain Rule

$F(x)$	$F'(x)$
$\frac{1}{n+1}\big[u(x)\big]^{n+1}$	$\big[u(x)\big]^n\, u'(x)$
$\sin\big[u(x)\big]$	$\cos\big[u(x)\big]\; u'(x)$
$\cos\big[u(x)\big]$	$-\sin\big[u(x)\big]\; u'(x)$
$\tan\big[u(x)\big]$	$\sec^2\big[u(x)\big]\; u'(x)$
$\cot\big[u(x)\big]$	$-\csc^2\big[u(x)\big]\; u'(x)$
$\sec\big[u(x)\big]$	$\sec\big[u(x)\big]\tan\big[u(x)\big]\; u'(x)$
$\csc\big[u(x)\big]$	$-\csc\big[u(x)\big]\cot\big[u(x)\big]\; u'(x)$

The u-Substitution

Original Integral	$u = g(x)$, $du = g'(x)dx$	New Integral
$\int [g(x)]^r g'(x)\,dx$	$\to$	$\int u^r du = \frac{u^{r+1}}{r+1} + C = \frac{[g(x)]^{r+1}}{r+1} + C \quad (r \neq -1)$
$\int \sin[g(x)]\,g'(x)\,dx$	$\to$	$\int \sin u\,du = -\cos u + C = -\cos[g(x)] + C$
$\int \cos[g(x)]\,g'(x)\,dx$	$\to$	$\int \cos u\,du = \sin u + C = \sin[g(x)] + C$
$\int \sec^2[g(x)]\,g'(x)\,dx$	$\to$	$\int \sec^2 u\,du = \tan u + C = \tan[g(x)] + C$
$\int \sec[g(x)]\tan[g(x)]g'(x)\,dx$	$\to$	$\int \sec u\ \tan u\,du = \sec u + C = \sec[g(x)] + C$
$\int \csc^2[g(x)]\,g'(x)\,dx$	$\to$	$\int \csc^2 u\,du = -\cot u + C = -\cot[g(x)] + C$
$\int \csc[g(x)]\cot[g(x)]\,g'(x)\,dx$	$\to$	$\int \csc u\cot u\,du = -\csc u + C = -\csc[g(x)] + C$

Examples
Examples 6. Calculate

1. $$\int (x^2-1)^4 2x\,dx$$
2. $$\int 3x^2\cos(x^3+2)\,dx$$
3. $$\int \sin x\cos x\,dx$$
4. $$\int \frac{dx}{(3+5x)^2}$$

Examples
Examples 7. Calculate

5. $$\int x^2\sqrt{4+x^3}\,dx$$
6. $$\int 2x^3\sec^2(x^4+1)\,dx$$
7. $$\int \sec^3 x\tan x\,dx$$
8. $$\int x(x-3)^5\,dx$$

Substitution in Definite Integrals

The Change of Variables Formula

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

We change the limits of integration to reflect the substitution.

Examples
Examples 8. Evaluate

1. $$\int_0^2 (x^2-1)(x^3-3x+2)^3\,dx$$
2. $$\int_0^{1/2} \cos^3\pi x\sin\pi x\,dx$$
3. $$\int_0^{\sqrt{3}} x^5\sqrt{x^2+1}\,dx$$