Lecture 20 Section 5.8 Properties of Definite Integral Section 5.9 Mean-Value Theorems for Integrals

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Test 3

- Tentative Dates for Test 3: Dec. 4-6 in CASA

Final Exam

- Final Exam: Dec. 14-17 in CASA

Review for Test 3

- Review for Test 3 by the College Success Program.
- Friday, November 21 2:30–3:30pm in the basement of the library by the C-site.

Help Session for Homework

- Homework Help Session by Prof. Morgan.
- Tonight 8:00 - 10:00pm in 100 SEC

Online Quizzes

- Online Quizzes are available on CourseWare.

Quiz 1
What is today?

a. Monday

b. Wednesday

c. Friday

d. None of these

1 Section 5.8 Additional Properties of the Definite Integral

Properties 1 and 2

$$1. \quad \int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$2. \quad \int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx, \qquad \alpha \text{ a constant}$$

In general,

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

where α and β are constants.

Quiz 2

Give the value of $\displaystyle\int_1^2 \frac{t^4+1}{t^2}\,dt$.

a. 2

b. 3

c. $\dfrac{17}{6}$

d. $\dfrac{21}{6}$

e. None of these

Property 3

$$3. \quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

for all choices of a, b and c from an interval on which f is continous.

$$\int_c^c f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Quiz 3

The graph of $y = f(x)$ is shown below, where Ω_1 has area 2, Ω_2 has area 3, Ω_3 has area 8, and Ω_4 has area 4. Give $\int_c^b f(x)\,dx$.

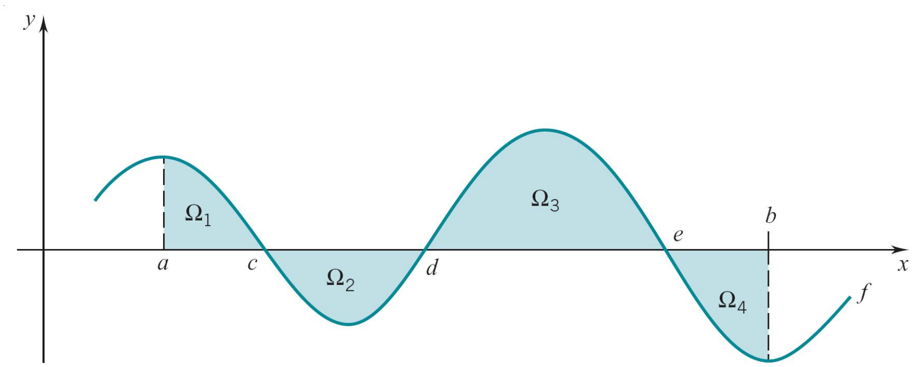
a. 1

b. 3

c. 12

d. 14

e. None of these



Quiz 4

The graph of $y = f(x)$ is shown below, where Ω_1 has area 2, Ω_2 has area 3, Ω_3 has area 8, and Ω_4 has area 4. Give $\int_d^c f(x)\,dx$.

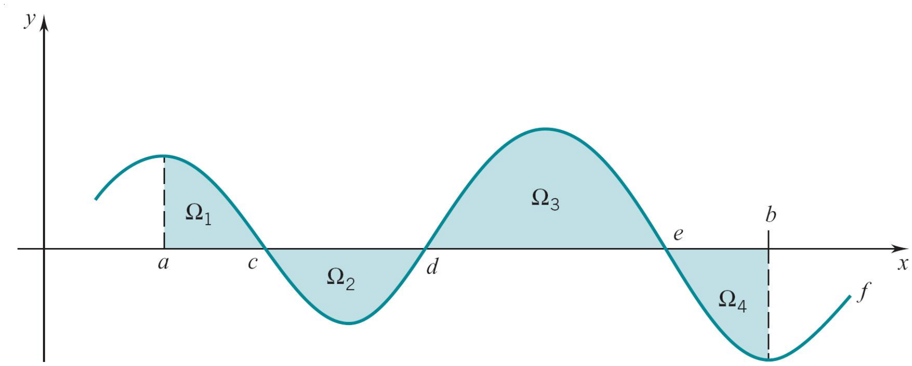
a. 1

b. 3

c. 12

d. 14

e. None of these



Properties 4 and 5

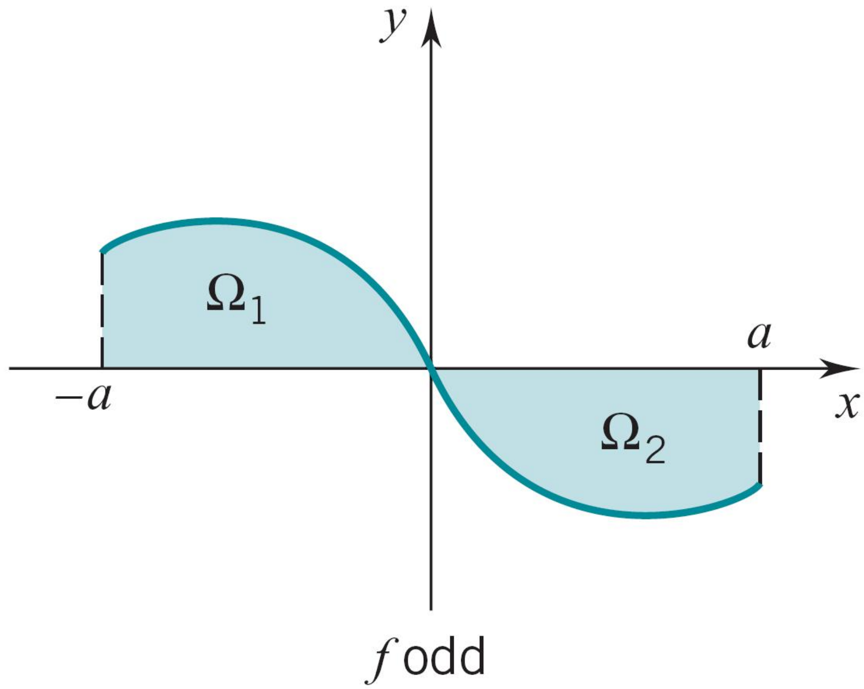
$$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(-x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \big[f(x) + f(-x)\big]\,dx$$

4. if f is odd on $[-a, a]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

5. if f is even on $[-a, a]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

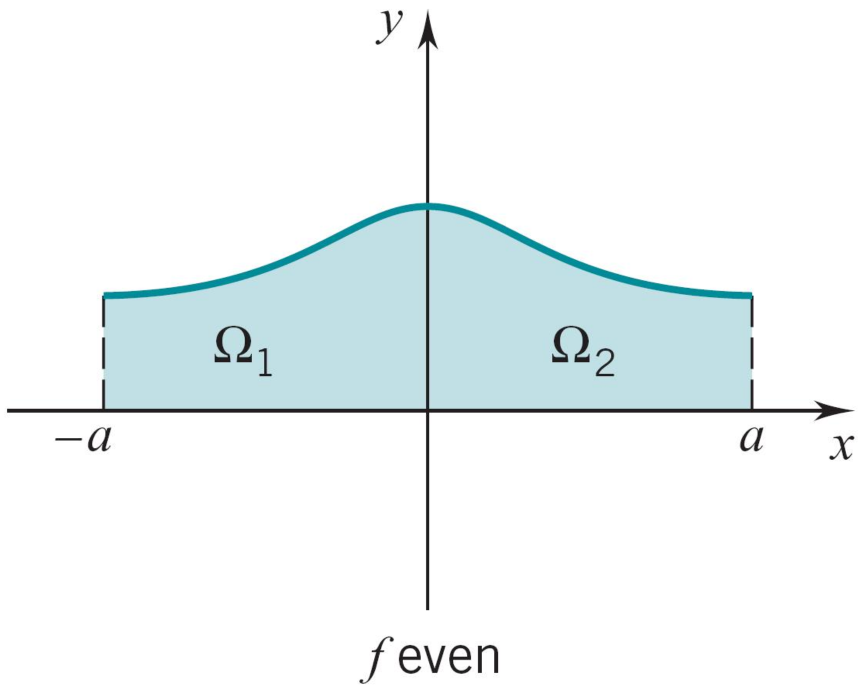
Definite Integrals of Odd and Even Functions



$$\int_{-a}^{a} f(x)\,dx$$

$$= \text{area of } \Omega_1 - \text{area of } \Omega_2$$

$$= 0.$$



$$\int_{-a}^{a} f(x)\,dx$$

$$= \text{ area of } \Omega_1 + \text{ area of } \Omega_2$$

$$= 2 \text{ area of } \Omega_2$$

$$= 2\int_{0}^{a} f(x)\,dx$$

Quiz 5

Give the value of $\displaystyle\int_{-\pi}^{\pi} (\sin x - x\cos x)^3\,dx$.

a. 0

b. π

c. 2π

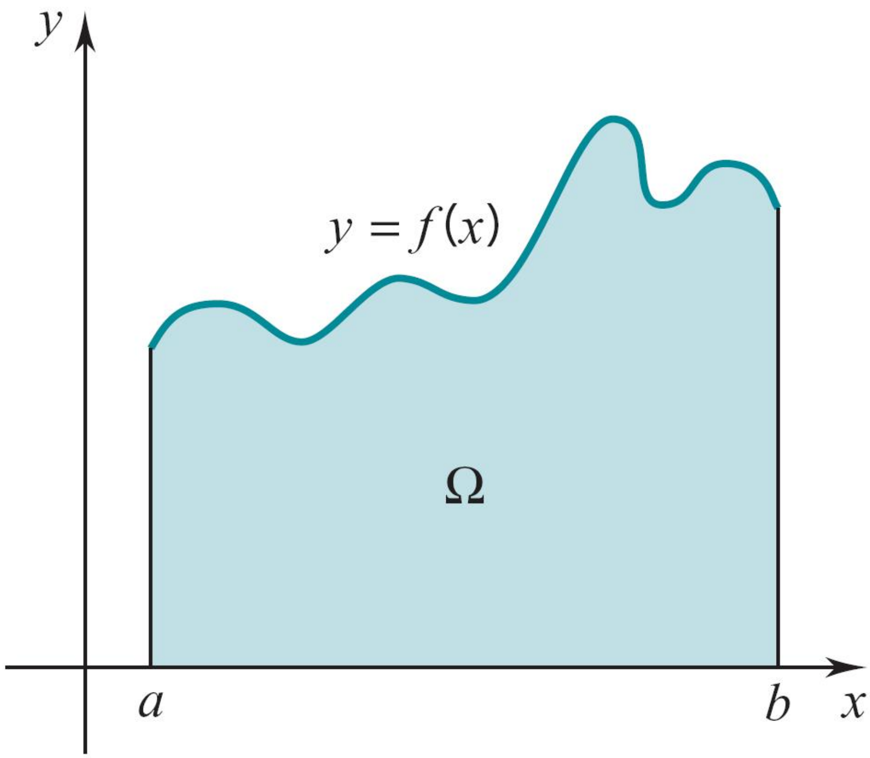
d. $-\pi$

e. None of these

Area below the graph of a Nonnegative f

$$f(x) \geq 0 \quad \text{for all } x \text{ in } [a, b].$$

$$\Omega = \text{ region below the graph of } f.$$



$$\text{Area of } \Omega = \int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

Properties 6 and 7

6. If $f \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.
7. If $f > 0$ on $[a, b]$, then $\int_a^b f(x)\,dx > 0$.

If $f \geq g$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

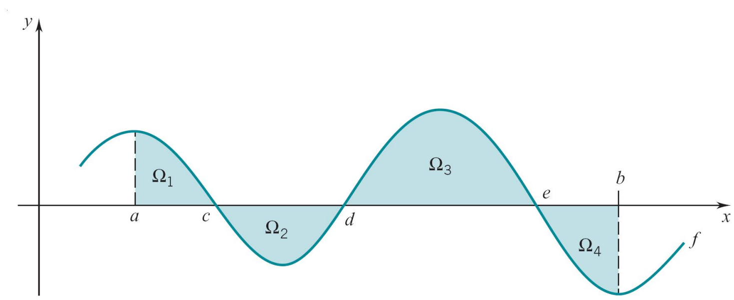
If $f > g$ on $[a,b]$, then $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$

$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$,

where m and M are the min and max of f on $[a,b]$ resp.

Property 8

$$8. \quad \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$



$\int_a^b f(x)\,dx =$ Area above the x-axis $-$ Area below the x-axis

$\int_a^b |f(x)|\,dx =$ Area above the x-axis $+$ Area below the x-axis

Quiz 6

The graph of $y = f(x)$ is shown below, where Ω_1 has area 2, Ω_2 has area 3, Ω_3 has area 8, and Ω_4 has area 4. Give $\int_d^b |f(x)|\,dx$.

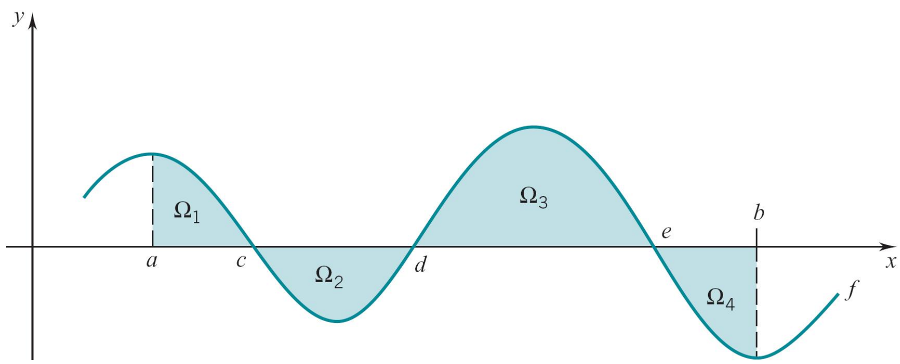
a. 1

b. 3

c. 12

d. 14

e. None of these



Quiz 7

The graph of $y = f(x)$ is shown below, where Ω_1 has area 2, Ω_2 has area 3, Ω_3 has area 8, and Ω_4 has area 4. Give the area bounded by the graph of f and the x-axis for $d \leq x \leq b$.

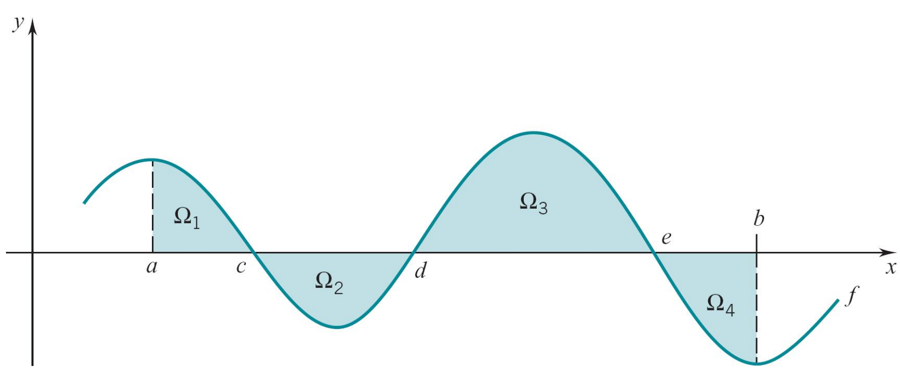
a. 1

b. 3

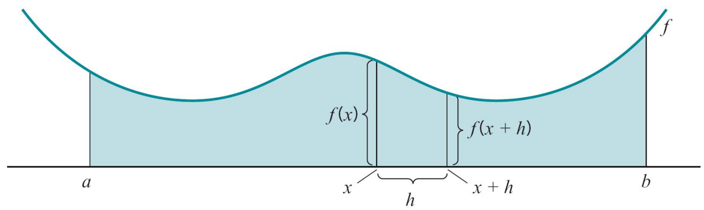
c. 12

d. 14

e. None of these



Definite Integral and Antiderivative



$F(x)$ = area from a to x and $F(x+h)$ = area from a to $x+h$.
Therefore $F(x+h) - F(x)$ = area from x to $x + h \cong f(x)\,h$ if h is small and

$$\frac{F(x+h)-F(x)}{h} \cong \frac{f(x)\,h}{h} = f(x).$$

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Examples

Examples 1. Find

1. $$\frac{d}{dx}\left(\int_0^x \frac{t}{(1+t^2)^2}\,dt\right)$$

2. $$\frac{d}{dx}\left(\int_{-3}^x \left(3t - \sin(t^2)\right)dt\right)$$

3. $$\frac{d}{dx}\left(\int_x^1 \left(3t - \sin(t^2)\right)dt\right)$$

4. $f(x)$ such that $\displaystyle\int_{-2}^x f(t)\,dt = \cos(2x) + 1.$

Property 9

9. $$\frac{d}{dx}\left(\int_a^u f(t)\,dt\right) = f(u)\frac{du}{dx}$$

$$\frac{d}{dx}\left(\int_v^b f(t)\,dt\right) = -f(v)\frac{dv}{dx}$$

$$\frac{d}{dx}\left(\int_v^u f(t)\,dt\right) = f(u)\frac{du}{dx} - f(v)\frac{dv}{dx}$$

Examples

Examples 2. Find

1. $$\frac{d}{dx}\left(\int_0^{x^3} \frac{dt}{1+t}\right)$$

2. $$\frac{d}{dx}\left(\int_{-3}^{x^2} \left(3t - \sin(t^2)\right) dt\right)$$

3. $$\frac{d}{dx}\left(\int_x^{2x} \frac{dt}{1+t^2}\right)$$

2 Section 5.9 Mean-Value Theorems for Integrals

Definite Integral and the Mean-Value

Mean-Value Theorems for Integrals: f = constant

$$\int_a^b f(x)\,dx = \text{ (the constant value of } f) \cdot (b-a).$$

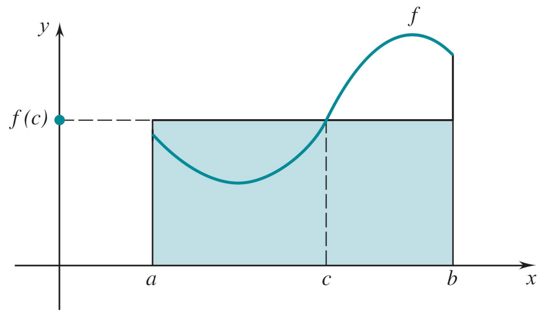
area = (the constant value of f)$\cdot$($b - a$)

Definite Integral and the Mean-Value

Mean-Value Theorems for Integrals

$$\int_a^b f(x)\,dx = \text{ (the average value of } f \text{ on } [a,b]) \cdot (b-a).$$

$$\text{area of } \Omega = \text{ (the average value of } f \text{ on } [a,b]) \cdot (b-a).$$

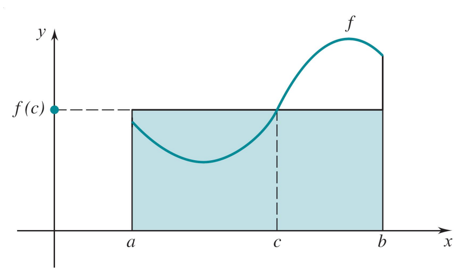


The Average value of f
Let f_{avg} denote the average or mean value of f on $[a, b]$. Then

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

The First Mean-Value Theorems for Integrals
If f is continous on $[a, b]$, then there is at least one number c in (a, b) for which

$$f(c) = f_{\text{avg}}.$$



Examples

Evaluate the average value of the function f on $[a, b]$:

1. $f(x) = k,\ k$ a constant.
2. $f(x) = x$
3. $f(x) = x^2$.

The Weighted Average value of f
Let $f_{g\text{-avg}}$ denote the g-weighted average of f on $[a, b]$, i.e,,

$$f_{g\text{-avg}} = \frac{1}{\int_a^b g(x)\,dx} \int_a^b f(x)g(x)\,dx$$

for g nonnegative.

The Second Mean-Value Theorems for Integrals
If f is continous on $[a, b]$, then there is at least one number c in (a, b) for which

$$f(c) = f_{g\text{-avg}}.$$

The center of mass x_M of a rod lying on the x-axis from a to b is

$$x_M = \frac{1}{\int_a^b \lambda(x)\,dx} \int_a^b x\,\lambda(x)\,dx, \qquad \lambda(x) \text{ the mass density.}$$