Review for Test 3

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1 Sections 10.4–10.6

1.1 Important Limits

Some Important Limits 1-4

$$\lim_{n\to\infty} \frac{1}{n^\alpha} = 0, \quad \alpha > 0.$$

$$\lim_{n\to\infty} x^{\frac{1}{n}} = 1, \quad x > 0.$$

$$\lim_{n\to\infty} x^n = 0 \quad \text{if } |x| < 1.$$

(The limit does not exist if $|x| > 1$ or $x = -1$.)

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Some Important Limits: 5-8

$$\lim_{n\to\infty} \frac{n^p}{x^n} = 0, \quad |x| > 1.$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$$

$$\lim_{n\to\infty} \frac{\ln n}{n^\alpha} = 0, \quad \alpha > 0.$$

Some Important Limits: 9-10

$$\lim_{n\to\infty} n\ln\left(1+\frac{x}{n}\right) = x.$$

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x.$$

1.2 Indeterminate Forms

L'Hôpital's Rule and Some Important Limits
L'Hôpital's Rule

Suppose that $\lim_{x\to\star} \frac{f(x)}{g(x)}$ is a $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then $\lim_{x\to\star} \frac{f(x)}{g(x)} = \lim_{x\to\star} \frac{f'(x)}{g'(x)}$

L'Hôpital's rule *does not apply* in cases where the numerator or the denominator has a finite non-zero limit!!!

$$\lim_{x\to 0^+} x^\alpha \ln x = 0, \quad \alpha > 0.$$

$$\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = 0, \quad \alpha > 0.$$

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

L'Hôpital's Rule and Some Important Limits
Exponential Forms: 1^∞, 0^0, and ∞^0

If $\lim_{x\to\star} \ln f(x) = L$, then $\lim_{x\to\star} f(x) = e^L$.

$$\lim_{x\to 0^+} x^x \ \left(0^0\right) \ = e^0 = 1$$

$$\lim_{x\to\infty} x^{1/x} \ \left(\infty^0\right) \ = e^0 = 1$$

$$\lim_{x\to 0^+} (1+x)^{1/x} \ \left(1^\infty\right) \ = e^1 = e$$

Peculiar Indeterminate Forms

Sometimes the application of L'Hôpital's rule leads to peculair situations.

$$\lim_{x\to\infty} \frac{2x}{\sqrt{1+x^2}} = \lim_{x\to\infty} \frac{2\sqrt{1+x^2}}{x} = \lim_{x\to\infty} \frac{2x}{\sqrt{1+x^2}}$$

Actually it is easy to calculate by algebraic methods.

$$\lim_{x\to\infty} \frac{2x}{\sqrt{1+x^2}} = \lim_{x\to\infty} \frac{2x}{x} \frac{1}{\sqrt{1+\left(\frac{1}{x}\right)^2}}$$

$$= 2\lim_{x\to\infty} \frac{1}{\sqrt{1+\left(\frac{1}{x}\right)^2}} == 2\lim_{u\to 0^+} \frac{1}{\sqrt{1+u^2}} = 2$$

1.3 Review Problems

Problem 1

Compute $\lim_{n\to\infty} (2n)^{\left(\frac{3}{n}\right)}$ [3ex] a) 0 [3ex] b) $\frac{2}{3}$ [3ex] c) Doesn't exist [3ex] d) 1 [3ex] e) $\frac{3}{2}$ [3ex] f) None of the above.

$$\lim_{u\to\infty} u^{\frac{1}{u}} = 1.$$

Set $u = 2n$.

$$\lim_{n\to\infty} (2n)^{\frac{1}{2n}} = \lim_{u\to\infty} u^{\frac{1}{u}} = 1.$$

Since $f(t) = t^6$ is continuous at 1,

$$\lim_{n\to\infty} (2n)^{\left(\frac{3}{n}\right)} = \lim_{n\to\infty} \left[(2n)^{\left(\frac{1}{2n}\right)}\right]^6 = 1$$

$$\lim_{n\to\infty} \frac{3}{n}\ln(2n) = \lim_{n\to\infty} \frac{3\ln 2}{n} + 3\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$\lim_{n\to\infty} (2n)^{\left(\frac{3}{n}\right)} = \lim_{n\to\infty} e^{\frac{3}{n}\ln(2n)} = 1$$

Problem 2

Compute $\lim_{n\to\infty} \left(\frac{\ln n^3}{\sqrt{n}}\right)$ [3ex] a) 1 [3ex] b) 2 [3ex] c) Doesn't exist [3ex] d) -1 [3ex] e) 0 [3ex] f) None of the above.

$$\lim_{n\to\infty} \frac{\ln n}{n^\alpha} = 0, \quad \alpha > 0.$$

$$\lim_{n\to\infty} \left(\frac{\ln n^3}{\sqrt{n}}\right) = 3\lim_{n\to\infty} \frac{\ln n}{n^{1/2}} = 0.$$

$$\lim_{x\to\infty} \left(\frac{\ln x^3}{\sqrt{x}}\right) = \lim_{x\to\infty} \left(\frac{\frac{1}{x^3}3x^2}{\frac{1}{2}x^{-1/2}}\right)$$

$$= \lim_{x\to\infty} \frac{6}{x^{1/2}} = 0$$

Problem 3

Compute $\lim_{x\to\infty}\left(\frac{\sqrt{x^2+3x+1}}{5x-1}\right)$ [3ex] a) 1 [3ex] b) $\frac{2}{5}$ [3ex] c) -2 [3ex] d) $\frac{1}{5}$ [3ex] e) $-\frac{1}{3}$ [3ex] f) None of the above.

The application of L'Hôpital's rule leads to peculair situations. Actually it is easy to calculate by algebraic methods.

$$\begin{aligned}
&\lim_{x\to\infty}\left(\frac{\sqrt{x^2+3x+1}}{5x-1}\right)\\
&= \lim_{x\to\infty}\left(\frac{x}{5x-1}\sqrt{1+3\left(\frac{1}{x}\right)+\left(\frac{1}{x}\right)^2}\right)\\
&= \lim_{x\to\infty}\frac{x}{5x-1}\cdot\lim_{x\to\infty}\sqrt{1+3\left(\frac{1}{x}\right)+\left(\frac{1}{x}\right)^2}\\
&= \frac{1}{5}
\end{aligned}$$

2 Section 10.7 Improper Integrals

2.1 Improper Integrals

Fundamental Theorem of Integral Calculus
Let $f(x) = F'(x)$ for $\forall x \geq a$. Then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx = \lim_{b\to\infty}\big(F(x)|_a^b\big) = \lim_{b\to\infty} F(b)-F(a) = F(x)\big|_a^\infty = \lim_{x\to\infty} F(x)-F(a)$$

Let $f(x) = F'(x)$ for $\forall x \leq b$. Then

$$\int_{-\infty}^b f(x)\,dx = F(x)\big|_{-\infty}^b = F(b) - \lim_{x\to-\infty} F(x)$$

Let $f(x) = F'(x)$ for $\forall x$. Then

$$\begin{aligned}
\int_{-\infty}^\infty f(x)\,dx &= \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx = F(x)\big|_{-\infty}^\infty = \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x)\\
&= \left(F(0) - \lim_{x\to-\infty} F(x)\right) + \left(\lim_{x\to\infty} F(x) - F(0)\right)
\end{aligned}$$

Examples

$$\int_1^\infty \frac{1}{x^3}\,dx = -\frac{2}{x^2}\bigg|_1^\infty = \lim_{x\to\infty}\left(-\frac{2}{x^2}\right) - (-2) = 2$$

$$\int_1^\infty \frac{1}{x}\,dx = \ln x|_1^\infty = \lim_{x\to\infty}(\ln x) - \ln 1 = \infty$$

$$\int_0^\infty e^{-2x}\,dx = -\frac{e^{-2x}}{2}\bigg|_0^\infty = \lim_{x\to\infty}\left(-\frac{e^{-2x}}{2}\right) - \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$\begin{aligned}\int_{-\infty}^\infty \frac{c}{c^2+x^2}\,dx &= \tan^{-1}\left(\frac{x}{c}\right)\bigg|_{-\infty}^\infty \\ &= \lim_{x\to\infty}\left(\tan^{-1}\left(\frac{x}{c}\right)\right) - \lim_{x\to-\infty}\left(\tan^{-1}\left(\frac{x}{c}\right)\right) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi\end{aligned}$$

Fundamental Theorem of Integral Calculus

Let $f(x) = F'(x)$ for $\forall x \in [a,b)$ and $f(x) \to \pm\infty$ as $x \to b^-$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx = \lim_{c\to b^-}\left(F(x)|_a^c\right) = \lim_{c\to b^-} F(c) - F(a) = F(x)\Big|_a^b = \lim_{x\to b^-} F(x) - F(a)$$

If $\lim_{x\to b^-} F(x) = \pm\infty$, then $\int_a^b f(x)\,dx$ *diverges.*

Let $f(x) = F'(x)$ for $\forall x \in (a,b]$ and $f(x) \to \pm\infty$ as $x \to a^+$. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - \lim_{x\to a^+} F(x)$$

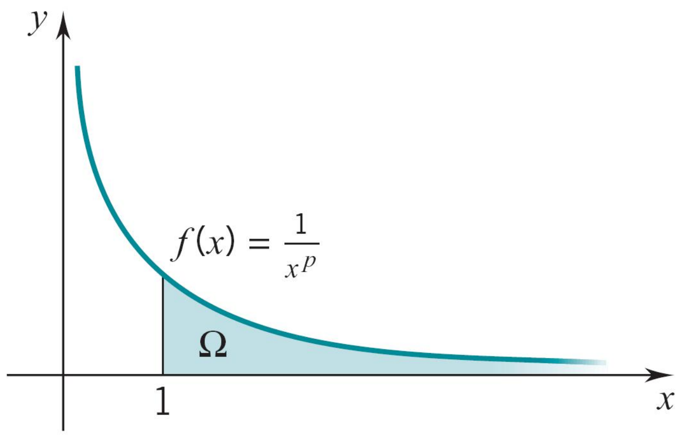
Examples

$$\int_0^1 \frac{1}{x^{1/3}}\,dx = \frac{3}{2}x^{2/3}\bigg|_0^1 = \frac{3}{2} - \frac{3}{2}\lim_{x\to 0^+}\left(x^{2/3}\right) = \frac{3}{2}$$

$$\int_0^1 \frac{1}{x}\,dx = \ln x|_0^1 = \ln 1 - \lim_{x\to 0^+}(\ln x) = \infty$$

$$\int_0^{\frac{\pi}{2}} \tan x\,dx = \ln\sec x|_0^{\frac{\pi}{2}} = \lim_{x\to\frac{\pi}{2}^-}(\ln\sec x) - \ln 1 = \infty$$

$\int_0^1 \frac{1}{x^\alpha}\,dx, \alpha > 0$



$$\int_0^1 \frac{1}{x^\alpha}\,dx = \begin{cases} \dfrac{1}{1-\alpha}, & \text{if } \alpha < 1, \\ \infty, & \text{if } \alpha \geq 1. \end{cases}$$

If $\alpha \neq 1$, then

$$\begin{aligned} \int_0^1 \frac{1}{x^\alpha}\,dx &= \frac{1}{1-\alpha} x^{1-\alpha}\bigg|_0^1 = \frac{1}{1-\alpha} - \frac{1}{1-\alpha}\lim_{x\to 0^+} x^{1-\alpha} \\ &= \begin{cases} \frac{1}{1-\alpha}, & \text{if } \alpha < 1, \\ \infty, & \text{if } \alpha > 1. \end{cases} \end{aligned}$$

If $\alpha = 1$, then

$$\int_0^1 \frac{1}{x}\,dx = \ln x|_0^1 = \ln 1 - \lim_{x\to 0^+}(\ln x) = \infty$$

2.2 Review Problems

Problem 1

- Explain why the definite integral below is an improper integral
$$\int_0^1 \frac{1}{(1-x)^3}\,dx$$
- Use proper limit notation to rewrite this improper integral as a limit of proper definite integrals.
- Solve the limit problem formed to determine whether the improper integral converges or diverges. If the improper integral converges, give its value.

$$\int_0^1 \frac{1}{(1-x)^3}\,dx = \int_0^1 \frac{1}{t^3}\,dt = \lim_{c\to 0^+}\int_c^1 \frac{1}{t^3}\,dt$$

$$= \lim_{c\to 0^+} -\frac{1}{2}t^{-2}\bigg|_c^1 = -\frac{1}{2} + \frac{1}{2}\lim_{c\to 0^+} c^{-2} = \infty$$

Problem 2

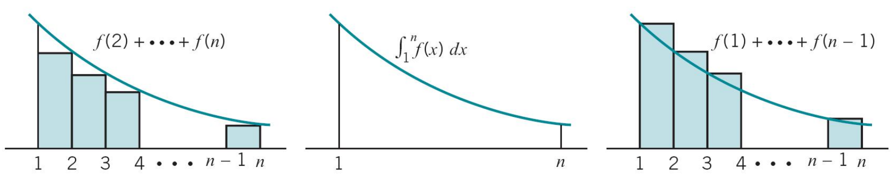
- Explain why the definite integral below is an improper integral
$$\int_0^3 \frac{3}{\sqrt{3-x}}\, dx$$
- Use proper limit notation to rewrite this improper integral as a limit of proper definite integrals.
- Solve the limit problem formed to determine whether the improper integral converges or diverges. If the improper integral converges, give its value.

$$\int_0^3 \frac{3}{\sqrt{3-x}}\, dx = \int_0^3 \frac{3}{t^{1/2}}\, dt = \lim_{c\to 0^+} \int_c^3 \frac{3}{t^{1/2}}\, dt$$
$$= \lim_{c\to 0^+} 6t^{1/2}\Big|_c^1 = 6 - 6\lim_{c\to 0^+} c^{1/2} = 6$$

3 Sections 11.1–11.3

3.1 The Integral Test

The Integral Test



Let $a_k = f(k)$, where f is continuous, *decreasing* and *positive* on $[1,\infty)$, then
$$\sum_{k=1}^{\infty} a_k \text{ converges} \quad \textit{iff} \quad \int_1^{\infty} f(x)dx \text{ converges}$$

(The p-Sereis)
$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \text{ converges} \quad \textit{iff} \quad p > 1.$$

Example

Show that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln k}$ diverges.

The related improper integral is
$$\int_2^{\infty} \frac{1}{x\ln x}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u}\, du = \ln u\Big|_{\ln 2}^{\infty} = \lim_{x\to\infty} \ln x - \ln\ln 2 = \infty.$$
Since this impproper integral *diverges*, so does the infinite series.

Show that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges.

The related improper integral is
$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u^2}\, du = -u^{-1}\Big|_{\ln 2}^{\infty} = \frac{1}{\ln 2} - \lim_{x\to\infty} x^{-1} = \frac{1}{\ln 2}.$$
Since this impproper integral *diverges*, so does the infinite series.

3.2 Convergence Tests

Basic Series that Converge or Diverge

Basic Series that Converge

$$\text{Geometric series:} \quad \sum x^k, \quad \text{if } |x| < 1$$
$$p\text{-series:} \quad \sum \frac{1}{k^p}, \quad \text{if } p > 1$$

Basic Series that Diverge

$$\text{Any series } \sum a_k \quad \text{for which } \lim_{k\to\infty} a_k \neq 0$$
$$p\text{-series:} \quad \sum \frac{1}{k^p}, \quad \text{if } p \leq 1$$

Convergence Tests (1)

Basic Test for Convergence
Keep in Mind that, if $a_k \nrightarrow 0$, then the series $\sum a_k$ *diverges*; therefore there is no reason to apply any special convergence test.

Examples 1. $\sum x^k$ with $|x| \geq 1$ (e.g, $\sum(-1)^k$) *diverge* since $x^k \nrightarrow 0$. [1ex] $\sum \dfrac{k}{k+1}$ *diverges* since $\frac{k}{k+1} \to 1 \neq 0$. [1ex] $\sum \left(1 - \dfrac{1}{k}\right)^k$ *diverges* since $a_k = \left(1 - \frac{1}{k}\right)^k \to e^{-1} \neq 0$.

Convergence Tests (2)

Comparison Tests
Rational terms are most easily handled by *basic comparison* or *limit comparison* with p-series $\sum 1/k^p$

Basic Comparison Test
$\sum \dfrac{1}{2k^3+1}$ converges by comparison with $\sum \dfrac{1}{k^3}$ $\sum \dfrac{k^3}{k^5+4k^4+7}$ converges by comparison with $\sum \dfrac{1}{k^2}$ $\sum \dfrac{1}{k^3-k^2}$ converges by comparison with $\sum \dfrac{2}{k^3}$ $\sum \dfrac{1}{3k+1}$ diverges by comparison with $\sum \dfrac{1}{3(k+1)}$ $\sum \dfrac{1}{\ln(k+6)}$ diverges by comparison with $\sum \dfrac{1}{k+6}$

Limit Comparison Test

$\sum \frac{1}{k^3-1}$ converges by comparison with $\sum \frac{1}{k^3}$. $\sum \frac{3k^2+2k+1}{k^3+1}$ diverges by comparison with $\sum \frac{3}{k}$ $\sum \frac{5\sqrt{k}+100}{2k^2\sqrt{k}-9\sqrt{k}}$ converges by comparison with $\sum \frac{5}{2k^2}$

Convergence Tests (3)

Root Test and Ratio Test
The *root test* is used only if *powers* are involved.

Root Test
$\sum \frac{k^2}{2^k}$ *converges*: $(a_k)^{1/k} = \frac{1}{2} \cdot \left[k^{1/k}\right]^2 \to \frac{1}{2} \cdot 1$ $\sum \frac{1}{(\ln k)^k}$ *converges*: $(a_k)^{1/k} = \frac{1}{\ln k} \to 0$ $\sum \left(1 - \frac{1}{k}\right)^{k^2}$ *converges*: $(a_k)^{1/k} = \left(1 + \frac{(-1)}{k}\right)^k \to e^{-1}$

Convergence Tests (4)
Root Test and Ratio Test
The *ratio test* is effective with *factorials* and with combinations of powers and factorials.

Ratio Comparison Test
$\sum \frac{k^2}{2^k}$ *converges*: $\frac{a_{k+1}}{a_k} = \frac{1}{2} \cdot \frac{(k+1)^2}{k^2} \to \frac{1}{2}$ $\sum \frac{1}{k!}$ *converges*: $\frac{a_{k+1}}{a_k} = \frac{1}{k+1} \to 0$
$\sum \frac{k}{10^k}$ *converges*: $\frac{a_{k+1}}{a_k} = \frac{1}{10} \cdot \frac{k+1}{k} \to \frac{1}{10}$ $\sum \frac{k^k}{k!}$ *diverges*: $\frac{a_{k+1}}{a_k} = \left(1 + \frac{1}{k}\right)^k \to e$
$\sum \frac{2^k}{3^k - 2^k}$ *converges*: $\frac{a_{k+1}}{a_k} = 2 \cdot \frac{1-(2/3)^k}{3-2(2/3)^k} \to 2 \cdot \frac{1}{3}$ $\sum \frac{1}{\sqrt{k!}}$ *converges*: $\frac{a_{k+1}}{a_k} = \sqrt{\frac{1}{k+1}} \to 0$

3.3 Absolute Convergence

Absolute Convergence
Absolute Convergence
A series $\sum a_k$ is said to *converge absolutely* if $\sum |a_k|$ converges.

Alternating p-Series with $p > 1$
$\sum \frac{(-1)^k}{k^p}$, $p > 1$, *converge absolutely* because $\sum \frac{1}{k^p}$ converges.

Conditional Convergence
A series $\sum a_k$ is said to *converge conditionally* if $\sum a_k$ converges while $\sum |a_k|$ diverges.

Alternating p-Series with $0 < p \leq 1$
$\sum \frac{(-1)^k}{k^p}$, $0 < p \leq 1$, *converge conditionally* because $\sum \frac{1}{k^p}$ diverges.

Examples

Alternating Series Test
Let $\{a_k\}$ be a *decreasing* sequence of *positive* numbers.

$$\text{If } a_k \to 0, \quad \text{then } \sum (-1)^k a_k \text{ converges.}$$

$\sum \frac{(-1)^k}{2k+1}$, *converge* since $f(x) = \frac{1}{2x+1}$ is *decreasing*, i.e., $f'(x) = -\frac{2}{(2x+1)^2} > 0$ for $\forall x > 0$, and $\lim_{x\to\infty} f(x) = 0$.

$\sum \frac{(-1)^k k}{k^2+10}$, *converge* since $f(x) = \frac{x}{x^2+10}$ is *decreasing*, i.e., $f'(x) = -\frac{x^2-10}{(x^2+10)^2} > 0$, for $\forall x > \sqrt{10}$, and $\lim_{x\to\infty} f(x) = 0$.

3.4 Review Problems

Problem 1

Determine $\sum_{n=0}^{\infty} \frac{2n(-1)^n}{3n^2+n+1}$ [3ex] a) converges conditionally [3ex] b) converges absolutely [3ex] c) diverges [3ex] d) None of the above.

$\sum \frac{2n(-1)^n}{3n^2+n+1}$, *converges* since [2ex] $f(x) = \frac{2x}{3x^2+x+1}$ is *decreasing*, i.e., [2ex] $f'(x) = -\frac{6x^2-2}{(3x^2+x+1)^2} < 0, \ \forall x > 1/\sqrt{3}$ [2ex] and $\lim_{x\to\infty} f(x) = 0$.

conditionally, since $\sum \frac{2n}{3n^2+n+1}$ diverges by comparison with $\sum \frac{2}{3n}$

Outline

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